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# ON THE NUMERICAL APPROXIMATION OF FIRST–ORDER HAMILTON–JACOBI EQUATIONS

RÉMI ABGRALL *, VINCENT PERRIER **

* Institut de Mathématiques de Bordeaux and INRIA project Scalapplix
Université Bordeaux I, 341 Cours de la Libération, 33 405 Talence
e-mail: remi.abgrall@math.u-bordeaux.fr

** Department of Applied Mathematics
Université Bordeaux I, 341 Cours de la Libération, 33 405 Talence Cedex, France
e-mail: perrier@math.u-bordeaux1.fr

Some methods for the numerical approximation of time-dependent and steady first-order Hamilton-Jacobi equations are reviewed. Most of the discussion focuses on conformal triangular-type meshes, but we show how to extend this to the most general meshes. We review some first-order monotone schemes and also high-order ones specially dedicated to steady problems.

**Keywords:** approximation of Hamilton-Jacobi equations, viscous solution, Cauchy-Dirichlet problem, triangular mesh

## 1. Introduction

This paper describes some of the schemes that are currently used to compute approximated solutions of first-order Hamilton-Jacobi equations, namely, the steady Dirichlet problem

$$\begin{aligned} H(x,u,Du) &= 0, \quad x \in \Omega \subset \mathbb{R}^d, \\ u &= \varphi, \quad x \in \partial\Omega, \end{aligned}$$

or the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} + H\big(x,u(x),Du\big) &= 0, \quad x \in \mathbb{R}^d, \quad t > 0, \\ u(x,0) &= u_0(x). \end{aligned}$$

In many problems of physical interest, one needs to compute the solution of such an equation. One of the simplest examples is the computation of a distance function, more sophisticated examples include evaluating the arrival time of a front (a wave front, a flame front, etc.) in a non-homogeneous medium. Similar problems also appear in control theory, thermodynamics, etc.

We will concentrate on the numerical approximation of these equations on conformal triangular-type meshes. This is a more general situation than for the standard Cartesian meshes, where this problem was considered originally, but still less general than the case of non-conformal meshes. However, we sketch how to extend the schemes we describe here to the most general case. Indeed, the situation we consider is an intermediate one, but it is general enough to be obliged to imagine solutions that are not too specific so that the structure of the mesh does not play too rigid a role.

In the first part, we briefly recall the notion of viscosity solutions for HJ equations, either for the Cauchy problem or for the steady one with Dirichlet boundary conditions. Then, in two particular cases, we recall the exact solution. The next section is devoted to the numerical approximation of the Cauchy problem. In the third section we discuss the approximation of the Dirichlet problem. The fourth section considers a general formulation for a high-order discretization. The bridge between Cartesian and non-conformal meshes is sketched in the fifth section. The last section is devoted to some numerical applications.

As we have already said, our point of view is quite biased. There are basically two classes of approximation techniques. The first one tries to directly use the notion of viscosity solutions, see Section 2, and this is our point of view. In the second class of methods, one tries to exploit a formal link between some systems of conservation laws

and HJ equations. The link is that if one differentiates the equation

$$\frac{\partial u}{\partial t} + H(Du) = 0$$

with respect to $x$ and $y$, then setting

$$p_i = \frac{\partial u}{\partial x_i}, \quad p = Du,$$

we have

$$\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial x}(p) = 0.$$

This is the point of view of the papers that extend finite volume or discontinuous Galerkin methods, see, e.g., (Osher and Shu, 1991).

## 2. Short Review of HJ Equations and Viscosity Solutions

We consider the Cauchy problem: find $u \in C^0(\Omega)$, the space of continuous functions on the open subset $\Omega \subset \mathbb{R}^d$, such that

$$\begin{aligned} H(x,u,Du) &= 0, \quad x \in \Omega \subset \mathbb{R}^d, \\ u &= g, \quad x \in \partial\Omega \end{aligned} \tag{1}$$

in the viscosity sense. In (1), $(x,s,p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \mapsto H(x,s,p)$ is uniformly continuous.

Before going further, let us briefly review the notion of the viscosity solution for (1). For any function $z$, we consider the upper semi-continuous (u.s.c) and lower semi-continuous (l.s.c) envelopes of $z$ with respect to all variables. They are respectively defined by

$$z^*(x) = \limsup_{x \to y} z(y), \quad z_*(x) = \liminf_{x \to y} z(y).$$

Following (Barles, 1994), we introduce the function $G$:

$$G(x,s,p) = \begin{cases} H(x,s,p), & x \in \Omega, \\ s - g(x), & x \in \partial\Omega. \end{cases}$$

The computation of $G_*$ and $G^*$ is easy and we have

$$\begin{cases} G_*(x,s,p) = G^*(x,s,p) = H(x,s,p) & \text{if } x \in \Omega, \\ G_*(x,s,p) = \min\big(H(x,s,p), s - g(x)\big) & \text{if } x \in \partial\Omega, \\ G^*(x,s,p) = \max\big(H(x,s,p), s - g(x)\big) & \text{if } x \in \partial\Omega. \end{cases} \tag{2}$$

A locally bounded u.s.c function $u$ defined on $\overline{\Omega}$ is a viscosity sub-solution of (1) if, and only if, for any $\phi \in C^1(\overline{\Omega})$, if $x_0 \in \overline{\Omega}$ is a local maximum of $u - \phi$, then

$$G_*\big(x_0, u(x_0), D\phi(x_0)\big) \le 0. \tag{3}$$

Similarly, $u$, a locally bounded, l.s.c. function defined on $\overline{\Omega}$ is a viscosity super-solution of (1) if, and only if, for any $\phi \in C^1(\overline{\Omega})$, if $x_0 \in \overline{\Omega}$ is a local minimum of $u - \phi$, then

$$G^*\big(x_0, u(x_0), D\phi(x_0)\big) \ge 0. \tag{4}$$

A viscosity solution is simultaneously a sub- and a super-solution of (1). This can be generalized to other types of boundary conditions such as Neumann, etc.

In the case of the Cauchy problem, we have

$$\begin{aligned} \frac{\partial u}{\partial t} + H\big(x, u(x), Du\big) &= 0, \quad x \in \mathbb{R}^d, \quad t > 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{5}$$

where $u_0$ belongs to the set of bounded and uniformly continuous functions, $BUC(\mathbb{R}^2)$. One can easily adapt the arguments raised for the steady problem. Here, $G$ is simply

$$\begin{aligned} G(x,s,p) = p_t + H(x,s,p_x), \quad x \in \mathbb{R}^d, \quad s \in \mathbb{R}^d, \\ p = (p_t, p_x) \in \mathbb{R} \times \mathbb{R}^d, \end{aligned}$$

so that $G_* = G^* = G$. Sub-solutions (resp. super-solutions) are elements of $BUC(\mathbb{R}^d \times [0,T])$ where $T > 0$, so that (3) (resp. (4)) holds.

