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DETERMINATION OF THE CRITICAL VELOCITY OF A STRAIGHT WING WITH A HIGH ASPECT RATIO

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An aerodynamic problem on an air flow around a large aspect ratio rectangular wing is investigated in this study. According to the theory of Vlasov, the wing is considered to be a thin rod. External loads are assumed to be proportional to the airfoil angle of attack related to the dimensionless coefficient of the lift and the pitching moment coefficient. These coefficients depend on the airfoil parameters and the Mach number M and are determined by experimental measurements for subsonic and supersonic velocities. In this case, to define the unstable cases of the wing, one bases on the Lyapunov stability theory. Equations of bending and torsional free vibrations have resulted. Based on the analysis of natural frequencies (eigenfrequencies), it is possible to determine the changing positions of the real part and the imaginary part of the characteristic equation solution. These positions can cause instabilities for the wing such as torsional divergence and flutter.

Key words: thin rod, divergence, linear aerodynamics, straight wing, high aspect ratio, critical velocity.

1. Introduction

Aeroelasticity studies the effect of aerodynamic forces on elastic bodies [1, 2, 3]. The classical theory of aeroelasticity deals with the stress and deformation of an elastic body under the action of external forces, usually the oncoming air flow, and their magnitude is determined by the deformations of the body. Due to a given configuration of an elastic body, the aerodynamic forces rapidly increase while the flow velocity grows. So, there may exist a critical flow velocity at which the structure becomes unstable [4, 5]. Such instability can cause excessive deformation and can lead to structural failure [6]. Therefore, the determination of the critical velocity is important for such branches of technology as the construction of structures, hydrofoils, and especially aviation technology [7, 8]. For these cases, the structures of the fuselage, the shells of the combustion chambers of rocket engines, and the wings are elastic bodies interacting with the air flow.

The determination of the critical velocity has been investigated since the 30s of the last century and is still widely studied in many countries over the world such as in the USA, Russia, and some countries of the EU. There has been a lot of works related to this problem [9, 10, 11, 12, 13, 14]. The challenging issue is to construct and solve a system of equations of state for the flow and the related factors. In [15, 16], the so-called piston model of supersonic flow is proposed, and the considered object is a thin plate of variable thickness. In [17], the wing is represented by a rod of constant cross-section, and external loads are also determined by the piston theory.

In this paper, the Lyapunov stability theory is used to define the unstable cases of the wing according to the Mach number M. By combining Vlasov’s theory, a new model is obtained. Based on the acquired model, we can analyze the natural frequencies of the solution of the characteristic equation to study the instability of the wing. The current study focuses on:

- Developing a new mathematical model for the bending and torsional free vibrations.
- Using Laplace transformation to solve the above model.

- Defining unstable cases of the wing depending on the Mach number M based on analyzing the eigenfrequencies of the above solution.

Many modern unmanned aircrafts have deployable wings with a sufficiently high aspect ratio (approximately 5 or higher) [18]. They have an asymmetrical profile with a small relative thickness. In this article, we propose to use the equations of the theory of rods to determine the aerodynamic loads and to model the behavior of wings such as the dependency of the real and imaginary parts of the solution of the characteristic equation on the Mach number M and compare it to the experimental results obtained in [19]. Since these graphs are applied for both subsonic and supersonic velocities, the capabilities of the model can be expanded.

2. Thin rod theory

In the theory of rods of Vlasov, to formulate the problem, one needs to take into account the warping of sections [20, 21, 22]. There are the following hypotheses:

(1) A flat section that is perpendicular to the rod axis with the initial state not deformed in its plane. In other words, it retains the transverse dimensions and shape of the contour, the contour plane remains perpendicular to the deformed axis of the rod, and the section plane undergoes warping.

(2) In the above case, the system of displacements includes transverse displacements corresponding to the translational motion of the section plane as a rigid body and its rotation around the direction vector of the axis, and the longitudinal displacements are determined by rotations of the section as a rigid body around the main central axes of inertia of the cross-section.