All this can be extended to the Cauchy–Dirichlet problem

$$\begin{aligned} \frac{\partial u}{\partial t} + H\big(x, u(x), Du\big) &= 0, && x \in \Omega \subset \mathbb{R}^d,\ t > 0 \\ u(x,0) &= u_0(x), && x \in \Omega, \\ u(x,t) &= g(x,t), && x \in \partial\Omega,\ t > 0. \end{aligned} \tag{6}$$

Under standard assumptions regarding the open subset $\Omega$, $g$ and $H$ and $u_0$, one can prove the existence and uniqueness of the viscosity solutions of (1), (5) and (6), see (Barles, 1994). In particular, this is true if the Hamiltonian $H$ is convex in $p \in \mathbb{R}^d$ and if $\partial\Omega$ is Lipschitz continuous.

In this paper, we assume that (1) has a uniqueness principle, that is, any sub-solution $u$ and any super-solution $v$ of (1) satisfy

$$u(x) \le v(x), \quad \forall x \in \Omega, \tag{7}$$

and

$$u(x,t) \le v(x,t), \quad \forall x \in \mathbb{R}^d, t > 0, \tag{8}$$

in the case of the Cauchy problem.

## 3. Some Exact Solutions

Two examples are considered. Either we look for the steady problem (1) with a convex Hamiltonian, or we look for the Cauchy problem (5) with either a convex (or concave) Hamiltonian or a convex (or concave) initial condition.

The main tool is the Legendre transform. If $f$ is a convex function such that

$$\lim_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = +\infty, \tag{9}$$

we define the Legendre transform of $f$ by

$$f^\star(p) = \sup_{y\in\mathbb{R}^d} \big(p\cdot y - f(y)\big).$$

If the supremum is reached at $y^\star$, we have the relation

$$f^\star(p) + f(y^\star) = p\cdot y^\star.$$

This shows that $f^\star(p)$ can be seen as the abscissa of the tangent of the graph of $f$ at $y^*$. This graphic interpretation helps us to see that, if $f$ is regular enough, the graph of $f$ is the envelope of its tangent, so that

$$(f^\star)^\star = f.$$

Of course, this relation is generalizable to convex $f$ when it satisfies (9).

All this generalizes to concave functions (since $-f$ is convex):

$$f^\star(p) = -(-f)^\star(p) = \inf_{y\in\mathbb{R}^d} \big(-y\cdot p - f(y)\big).$$

**3.1. Steady Problem.** We assume that the Hamiltonian is given by

$$H(x,u,p) = \sup_{v\in V} \left\{-b(x,v)\cdot p + \lambda u - f(x,v)\right\},$$

where the space of controls $V$ is compact, and we have standard assumptions on $b$, $f$ and $\lambda > 0$, see (Barles, 1994). For the Dirichlet condition, the solution of (1) is given by the dynamic programming principle. For any $T > 0$, we get

$$\begin{aligned} u(x) &= \inf_{v(\cdot)} \Bigg[ \int_0^{\min(T,\tau)} f\big(y_x(t),v(t)\big)e^{-\lambda t}\mathrm{d}t \\ &\quad + 1_{\{T<\tau\}} u\big(y_x(T)\big)e^{-\lambda T} + 1_{\{T\geq\tau\}}\varphi\big(y_x(\tau)\big)e^{-\lambda\tau}\Bigg]. \end{aligned} \tag{10}$$

The trajectory $y_x(\cdot)$ satisfies $y_x(0) = x \in \Omega$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} y_x(t) = b\big(y_x(t),v(t)\big) \quad \text{for } t > 0.$$

They are defined if $f$ is regular enough, e.g., Lipschitz continuous. The exit time $\tau$ is

$$\tau = \inf\big\{t \geq 0, y_x(t) \notin \Omega\big\}.$$

Details can be found in (Lions, 1982; Barles, 1994).

**3.2. Cauchy Problem.** Analytical formulas for the solution of (5), when $H$ only depends on $p \in \mathbb{R}^d$, are given in (Bardi and Evans, 1984).

1. When $H$ is uniformly Lipschitz continuous and $u_0$ convex, we have

$$u(x,t) = \sup_{p\in\mathbb{R}^d} \Big[x\cdot p - u_0^\star(p) - tH(p)\Big], \tag{11}$$

and when $u_0$ is concave, we get

$$u(x,t) = \inf_{p\in\mathbb{R}^d} \Big[-x\cdot p + u_0^\star(p) - tH(p)\Big]. \tag{12}$$

2. When $u_0$ is uniformly continuous, we have for a convex Hamiltonian

$$u(x,t) = \inf_{y\in\mathbb{R}^d} \left[u_0(y) + tH^\star\left(\frac{y-x}{t}\right)\right], \tag{13}$$

and for a concave Hamiltonian

$$u(x,t) = \sup_{y\in\mathbb{R}^d} \left[u_0(y) + tH^\star\left(\frac{x-y}{t}\right)\right]. \tag{14}$$

The formulas (11) and (12) reflect Huygens's principle, while (13) and (14) are consequences of the dynamic programing principle (10).

Note that if $u_0$ is linear in $x$, then $u_0(x) = A + p\cdot x$, and we have

$$u(x,t) = u_0(x) - tH(p).$$

These results are only valid for special initial conditions or a particular Hamiltonian. We have the following more general results:

**Lemma 1.** (Bardi and Osher, 1991) *If $u_0 = u_0^{\mathrm{conv}} + u_0^{\mathrm{conc}}$, where $u_0^{\mathrm{conv}}$(resp. $u_0^{\mathrm{conc}}$) is convex (resp. concave), then the solution $u$ of* (5) *satisfies*

$$\psi_2(x,t) \leq u(x,t) \leq \psi_1(x,t), \quad \forall (x,t) \in \mathbb{R}^d \times [0,T],$$

*with*

$$\begin{aligned} \psi_1(x,t) = \inf_{q\in\mathbb{R}^d} \sup_{p\in\mathbb{R}^d} \Big(&x\cdot p - \big(u_0^{\mathrm{conv}}\big)^\star(p) \\ &- \big(u_0^{\mathrm{conc}}\big)^\star(q) - tH(p-q)\Big) \end{aligned}$$

*and*

$$\begin{aligned} \psi_2(x,t) = \sup_{p\in\mathbb{R}^d} \inf_{q\in\mathbb{R}^d} \Big(&-x\cdot q - \big(u_0^{\mathrm{conv}}\big)^\star(p) \\ &- \big(u_0^{\mathrm{conc}}\big)^\star(q) - tH(q-p)\Big). \end{aligned}$$

*Proof.* The proof uses the fact that

$$u_0^{\rm conc}(x) = \sup_{p\in\mathbb{R}^d}\left(x\cdot p - \left(u_0^{\rm conc}\right)^\star(p)\right)$$

so that for any $p$

$$u_0^{\rm conc}(x) \geq v_{p,0}(x) := x\cdot p - \left(u_0^{\rm conc}\right)^\star(p).$$

Then we solve the Cauchy problem for $v_0 + u_0$ which is convex, use the comparison principle (8), and take the maximum. This gives the first inequality. The second one is obtained in a similar way. ■

**Lemma 2.** (Abgrall, 1996) *If $H = H_{\rm conv} + H_{\rm conc}$, where $H_{\rm conv}$ (resp. $H_{\rm conc}$) is convex (resp. concave) uniformly continuous, the solution of the Cauchy problem* (5) *satisfies*

$$\Phi_2(x,t) \leq u(x,t) \leq \Phi_1(x,t)$$

*with*

$$\Phi_1(x,t) = \inf_{q\in\mathbb{R}^d}\sup_{y\in\mathbb{R}^d}\left[u_0(y)+tH^\star_{\rm conv}(q)+tH^\star_{\rm conc}\left(\frac{y-x}{t}+q\right)\right],$$

$$\Phi_2(x,t) = \sup_{p\in\mathbb{R}^d}\inf_{y\in\mathbb{R}^d}\left[u_0(y)+tH^\star_{\rm conc}(q)+tH^\star_{\rm conv}\left(\frac{x-y}{t}+q\right)\right].$$

The proof is similar and is given in (Abgrall, 1996).

## 4. First-Order Approximation of the Cauchy Problem

For simplicity, from now on we assume that $d = 2$, but all the results can be easily generalized to other dimensions, in particular $d = 3$. We consider a triangulation of $\mathbb{R}^2$ for which the vertices are $\{M_i\}_{i=1,n_s}$ and the triangles are $\{T_j\}_{j=1,n_t}$. We denote by $T$ a generic triangle. The vertices of $T_k$ are $M_{i_1}$, $M_{i_2}$ and $M_{i_3}$. For simplicity, we often denote them by $i_1$, $i_2$, $i_3$ or by 1, 2, 3 when there is no ambiguity. The family of triangulations we consider is shape regular.

Up to our knowledge, the first paper to discuss in detail the approximation of (5) is (Crandall and Lions, 1984). As in this reference, (5) is approximated by

$$\begin{aligned} u_i^{n+1} &= u_i^n - \Delta t\mathcal{H}_i, \quad i = 1,\dots n_s, \quad n\in\mathbb{N}^*,\\ u_i^0 &= u_0(M_i), \end{aligned} \tag{15}$$

where $\Delta t > 0$ is the time step and $u_i^n$ is an approximation of $u(M_i, n\Delta t)$, and the numerical Hamiltonian $\mathcal{H}_i$ depends on $u_i^n$ , the values of $u_j^n$ where $j\in\mathcal{V}_i$ ($\mathcal{V}_i$ is the set of neighbors of $M_i$ including $M_i$ by convention), and if necessary on $M_i$,

$$\mathcal{H}_i := \mathcal{H}\big(M_i, u_i^n, \{u_i^n\}_{j\in\mathcal{V}_i}\big). \tag{16}$$

In this reference the notion of consistency is introduced. The numerical Hamiltonian $\mathcal{H}_i$ is consistent when, if $v_i = A + p\cdot\vec{OM}_i$, then for any $M_j$ and $s\in\mathbb{R}$, we have

$$\mathcal{H}\big(M_j, s, \{v_i\}_{j\in\mathcal{V}_i}\big) = H(M,s,p). \tag{17}$$

A less restrictive definition, which is helpful for the proof, is given in (Barles and Souganidis, 1991),

**Definition 1.** We say that the Hamiltonian $\mathcal{H}$ is *weakly consistent* if for all $x\in\overline{\Omega}$ and $\phi\in C_b^\infty(\overline{\Omega})$ (the set of $C^\infty$ bounded functions)

$$\limsup_{h\to0,y\to x,\xi\to0}\mathcal{H}\big(y,\phi(y)+\xi,\phi+\xi\big) \leq G^*\big(x,\phi(x),D\phi(x)\big) \tag{18}$$

and

$$\liminf_{\rho\to0,y\to x,\xi\to0}\mathcal{H}\big(y,\phi(y)+\xi,\phi+\xi\big) \geq G_*\big(x,\phi(x),D\phi(x)\big). \tag{19}$$

A scheme that satisfies (17) is said to be strongly consistent. A strongly consistent scheme is weakly consistent.

The structure of the solution of (5) was first used in (Bardi and Osher, 1991; Osher and Shu, 1991). In particular, the results of Lemma 1 were used to define a Godunov-like scheme. In (Abgrall, 1996), it is shown that, in general, for non-structured meshes, the generalization of the Godunov-like scheme of (Bardi and Osher, 1991) leads to a non-consistent Hamiltonian. However, Lemma 2 provides a solution.

Assume that $\{u_j^n\}_{j=1,\dots,n_s}$ is known and denote by $u_h^n$ the piecewise linear interpolation of these data. For any mesh point $M_i$, we consider the set $\{\Omega_i\}_{i=1,\dots,\omega_i}$ of angular sectors at $M_i$, see Fig. 1. Each angular sector $\Omega_i$ corresponds to one of the triangles that share $M_i$, and we denote by $U_i$ the gradient of $u_h^n$ in that triangle. The

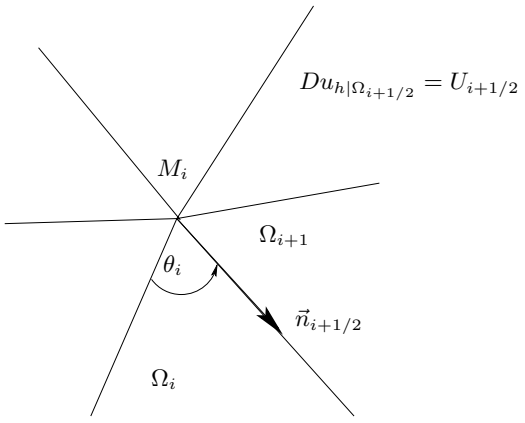



Fig. 1. Illustration of the angular sectors $\Omega_i$, $\theta_i$ and the vectors $\vec{n}_{i+1/2}$ that are needed in numerical Hamiltonians definitions.