These hypotheses lead to the following expressions for the displacements of points of the rod:

$$\begin{cases} u(x,y,z,t) = u(z,t) + \varphi(z,t) r \sin(\psi), \\ v(x,y,z,t) = v(z,t) + \varphi(z,t) r \cos(\psi), \\ r^2 = x^2 + y^2, \qquad \cos(\psi) = \dfrac{x}{r}, \qquad \sin(\psi) = \dfrac{y}{r}, \\ w(x,y,z,t) = w(z,t) - \theta_y(z,t)x - \theta_z(z,t)y + d(z,t)xy. \end{cases} \tag{2.1}$$

where $u(z,t)$, $v(z,t)$, $w(z,t)$ are components of the displacement vector of the center of mass of the section; $\theta_y(z,t), \theta_z(z,t)$ are rotation angles around the y and x axes; $\varphi(z,t)$ is an angle of twisting; r is the distance from a point of the section to its center of mass; $d(z,t)$ is a measure of warping. The warping function is adopted by Vlasov in the form of a product xy. The consequences are obtained from the kinematic hypothesis of Vlasov and the hypothesis of Bernoulli. The difference is that the warping measure $d(z,t)$ is unknown.

Because we only take into account the inertia of translational motion and rotation around the tangent to the axis, the mathematical model is formulated based on components of the displacement vector $u(z,t)$, $v(z,t)$, $w(z,t)$ and the angle of twist $\varphi(z,t)$. The system of differential equations has the following form:

$$\begin{cases} \dfrac{\partial^2 w}{\partial z^2} + \rho A \dfrac{\partial^2 w}{\partial t^2} = q_z, \\ \dfrac{\partial^4 u}{\partial z^4} + EJ_{x2y} \dfrac{J_x + J_y}{J_y - J_x} \dfrac{\partial^4 \varphi}{\partial z^4} + \rho A \dfrac{\partial^2 u}{\partial t^2} = q_x, \\ \dfrac{\partial^4 v}{\partial z^4} + EJ_{xy2} \dfrac{J_x + J_y}{J_y - J_x} \dfrac{\partial^4 \varphi}{\partial z^4} + \rho A \dfrac{\partial^2 v}{\partial t^2} = q_y, \\ EJ_\varphi \dfrac{\partial^4 \varphi}{\partial z^4} - G \dfrac{4 J_x J_y}{J_x + J_y} \dfrac{\partial^2 \varphi}{\partial z^2} - \rho \left(J_x + J_y \right) \dfrac{\partial^2 \varphi}{\partial t^2} = m_z(z,t), \\ J_{x2y} = \int\limits_A x^2 y dA; \quad J_{xy2} = \int\limits_A x y^2 dA; \quad J_\varphi = \int\limits_A x^2 y^2 dA; \quad J_d = \dfrac{4 J_x J_y}{J_x + J_y}. \end{cases} \tag{2.2}$$

where A is the cross-sectional area; J_x, J_y are the principal central moments of inertia of the section; q_x, q_y, q_z are components of the load that is distributed along the corresponding axes, $m_z(z,t)$ is the torsion torque distributed along the z axis; E is Young's modulus, G is shear modulus, ρ is the density of air. We can eliminate the warping measure $d(z,t)$ in the system of Eq.(2.2) by using Eq.(2.3):

$$G\left(J_y - J_x\right)\frac{\partial d}{\partial z} + G\left(J_x + J_y\right)\frac{\partial^2 \varphi}{\partial z^2} + m_z(z,t) = 0. \tag{2.3}$$

Equation (2.3) is the equilibrium condition for the torques, i.e., the projections of the principal moment onto the axis of the rod.

3. Unstable cases of the wing

3.1. The proposed load model

In aerodynamics [19], the lifting force Y and the pitching moment M_z can be expressed in terms of the dimensionless coefficients of the lift C_y and the pitching moment M_z. By the linear theory [23, 24], they can be represented by linear functions of the angle of attack of the airfoil α:

$$Y = C_y^\alpha \alpha \frac{\rho V^2}{2} S; \qquad M_z = M_z^\alpha \alpha \frac{\rho V^2}{2} b_A S. \tag{3.1}$$

where C_y^α, M_z^α are the derivatives of the lift coefficient and the pitching moment of the wing profile by α; V is the velocity of the undisturbed airflow; b_A is the mean aerodynamic chord of wing; S is the wing console area.