functions $\Phi_1$ and $\Phi_2$ evaluated at $x = M_i$ are

$$\Phi_1(M_i,\Delta t) = u_i^n - \Delta t \min_{q\in\mathbb{R}^2} \max_{1\le k\le \omega_i} \sup_{z\in\Omega_k} \Big( U_i\cdot z - H_1^*(z-q) - H_2^*(q)\Big),$$

$$\Phi_2(M_i,\Delta t) = u_i^n - \Delta t \max_{q\in\mathbb{R}^2} \min_{1\le k\le \omega_i} \inf_{z\in\Omega_k} \Big( U_i\cdot z - H_1^*(z-q) - H_2^*(q)\Big).$$

A key observation is that any of the terms multiplied by $\Delta t$, e.g., $\min_{q\in\mathbb{R}^2} \max_{1\le k\le \omega_k} \sup_{z\in\Omega_k} (U_i\cdot z - H_1^*(z-q) - H_2^*(q))$ reduces to $H(p)$ when $U_i \equiv p$, $\forall i$. Hence any of these terms defines a consistent numerical Hamiltonian, for example,

$$\mathcal{H}_i := \max_{q\in\mathbb{R}^2} \min_{1\le k\le \omega_i} \inf_{z\in\Omega_k} \big( U_i\cdot z - H_1^*(z-q) - H_2^*(q)\big). \tag{20}$$

The dependence on $u_j^n$, $j \in \mathcal{V}_i$ appears here in the gradients $U_i$. This formula can be easily extended to the more general case $H = H(x, u, Du)$ and simplifies when $H$ is convex, e.g.,

$$\mathcal{H}_i = \max_{1\le k\le \omega_i} \max_{z\in\Omega_i} \big( U_i\cdot z - H^*(z)\big). \tag{21}$$

Another observation is that, by construction, $\mathcal{H}_i$ defined by (20) or (21) is monotone.

**Definition 2.** We say that $\mathcal{H}$ is *monotone* if, whatever $M_i \in \Sigma$, $u_j \le v_j$, and for any $s \in \mathbb{R}$

$$\mathcal{H}\big(M_i, s, \{u_j\}_{j\in\mathcal{V}_i}\big) \ge \mathcal{H}\big(M_i, s, \{v_j\}_{j\in\mathcal{V}_i}\big).$$

The Hamiltonian (20) is monotone by construction if $\Delta t/h \max_{\|Du_{|T}\|_\infty \le L} \|D_p H(p)\|_\infty \le 1/2$: this is a consequence of (8) and the fact that $\Phi_2$ (and $\Phi_1$) can be interpreted as the solution of an Hamilton Jacobi equation that is introduced in the proof of Lemma 2, see (Abgrall, 1996) for details.

Another key observation is that the value of $\mathcal{H}_i$ defined by (20) or (21) does not depend on the structure of the mesh, but on the interpolant $u^n$. In other words, if one splits an angular sector $\Omega_k$ in two, *without changing the value of* $U_k$, the numerical Hamiltonian is not modified. We say that the scheme is *intrinsic* and we have the following error estimate:

**Theorem 1.** (Abgrall, 1996) *Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be continuous and $u_0 \in BUC(\mathbb{R}^2)$ be Lipschitz continuous (with constant $L$). Let $\mathcal{T}$ be a triangulation where $h$ is the largest radius of the circles with centers $M_i$, $i = 1,\dots,n_s$ and contained in all the triangles having the $M_i$s as vertices. We assume that the mesh is shape regular, i.e., the minimum angle $\alpha$ of the triangles $T$ is uniformly bounded from below.*

*Let $u$ be the viscosity solution of* (5) *and $\{u_i^n\}_{j=1,\dots,n_s}$ be defined by* (15)*. Then there exists a constant $c$ which depends only on $\alpha$, $L$, $T > 0$ and $\mathcal{H}$ such that for any $M_i$ and $n$ with $0 \le n\Delta t \le T$*

$$|u_i^n - u(M_i, n\Delta t)| \le c\sqrt{\Delta t}$$

The proof is an adaptation of the main result of (Crandall and Lions, 1984) with some technicalities (in particular, for showing that the time step only depends on $u_0$) specific to unstructured meshes. We refer the reader to (Abgrall, 1996).

The practical evaluation of the Legendre transform is not always an easy task, so other numerical Hamiltonians exist. The simplest one is the Lax-Friedrichs one, which is inspired by the Lax-Friedrichs scheme for conservation laws. It has several versions. The first one is

$$\mathcal{H}_i^{LF}(Du_{h|\Omega_1},\dots,Du_{h|\Omega_{k_i}}) = H(\bar{U}) - \frac{\epsilon}{h}\oint_{C_h} \big[u_h(M) - u_h(M_i)\big]\,\mathrm{d}l, \tag{22a}$$

where $C_h$ (resp. $D_h$) is a circle (disk) with center $M_i$ and radius $h$,

$$\widehat{U} = \frac{\displaystyle\int_{D_h} Du_h \,\mathrm{d}x\,\mathrm{d}y}{\pi h^2},$$

and $\epsilon$ is greater than any Lipschitz constant of $H$ divided by $2\pi$. This defines a monotone scheme provided that $\Delta t/h \le \varepsilon/2\pi$.

A different version of the Lax Friedrichs Hamiltonian, which is monotone under the same constraint, is the following:

$$\mathcal{H}_h^{LF}(Du_{h|\Omega_1},\dots,Du_{h|\Omega_{k_i}}) = \frac{\displaystyle\int_{D_h} H(Du_h)}{\pi h^2} - \frac{\epsilon}{h}\oint_{C_h} \big[u_h(M) - u_h(M_i)\big]\,\mathrm{d}l. \tag{22b}$$

This version can be rewritten as

$$\mathcal{H}_h^{LF}(Du_{h|\Omega_1},\dots,Du_{h|\Omega_{k_i}}) = \sum_{0\le l\le k_i} \frac{\theta_l^i}{2\pi} H(Du_{h|\Omega_l^i}) + \varepsilon \sum_{0\le l\le k_i} \tan\theta_l^i \frac{\vec{n}_{l-1/2}^i + \vec{n}_{l+1/2}^i}{2}\cdot Du_{h|\Omega_l^i}.$$

The vector $\vec{n}_{l+1/2}$ is the unit vector of the edge that separates the angular sectors $\Omega_l$ and $\Omega_{l+1}$, the angle $\theta_l^i$ is the angle of the angular sector at $M_i$, see Fig. 1. The parameter $\varepsilon$ is the same as in the previous version.

The third version is

$$\mathcal{H}^{LF}(Du_{h|\Omega_1},\dots,Du_{h|\Omega_{k_i}}) = \frac{\sum_{T\ni M_i}|T|H(Du_{h|T})+\alpha\sum_{M_j\in T}(u_i-u_j)}{\sum_{T\ni M_i}|T|}, \quad (22c)$$

where $\alpha \geq h_T \max_p \|D_pH\|$, $h_T$ being the length of the longest edge of $T$.

The main difference between these different formulas is that (22a) and (22b) are intrinsic in the sense given in (Abgrall, 1996), while (22c) is not. Hence, following the same reference, (22a) and (22b) are convergent and the error estimate is $\mathcal{O}(h^{1/2})$. For (22c), such an error estimate is not available (at least when following the technique of (Abgrall, 1996)), but the scheme is convergent: this is a straightforward application of (Barles and Souganidis, 1991).

The advantage of (22c) over the other two versions is its simplicity in coding compared with (22a) and (22b).

## 5. Dirichlet Problem

The approximation of the Dirichlet problem is not as simple as it may seem. An illustration is the problem of finding $u : [0,1] \to \mathbb{R}$ such that

$$|u'| - 1 = 0 \text{ in } x \in [0,1], \quad u(0) = 1, \quad u(1) = 2,$$

which has no classical solution, but whose viscosity solution, defined only in $[0,1[$, is $u(x) = x$. We have $\lim_{x\to 1^-} u(x) = 1 \neq 2$. In other cases, e.g., $u(0) = u(1) = 0$, we have $u(x) = |x - 1/2|$, which matches strongly the boundary conditions.

In order to define a scheme, we start from (10), and consider a triangulation of $\Omega$. First we assume that $M_i \in \partial\Omega$. In (10), the set of controls can be splitted into two parts: the set $V_1$ for which $T < \tau$, and $V_2$ for which $T \geq \tau$. Hence

$$u(x) = \min\left(\inf_{v\in V_1}[\cdots], \inf_{v\in V_2}[\cdots]\right).$$

Let $\vec{n}$ be the interior normal to $\Omega$ at $x \in \overline{\Omega}$. Since $T$ is arbitrary, it can be chosen as small as possible. In the limit $T \to 0$, the set $V_1$ would be the set of controls for which $b(x,v)\cdot\vec{n} > 0$, i.e., the control for which the trajectory goes into $\Omega$. The dynamic programing principle $\inf_{v\in V_1}[\cdots] - u(x) = 0$ corresponds to the Hamiltonian

$$H_b(x,t,p) = \sup_{v\in V_1}\{b(x,v)\,.\,p + \lambda t - f(x,v)\}.$$

We also have the relation $H_b \leq H$.

If $T$ is small, the "inf" on $V_2$ can be approximated by $\varphi(y_x(\tau))$. Since $T \leq \tau$, provided that we can choose controls for which $T \simeq \tau$, we get

$$\varphi\big(y_x(\tau)\big) \simeq \varphi(x)$$

because $\varphi$ is continuous. We see that, at a boundary point, (10) can be approximated by

$$0 = \max\big(\mathcal{H}_i^b, u(x) - \varphi(x)\big),$$

where $\mathcal{H}^b$ is a consistent approximation of $H_b$.

When $M_i \notin \partial\Omega$, taking $T$ small enough, we can see formally that the boundary plays no role, so that we can take any consistent Hamiltonian, e.g., those defined in the previous section.

The scheme is then

$$\mathcal{S}(M_i, u_i, \{u_j\}_{j\in\mathcal{V}_i}) = 0, \quad \forall i \quad (23)$$

with

$$\mathcal{S}(x,s,\{u_j\}_{j\in\mathcal{V}_i}) = \begin{cases}\mathcal{H}\big(x,s,\{u_j\}_{j\in\mathcal{V}_i}\big) & \text{if } x\notin\partial\Omega,\\ \max\big(\mathcal{H}_b(x,s,\{u_j\}_{j\in\mathcal{V}_i}), s-\varphi(x)\big) & \text{otherwise.}\end{cases} \quad (24)$$

The scheme (23)–(24) can be extended to other types of boundary conditions. There is an implicit dependence of $\mathcal{S}$ with respect to $h$. We extend the definition of $\mathcal{S}$ to any $y \in \Omega$ by saying that $\mathcal{S}(x,s,\{u_j\}_{j\in\mathcal{V}_j}) = \mathcal{S}(M_i,s,\{u_j\}_{j\in\mathcal{V}_j})$ if $x$ belongs to the dual control volume associated with $M_i$.

**Theorem 2.** (Abgrall, 2004) *Assume that*

1. $H_b \leq H$,
2. $\mathcal{H}$, $\mathcal{H}^b$ *are monotone and the solutions of* (23) *are uniformly bounded,*
3. *for all* $\phi \in C_b^\infty(\overline{\Omega})$, *we have the following:*
   (a) *For any* $x \in \overline{\Omega}$,
   $$\lim_{h\to 0, y\to x,\xi\to 0}\mathcal{H}\big(y,\varphi(y)+\xi,\varphi+\xi\big) = H\big(x,\varphi(x),D\varphi(x)\big), \quad (25a)$$
   (b) *For any x in a neighborhood of* $\partial\Omega$,
   $$\lim_{h\to 0, y\to x,\xi\to 0}\mathcal{H}^b(y,\varphi(y)+\xi,\varphi+\xi) = H_b\big(x,\varphi(x),D\varphi(x)\big), \quad (25b)$$
4. *Equation (1) satisfies a uniqueness principle.*

*Then the family* $u_h$ *defined by (24) converges locally uniformly to the solution of (1) in* $\Omega$.

*Proof.* The key argument of the proof is a convergence result by Barles and Souganidis (1991). ■

Unfortunately, this results is not sufficient to guarantee a "good" convergence. Take the example from the

beginning of the section, a regular mesh ($1/\Delta x = N+1$); the Godunov scheme that reduces here to

$$\begin{aligned}\mathcal{H}_i &= \mathcal{H}(u_{i-1}, u_i, u_{i+1}) \\ &= \max\left(\frac{|u_{i+1}-u_i|}{\Delta x}, \frac{|u_{i-1}-u_i|}{\Delta x}\right) - 1\end{aligned}$$

and $\mathcal{H}_b = -\infty$. This amounts to setting $u_0 = 0$ and $u_N = 2$. Theorem 2 applies but numerical experiments indicate that the gradient of the solution is not bounded, so that there is no hope to have a convergence like $\Delta x^\alpha$ with $\alpha > 0$ reasonable.