The derivatives of the coefficients $C_y^{\alpha}, M_z^{\alpha}$ mainly depend on the Mach number M and the shape of the wings in the plane. The shape of the wings is characterized for trapezoidal wings by the aspect ratio λ, the narrowing η, the relative thickness c, and the sweep angle χ. In this way,

$$C_y^{\alpha} = \mathrm{f}_1\left(M, \lambda, \eta, \chi, c\right), M_z^{\alpha} = \mathrm{f}_2\left(M, \lambda, \eta, \chi, c\right). \tag{3.2}$$

Note that for the wing rod model, the lift is the shear load that is applied to the center of mass of the airfoil, and the pitching moment is the torque distributed along the length.

The wings of unmanned aircraft are characterized by symmetrical profiles. Therefore, from the complete system of equations of the rod state (2.2), we use two equations of the fourth-order to simulate bending across the symmetric and torsional planes:

$$EJ_x \frac{\partial^4 v(z,t)}{\partial z^4} + \rho A \frac{\partial^2 v(z,t)}{\partial t^2} - \rho J_x \frac{\partial^4 v(z,t)}{\partial z^2 \partial t^2} = C_y^{\alpha}(M) \cdot \left(\alpha + \theta(z,t)\right) \cdot \frac{S}{L} \cdot \frac{\rho \cdot V^2}{2}, \tag{3.3}$$

$$\begin{gathered} EJ_{\varphi} \frac{\partial^4 \theta(z,t)}{\partial z^4} - GJ_d \frac{\partial^2 \theta(z,t)}{\partial z^2} - \rho J_{\omega} \frac{\partial^4 \theta(z,t)}{\partial z^2 \partial t^2} + rA \frac{\partial^2 \theta(z,t)}{\partial t^2} = \\ = M_z^{\alpha}(M)\left[\alpha + \theta(z,t)\right] \frac{Sb}{L} \frac{\rho V^2}{2} \end{gathered} \tag{3.4}$$

where $S = Lb$ is the wing console area, L – the console length, b – the side chord.

To transform torsional vibrations (3.4) to a dimensionless form, we introduce a dimensionless longitudinal coordinate $\zeta = \frac{z}{L}$ and divide the left and right sides of Eq. (3.4) by the product EJ_{φ}:

$$\begin{gathered} \frac{\partial^4 \theta(z,t)}{\partial \zeta^4} - \frac{GJ_d}{L^2} \frac{L^4}{EJ_{\varphi}} \frac{\partial^2 \theta(z,t)}{\partial \zeta^2} - \frac{\rho J_{\omega}}{L^2} \frac{L^4}{EJ_{\varphi}} \frac{\partial^4 \theta(z,t)}{\partial \zeta^2 \partial t^2} + \rho r^2 A \frac{L^4}{EJ_{\varphi}} \frac{\partial^2 \theta(z,t)}{\partial t^2} = \\ = M_z^{\alpha}(M) \cdot \left(\alpha + \theta(\zeta,t)\right) \frac{Sb}{L} \frac{\rho V^2}{2} \frac{L^4}{EJ_{\varphi}}. \end{gathered} \tag{3.5}$$

We have:

$$\frac{GJ_d}{L^2} \frac{L^4}{EJ_{\varphi}} = \frac{EJ_d}{2(1+v)L^2} \frac{L^4}{EJ_{\varphi}} = \frac{J_d L^2}{2J_{\varphi}(1+v)},$$

$$J_{d\varphi} = \frac{J_d L^2}{2J_{\varphi}(1+v)}, \qquad r^2 = \frac{J_x + J_y}{A}, \qquad L^2 \frac{J_x + J_y}{J_{\varphi}} = J_{xy\varphi},$$

$$\frac{Sb}{L} \frac{\rho V^2}{2} \frac{L^4}{EJ_{\varphi}} = \frac{Sb\rho L^3 V_s^2}{2EJ_{\varphi}} \left(\frac{V}{V_s}\right)^2 = \left[Let\ M = \frac{V}{V_s}, K_t = \frac{Sb\rho L^3 V_s^2}{2EJ_{\varphi}}\right] = K_t M^2$$

where V_s is the velocity of sound in the wing material, v is Poisson's ratio. Then, the torsion equation in a dimensionless form can be presented as follows:

$$\frac{\partial^4\theta(\zeta,\tau)}{\partial\zeta^4} - J_{d\varphi}\frac{\partial^2\theta(\zeta,\tau)}{\partial\zeta^2} - \frac{\partial^4\theta(\zeta,\tau)}{\partial\zeta^2\partial\tau^2} + J_{xy\varphi}\frac{\partial^2\theta(\zeta,\tau)}{\partial\tau^2} = M_z^{\alpha}(M)\big(\alpha+\theta(\zeta,\tau)\big)K_t M^2. \tag{3.6}$$

where

$$\tau = \frac{t}{T_1}.$$

Let the initial setup of the wing angle $\alpha = 0$. We consider the equation of free torsional vibrations in the case of harmonic vibrations:

$$\theta(\zeta,\tau) = \theta_a(\zeta)\cdot e^{i\Omega\tau}. \tag{3.7}$$

where Ω is the dimensionless frequency of free vibrations.
In this case, the equation of free torsional vibrations (3.6) will have the following form:

$$\frac{\partial^4\theta_a(\zeta)}{\partial\zeta^4} - J_{d\varphi}\frac{\partial^2\theta_a(\zeta)}{\partial\zeta^2} + \Omega^2\frac{\partial^2\theta_a(\zeta)}{\partial\zeta^2} - \Omega^2 J_{xy\varphi}\theta_a(\zeta) = M_z^{\alpha}(M)\theta_a(\zeta)K_t M^2. \tag{3.8}$$

3.2. Solution of the proposed load model

Now, we group similar terms and obtain:

$$\frac{\partial^4\theta_a(\zeta)}{\partial\zeta^4} + \left(\Omega^2 - J_{d\varphi}\right)\frac{\partial^2\theta_a(\zeta)}{\partial\zeta^2} - \left(M_z^{\alpha}(M)K_t M^2 + \Omega^2 J_{xy\varphi}\right)\theta_a(\zeta) = 0 \tag{3.9}$$

and denote

$$2a(\Omega) = \Omega^2 - J_{d\phi}, \quad b(\Omega, M) = M_z^{\alpha}(M)K_t M^2 + \Omega^2 J_{xy\varphi}.$$

The equation of free vibrations is an ordinary linear equation of the fourth-order with constant coefficients

$$\frac{\partial^4\theta_a(\zeta)}{\partial\zeta^4} - 2a\frac{\partial^2\theta_a(\zeta)}{\partial\zeta^2} - b\theta_a(\zeta) = 0. \tag{3.10}$$

We can transform Eq. (3.10) to a system of the first-order equations as bellow:

$$\begin{cases} \dfrac{\partial}{\partial \zeta}\theta_a(\zeta) = \phi_a(\zeta), \\[2ex] \dfrac{\partial}{\partial \zeta}\phi_a(\zeta) = \beta_a(\zeta), \\[2ex] \dfrac{\partial}{\partial \zeta}\beta_a(\zeta) = \eta_a(\zeta), \\[2ex] \dfrac{\partial}{\partial \zeta}\eta_a(\zeta) = 2a\beta_a(\zeta) + \mathrm{b}\,\theta_a(\zeta). \end{cases}$$

Now, we introduce a state vector [25]:

$$\psi(\zeta) = \left(\theta_a(\zeta) \quad \phi_a(\zeta) \quad \beta_a(\zeta) \quad \eta_a(\zeta)\right)^T. \tag{3.11}$$

The dimensionless equation for the amplitudes ψ takes the form:

$$\psi' = A_M(M, \Omega, a, b)\psi \tag{3.12}$$

where

$$A_M(M, \Omega, a, b) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mathrm{b} & 0 & 2a & 0 \end{bmatrix}.$$

This ordinary differential equation is solved analytically by the method of initial parameters [25]:

$$\psi(\zeta) = \mathrm{V}(\zeta, \Omega)\,\psi_0(\Omega)$$

where $\psi_0(\Omega)$ is the value of the state vector at $\zeta = 0$ (vector of initial parameters), the normalized matrix of fundamental solutions $\mathrm{V}(\zeta, \Omega)$ is defined as the original of the matrix defined by the Laplace transform of Eq.(3.12) in the coordinate ζ [25]:

$$V^*(p) = \left[pI - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mathrm{b} & 0 & 2a & 0 \end{pmatrix} \right]^{-1}, \qquad \mathrm{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.13}$$