In (Abgrall, 2004), this problem is studied and it is shown that if $H$ is convex, if the Godunov scheme constructed on the boundary Hamiltonian $H_b$ is constructed, and if a coercivity assumption holds for $H$, $H_b$ and the associated numerical Hamiltonian, then one can control $Du$, and in (Abgrall and Perrier, 2007) it is shown that the error behaves like $h^{-1/2}$. Similar error estimates (for Cartesian meshes) were obtained in (Deckelnick and Elliot, 2004).

## 6. High-Order Extension

Up to now, all the examples have been only first-order accurate schemes. There are several ways to construct high-order schemes.

One possible construction is a consequence of the following fact: The "$Du$" dependence in the Hamiltonian comes from the term "$\{u_j, j \in \mathcal{V}_i\}$" in (16). More precisely, in all known examples, this dependence occurs through differences, $u_j - u_i$ for $j \in \mathcal{V}_j$. These quantities can be rewritten in terms of the gradients of $u$ in the triangles surrounding $M_i$ (this remark has already been used in (22a)–(22c)). One can exploit this remark, as, e.g., in (Osher and Shu, 1991), by modifying the evaluation of the gradients in the triangles. Instead of a linear interpolant, one can use higher-degree polynomials thanks to the ENO/WENO methodology (Abgrall, 1996; Qiu and Shu, 2005; Zhang and Shu, 2003).

Another solution is the discontinuous Galerkin strategy (Hu and Shu, 1999; Li and Shu, 2005; Augoula and Abgrall, 2000). We do not detail this technique here.

The last method is a mixed strategy (Abgrall, 2007). The idea is to combine a low-order, monotone Hamiltonian ($\mathcal{H}^M$) with a high-order consistent Hamiltonian ($\mathcal{H}^H$). By "high-order" we mean that if $u$ is a smooth solution to (1), then

$$\mathcal{H}^H\big(M_i, u_i, \{u_j\}_{j\in\mathcal{V}_i}\big) = O(h^k) \tag{26}$$

for $k > 1$. The scheme is

$$\begin{aligned}&\mathcal{H}\big(M_i, u_i, \{u_j\}_{j\in\mathcal{V}_i}\big) \\ &\quad= \ell_i \mathcal{H}^M(M_i, u_i, \{u_j\}_{j\in\mathcal{V}_i}) \\ &\qquad+ (1-\ell_i)\mathcal{H}^H\big(M_i, u_i, \{u_j\}_{j\in\mathcal{V}_i}\big) + \varepsilon(h),\end{aligned} \tag{27}$$

where $\varepsilon(h) = Ch^k$ for some positive constant $C$ and $\ell_i$ is chosen such that, if $r_i := \mathcal{H}_i^H/\mathcal{H}_i^M$, we have

$$\ell_i + (1-\ell_i)r_i \geq \varepsilon'(h), \tag{28}$$

where $\varepsilon'(h)^{-1}\varepsilon(h) = o(1)$. We have the following lemma whose proof is immediate:

**Lemma 3.** *If $\mathcal{H}^M$ and $\mathcal{H}^H$ are strongly consistent, $\mathcal{H}$ defined by* (27) *is weakly consistent.*

The justification of (28) comes from the simple relation

$$\begin{aligned}&\mathcal{H}\big(M_i, u_i, \{u_j\}_{j\in\mathcal{V}_i}\big) \\ &\quad= \Big(\ell_i + (1-\ell_i)r_i\Big)\mathcal{H}^M(M_i, u_i\{u_j\}_{j\in\mathcal{V}_i}) + \varepsilon(h),\end{aligned} \tag{29}$$

from which, using once more the technique of the convergence result of (Barles and Souganidis, 1991), one can show the following result for the scheme (23)–(24), where $\mathcal{H}$ is given by (27):

**Theorem 3.** (Abgrall, 2007) *We consider the scheme* (23)*, where $\mathcal{H}$ in* (24) *is defined by* (27)*. We assume that*

1. *$\mathcal{H}^M$, $\mathcal{H}^H$ and $\mathcal{H}_b$ are strongly consistent,*
2. *$\mathcal{H}^M$ and $\mathcal{H}_b$ are monotone Hamiltonians,*
3. *$H_b \leq H$,*
4. *the mixture parameter $\ell$ belongs to $[0,1]$ and satisfies*

$$r = \frac{\mathcal{H}^H(x, u_h(x), u_h)}{\mathcal{H}^M(x, u_h(x), u_h)}, \quad \ell(x) + \big(1-\ell(x)\big)r \geq \varepsilon'(h),$$

*where the parameters $\varepsilon(h)$ and $\varepsilon'(h)$ satisfy $\varepsilon'(h)^{-1}\varepsilon(h) = o(1)$,*

5. *there exists a unique solution $u_h$ of* (23) *that satisfies an $L^\infty$ bound that is uniform in $h$,*
6. *Equation (1) has a uniqueness principle.*

*Then the family $u_h$ defined by the scheme converges locally uniformly to the solution of (1) in $\Omega$.*

Given constants $\alpha_- \geq 1$, $\alpha_+ > 0$ and $\beta > 0$, an exemplary mixture parameter is

$$\ell = \begin{cases} \min(1, \alpha_-|r|) & \text{if } r \leq 0, \\ 0 & \text{if } 0 \leq r \leq \beta, \\ \min\big(1, \alpha_+(r-\beta)\big) & \text{otherwise.} \end{cases} \tag{30}$$

This is the one we have chosen in practical applications with $\beta = 0$ and $\alpha_+ = 1$. Implementation details can be found in (Abgrall, 2007).

## 7. Links Between Cartesian and Non-Conformal Meshes

It is not difficult to construct numerical Hamiltonians that work on general non-conformal meshes. The only key point is to construct *monotone* Hamiltonians. The convergence results of (Barles and Souganidis, 1991) and Theorem 2 can be easily adapted: a close inspection of the proof shows that the structure of the mesh plays *no* role. What matters is the problem of how to define, for any mesh point $M_i$, a local interpolation $\pi$ that operates on $U_i := \{u_j\}_{j\in\mathcal{V}_j}$ onto the space of piecewise linear functions, and such that if $u_j \leq v_j$, $j \in \mathcal{V}_j$, then $\pi(U_i) \leq \pi(V_i)$.