$$V^*(p) = \frac{-1}{D(p,a,b)} \begin{bmatrix} -2ap + p^3 & -2a + p^2 & p & 1 \\ b & -2ap + p^3 & p^2 & p \\ bp & b & p^3 & p^2 \\ bp^2 & bp & b + 2ap^2 & p^3 \end{bmatrix}, \tag{3.14}$$

$$D(p,a,b) = 2ap^2 + b - p^4 .$$

Since $D(p,a,b)$ is a biquadratic polynomial, its solutions are easily determined and the original of the matrix of fundamental solutions is found analytically:

$$V(\zeta) = \begin{bmatrix} V(\zeta)_{11} & V(\zeta)_{12} & V(\zeta)_{13} & V(\zeta)_{14} \\ V(\zeta)_{21} & V(\zeta)_{22} & V(\zeta)_{23} & V(\zeta)_{24} \\ V(\zeta)_{31} & V(\zeta)_{32} & V(\zeta)_{33} & V(\zeta)_{34} \\ V(\zeta)_{41} & V(\zeta)_{42} & V(\zeta)_{43} & V(\zeta)_{44} \end{bmatrix}$$

where

$$V(\zeta)_{11} = \frac{1}{Z}\left(ch(\zeta \cdot p_1) \cdot p_2^2 + \cos(\zeta \cdot p_2) \cdot p_1^2 \right);$$

$$V(\zeta)_{12} = \frac{1}{Z\sqrt{b}}\left(sh(\zeta \cdot p_1) \cdot p_2^3 + \sin(\zeta \cdot p_2) \cdot p_1^3 \right);$$

$$V(\zeta)_{13} = \frac{-1}{Z}\left(ch(\zeta \cdot p_1) - \cos(\zeta \cdot p_2) \right);$$

$$V(\zeta)_{14} = \frac{-1}{Z\sqrt{b}}\left(p_2 sh(\zeta \cdot p_1) - p_1 \sin(\zeta \cdot p_2) \right);$$

$$V(\zeta)_{21} = bV(\zeta)_{14};$$

$$V(\zeta)_{22} = V(\zeta)_{11};$$

$$V(\zeta)_{23} = -\frac{p_1 sh(\zeta \cdot p_1) + p_2 \sin(\zeta \cdot p_2)}{Z};$$

$$V(\zeta)_{24} = V(\zeta)_{13};$$

$$V(\zeta)_{31} = bV(\zeta)_{13};$$

$$V(\zeta)_{32} = V(\zeta)_{21};$$

$$V(\zeta)_{33} = -\frac{p_1^2 ch(\zeta \cdot p_1) + p_2^2 \cos(\zeta \cdot p_2)}{Z};$$

$$V(\zeta)_{34} = V(\zeta)_{23}; \qquad V(\zeta)_{41} = bV(\zeta)_{23}; \qquad V(\zeta)_{42} = V(\zeta)_{31};$$

$$V(\zeta)_{43} = \frac{-\mathrm{p}_2\left(2ap_1^2 + b\right) \cdot sh(\zeta \cdot p_1) + p_1 \sin(\zeta \cdot p_2)\left(b - 2ap_2^2\right)}{\sqrt{b} \cdot Z};$$

$$V(\zeta)_{44} = V(\zeta)_{33};$$

$$Z = p_1^2 + p_2^2; \qquad p_1 = \sqrt{\sqrt{a^2 + b} + a}; \qquad p_2 = \sqrt{\sqrt{a^2 + b} - a}\ .$$

Satisfying the boundary conditions at $\zeta = 0$, we set two initial parameters $\theta_{a0} = 0$, and $\phi_{a0} = 0$, and for the existence of a nontrivial solution, we set the main determinant of the homogeneous system of equations for the boundary conditions at the end of the console to zero:

$$\begin{bmatrix} V(1,M)_{33} & V(1,M)_{34} \\ V(1,M)_{43} & V(1,M)_{44} \end{bmatrix} \begin{Bmatrix} \beta_{a0} \\ \eta_{a0} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{3.15}$$

The main determinant is also expressed analytically:

$$\begin{gathered} D(\Omega, M) = \begin{vmatrix} V(1,M)_{33} & V(1,M)_{34} \\ V(1,M)_{43} & V(1,M)_{44} \end{vmatrix} = \\ = \frac{ch(p_1)}{2\left(a^2 + b\right)} \left[\frac{2a^2 + b}{ch(p_1)} + b\cos(p_2) - a\sqrt{b} th(p_1)\sin(p_2) \right]. \end{gathered} \tag{3.16}$$

The transcendental equation $D(\Omega, M) = 0$ is solved with respect to Ω, for example, by the method of half division for different M, a, b. The value of M, at which $\Omega_k(M) = 0$, $k=1,2,3\ldots$, is the critical dimensionless flow velocity around a straight wing, or divergence velocity.
To represent the solution of problem (3.12), we use an expansion of the forms of free vibrations (modal expansion):

$$y(\xi, \tau) = \sum_{k=1}^{4} y_k(x) e^{\Omega_k \tau}$$

where Ω_k are the frequencies of free torsional vibrations.

3.3. Analysis of wing stability

Let us show the possibility of a loss of wing stability by analyzing the solutions of the characteristic equation with respect to p:

$$D\left(p,a,b\right)=2ap^2+b-p^4=0\,. \tag{3.17}$$

We must solve the problem of free vibrations of a steel wing of rectangular section, i.e., Eq.(3.15) with $L=1m,\ b=0.1m,$ maximum thickness $h=0.01m,\ E=2\times 10^{11},\ \nu=0.03,\ \rho=7850.$ Dimensionless wing parameters are as follows:

$\lambda=\dfrac{L}{b}=10$ - aspect ratio of the wing;

$c=\dfrac{h}{b}=0{,}1$ - relative thickness;

$S=\lambda b^2=0{,}1$ - wing console area;

$A=cb^2=1\times 10^{-3}$ - cross-sectional area;

$$J_y=\frac{b^4c}{12}=8{,}333\times 10^{-7},\qquad J_x=\frac{b^4c^3}{12}=8{,}333\times 10^{-9},$$

$$J_d=\frac{b^4c^3}{3\cdot\left(c^2+1\right)}=3.3\times 10^{-8},\qquad J_\varphi=\frac{b^6c^3}{144}=6.944\times 10^{-12},$$

$$J_{d\varphi}=\frac{J_dL^2}{2J_{\varphi t}\left(\vartheta+1\right)}=1.828\times 10^{3},$$

$$J_{xy\varphi}=L^2\frac{J_x\,J_y}{J_\varphi}=1.212\times 10^{5},\quad K_t=\frac{SbL^3}{2\,J_\varphi}\frac{340^2\rho}{E}=3.267\times 10^{6}.$$

The pitching moment coefficient can be defined based on the following function:

$$M_{z\alpha t}\left(M\right)=\begin{cases}0.03 & \text{if}\quad M\le 1,\\ -0.015\cdot M-\dfrac{0{,}015}{M} & \text{if}\quad M>1.\end{cases}$$

Figures 1 and 2 showed the dependences of the real and imaginary parts of the solution of equation (3.17) on the Mach number M at $\Omega=1$, i.e., we set the frequency of free vibrations equal to 1.

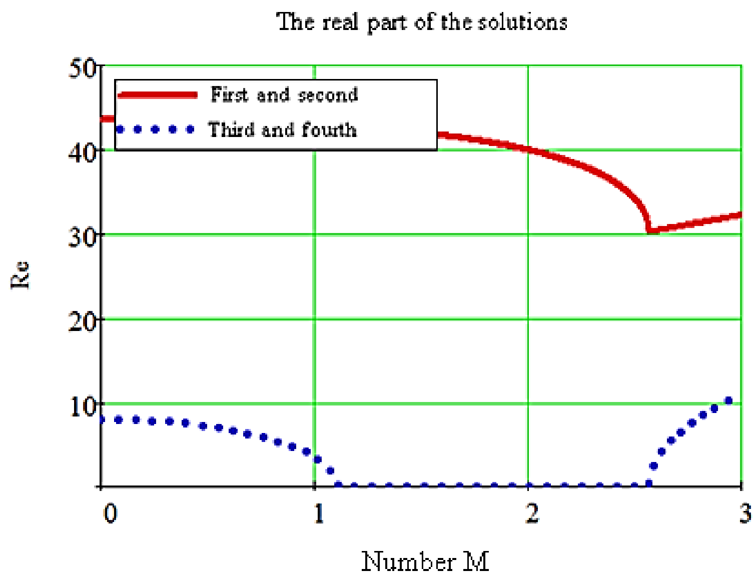

Fig.1. Dependence of the real part of the solutions of the characteristic equation on the Mach number *M* of the incident flow.