Consider Fig. 2. The neighbors of $M_i$ are $\{P_i\}_{i=1,\ldots,8}$, from which we construct a local triangulation (dotted lines) that is used to define a piecewise linear interpolant. It does not need to be continuous. Then

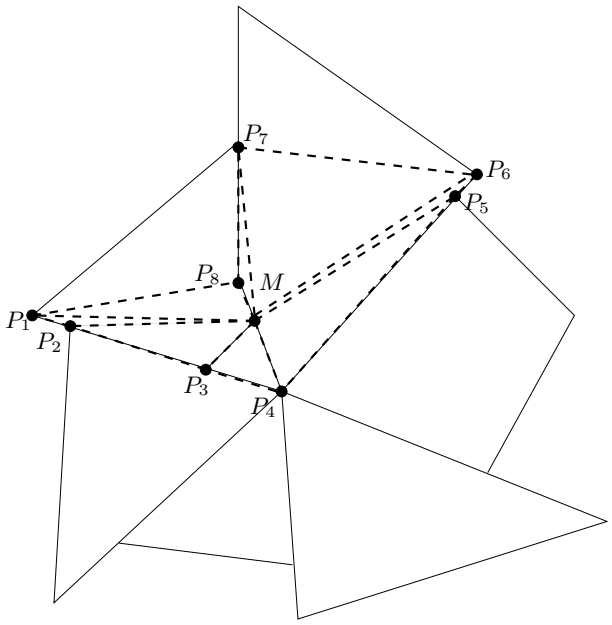



Fig. 2. Case of a non-conformal mesh.

we can use our Hamiltonians to define schemes that are clearly consistent and monotone. The tricky part is the choice of the neighbors. Figure 2 shows an extreme case. A probably better choice would be to choose only $\{P_2, P_3, P_4, P_5, P_7\}$ because the aspect ratio of the triangles is larger.

Note that the Hamiltonians of (Osher and Shu, 1991), thanks to this set of remarks, are particular cases of our formula.

## 8. Numerical Results

In general, it is difficult to compute analytically the solution of a first-order Hamilton-Jacobi equation, and the

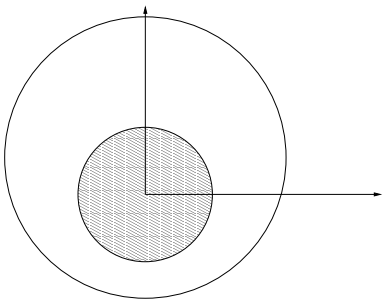


Fig. 3. Computational domain for the problem (31). $\Gamma_1$ is the inner circle with center $(0, 0)$ and radius $r = 1$, $\Gamma_2$ is the outer circle (center $(0, 0.5)$, radius $r = 3$).

situation is even worse when the Hamiltonian is not convex (nor concave) because the analogy with hyperbolic systems becomes looser in general. Hence it becomes more difficult to assess the quality of numerical results. To overcome this difficulty in a special case, we consider $H(p) = (\|p\| - 1)^3$ and the problem

$$\begin{aligned} H(Du) &= 0 \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1, \\ u &= 10 \quad \text{on } \Gamma_2 \end{aligned} \tag{31}$$

where $\Omega$ is displayed in Fig. 3. Since $t \mapsto t^3$ is monotone increasing, $u$ is a solution of (31) if and only if it is a solution of

$$\begin{aligned} \|Dv\| - 1 &= 0 \quad \text{on } \Omega, \\ v &= 0 \quad \text{on } \Gamma_1, \\ v &= 10 \quad \text{on } \Gamma_2. \end{aligned} \tag{32}$$

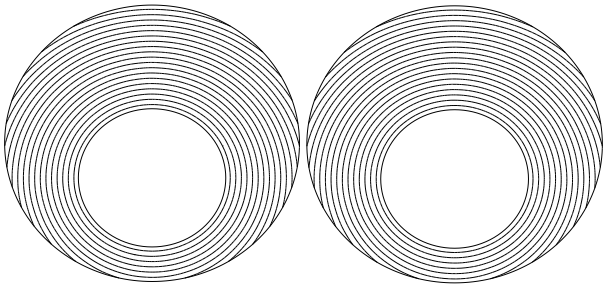
The solution of (31) and (32) is the distance to $\Gamma_1$.

In order to discretize (31), we write $H = H_1 + H_2$ with $H_1(p) = \max(\|p\| - 1, 0)^3$ and $H_2(p) = \min(\|p\| - 1, 0)^3$. These functions are respectively convex and concave. The numerical Hamiltonian is the Lax-Friedrich one and the boundary Hamiltonian is Godunov's. The numerical solution is displayed in Fig. 4(a). The solution of (32) with the Godunov Hamiltonian is provided in Fig. 4(b). A comparison reveals that they are (almost) identical.

Another application of the boundary conditions is given by the approximation to the following problem on the same geometry:

$$\begin{aligned} H(Du) &= 0 && \text{on } \Omega, \\ u(x, y) &= 0 && (x, y) \in \Gamma_1, \\ u(x, y) &= 3\cos(2\pi x) && (x, y) \in \Gamma_2. \end{aligned} \tag{33}$$

Since $H$ is not convex, it is difficult to assert *a priori* what would be the value of the solution on the boundary. The computed solution is given in Fig. 5(a). It can be seen that the solution satisfies strongly the boundary condition on



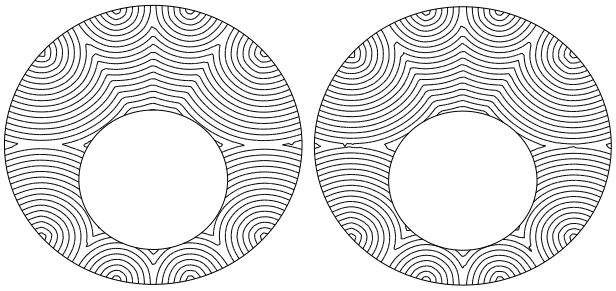
(a) (b)

Fig. 4. Computed solutions:
(a) Problem (31), $\min = 0$, $\max = 1.48$,
(b) Problem (32), $\min = 0$, $\max = 1.504$.

$\Gamma_2$ and only weakly on $\Gamma_1$ (in contrast to the previous example). Note, however, that they have been numerically *weakly* imposed on $\Gamma_1$ and $\Gamma_2$. The solution is also in very good agreement with the one obtained from the discretization of

$$\begin{aligned} \|Dv\| - 1 &= 0 && \text{on } \Omega, \\ v(x,y) &= 0 && (x,y) \in \Gamma_1, \\ v(x,y) &= 3\cos(2\pi x) && (x,y) \in \Gamma_2, \end{aligned} \tag{34}$$

which is displayed in Fig. 5(b).