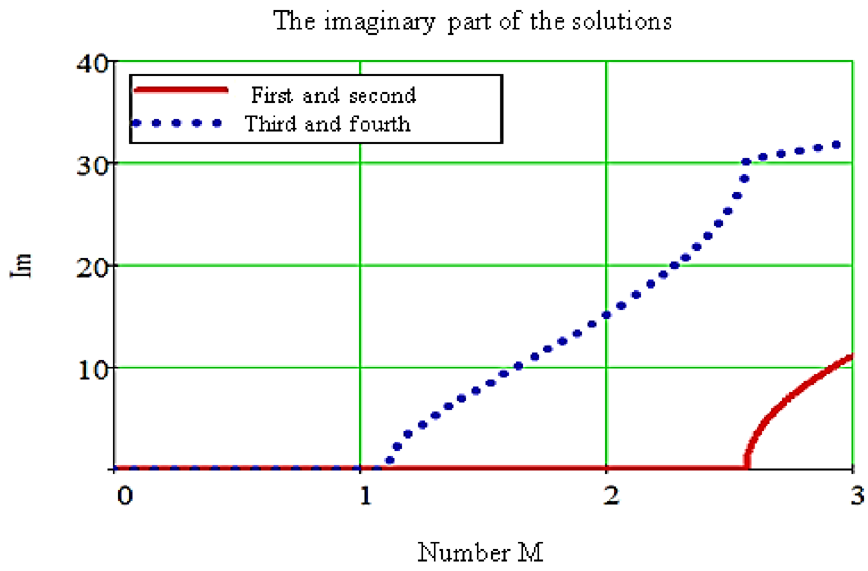

Fig.2. Dependence of the imaginary part of the solutions of the characteristic equation on the Mach number *M* of the incident flow.

4. Conclusions

In this article, we have presented a method to consider instability of the wing. While all the solutions of the characteristic equation (20) are real, free Lyapunov stable torsional vibrations will happen. At a given value of the Mach number *M*, a purely imaginary solution appears, one of the forms of critical velocity begins to develop aperiodically (with $M = 1.125$), stable vibrations in shape occur around this form, corresponding to another pair of solutions (see the graphs of the first and second solutions). The second critical velocity (with $M = 2.58$) corresponds to the appearance of two pairs of complex solutions, and the second and fourth solutions correspond to developing harmonic vibrations. The smaller solution corresponds to the torsional divergence, and the larger one corresponds to the torsional flutter. The divergence interval is $1.125 \le M \le 2.58$, flutter interval whereas the $2.58 < M < \infty$.

Thus, it is shown that at an arbitrarily assigned frequency of free vibrations there are several values of the Mach number M at which aperiodic components appear in the expansion in terms of the forms of free vibrations, therefore, the loss of stability is described by the proposed model.

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Nomenclature

A	– cross-sectional area
b	– side chord
b_A	– average aerodynamic chord of wing
C_y	– lift coefficient
C_y^α	– derivatives of the lift coefficient of the wing profile by α
c	– relative thickness
E	– Young's modulus
G	– shear modulus
i	– imaginary unit
J_x, J_y	– principal central moments of inertia of the section
L	– console length
M	– Mach number
M_z	– pitching moment
M_z^α	– derivatives of the pitching moment of the wing profile by α
$m_z(z,t)$	– torsion torque distributed along the z axis
q_x, q_y, q_z	– components of the load that is distributed along the corresponding axes
r	– distance from a point of the section to its center of mass; $d(z,t)$ is a measure of warping
S	– wing console area
$u(z,t), v(z,t), w(z,t)$	– components of the displacement vector of the center of mass of the section
V	– velocity of the undisturbed airflow
V_s	– velocity of sound in the wing material
Y	– lifting force
α	– airfoil
η	– narrowing
$\theta_y(z,t), \theta_z(z,t)$	– rotation angles around the y and x axes
λ	– aspect ratio
$\varphi(z,t)$	– angle of twisting
ρ	– density of air
χ	– sweep angle

ϑ – Poisson's ratio

Ω – dimensionless frequency of free vibrations

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