(a) (b)

Fig. 5. Computed solutions:
(a) Problem (33), $\min = -3$, $\max = -1.53$,
(b) Problem (34), $\min = -3$, $\max = -1.47$.

We also show how the high-order extension of Section 6 works when the Godunov solver is employed with $P^2$ interpolation for the high-order scheme. The zoom is displayed in Fig. 6. Clearly, a very large overshoot exists where $u$ is not $C^1$. There the solution of the blended scheme is monotone and very similar to the first-order one. In the smooth part of the solution, the second-order and mixed schemes are very similar (the results by the mixed scheme are slightly more dissipative than those of the second-order unlimited scheme).

The last figure shows that our implementation of the boundary conditions is effectual if we impose strongly the boundary conditions, as in Fig. 7(a). This has to be compared with Fig. 5. Figure 7 shows that there is a strong boundary layer in parts of the outer boundary (where the isolines bunch). This is not true for Fig. 7(a). In fact, in some parts of the outer boundary, the compatibility condition of (Lions, 1982) is true, so that one can impose the boundary conditions strongly, and in other parts this is not true and we have to apply them weakly. This partition of the boundary is not known *a priori*: our implementation takes this into account automatically.

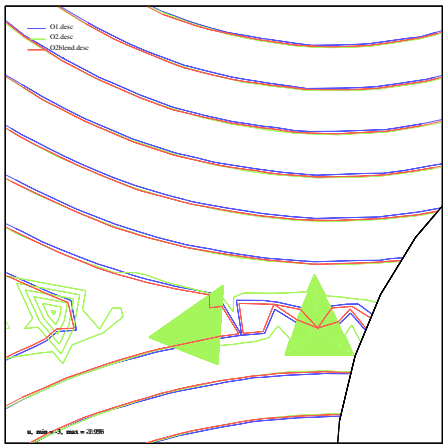
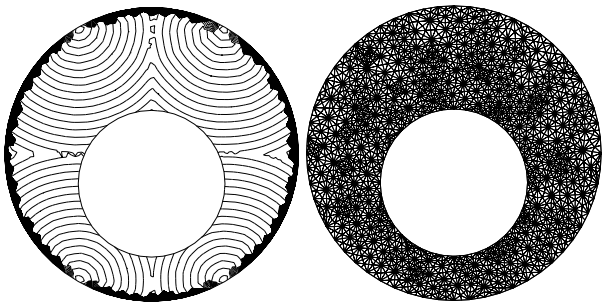



Fig. 6. Comparison of the solution of the problem (34) with several schemes. (Blue: first-order scheme, $\min = -3$, $\max = -1.993$, green: second-order unlimited scheme, $\min = -3$, $\max = 23.25$, red: second-order mixed scheme, $\min = -3$, $\max = -1.996$.)



(a) (b)

Fig. 7. (a) Solution of the problem (34) when the boundary conditions are imposed strongly, (b) mesh.

## 9. Conclusions

We have described several techniques for the solution of first-order Hamilton-Jacobi equations. We attempted to explain the hidden details and the origins of the schemes. Several theoretical results were provided and the proofs were given in the references. Once more, there are many other methods for solving these problems. Some are very

general and some are specially tuned for a specific example such as computing a distance function, which is one of the key ingredients of the level set method.

## References


Abgrall R. (1996): *Numerical Discretization of First Order Hamilton-Jacobi Equations on Triangular Meshes*. Communications on Pure and Applied Mathematics, Vol. XLIX, No. 12, pp. 1339–1373.

Abgrall R. (2004): *Numerical discretization of boundary conditions for first order Hamilton Jacobi equations*. SIAM Journal on Numerical Analyis, Vol. 41, No. 6, pp. 2233–2261.

Abgrall R. (2007): *Construction of simple, stable and convergent high order schemes for steady first order Hamilton Jacobi equations*. (in revision).

Abgrall R. and Perrier V. (2007): *Error estimates for Hamilton-Jacobi equations with boundary conditions.* (in preparation).

Augoula S. and Abgrall R. (2000): *High order numerical discretization for Hamilton-Jacobi equations on triangular meshes.* Journal of Scientific Computing, Vol. 15, No. 2, pp. 197–229.

Bardi M. and Evans L.C. (1984): *On Hopf's formula for solutions of first order Hamilton-Jacobi equations.* Nonlinear Analysis Theory: Methods and Applications, Vol. 8, No. 11, pp. 1373–1381.

Bardi M. and Osher S. (1991): *The nonconvex multi-dimensional Riemann problem for Hamilton-Jacobi equations*. SIAM Journal on Mathematical Analysis, Vol. 22, No. 2, pp. 344–351.

Barles G. (1994): *Solutions de viscosité des équations de Hamilton-Jacobi*. Paris: Springer.

Barles G. and Souganidis P.E. (1991): *Convergence of approximation schemes for fully nonlinear second order equations*. Asymptotic Analysis, Vol. 4, No. 3, pp. 271–283.

Crandall M.G. and Lions P.L. (1984): *Two approximations of solutions of Hamilton-Jacobi equations*. Mathematics of Computation, Vol. 43, No. 167, pp. 1–19.

Deckelnick K. and Elliot C.M. (2004): *Uniqueness and error analysis for Hamilton-Jacobi equations with discontinuities.* Interfaces and Boundary, Vol. 6, No. 3, pp. 329–349.

Hu C. and Shu C.W. (1999): *A discontinuous Galerkin finite element method for Hamilton Jacobi equations*. SIAM Journal on Scientific Computing, Vol. 21, No. 2, pp. 666–690.

Li F. and Shu C.W. (2005): *Reinterpretation and simplified implementation of a discontinuous Galerkin method for Hamilton-Jacobi equations.* Applied Mathematics Letters, Vol. 18, No. 11, pp. 1204–1209.

Lions P.-L. (1982): *Generalized Solutions of Hamilton-Jacobi Equations.* Boston: Pitman.

Osher S. and Shu C. W. (1991): *High–order essentially nonoscillatory schemes for Hamilton-Jacobi equations.* SIAM Journal on Numerical Analysis, Vol. 28, No. 4, pp. 907–922.

Qiu J. and Shu C.W. (2005): *Hermite WENO schemes for Hamilton-Jacobi equations.* Journal of Computational Physics, Vol. 204, No. 1, pp. 82–99.

Zhang Y.T. and Shu C.W. (2003): *High-order WENO schemes for Hamilton-Jacobi equations on triangular meshes.* SIAM Journal on Scientific Computing, Vol. 24, No. 3, pp. 1005–1030.