

## OBLIQUE WATER WAVE DIFFRACTION BY A STEP

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This paper is concerned with the problem of diffraction of an obliquely incident surface water wave train on an obstacle in the form of a finite step. Havelock expansions of water wave potentials are used in the mathematical analysis to obtain the physical parameters reflection and transmission coefficients in terms of integrals. Appropriate multi-term Galerkin approximations involving ultraspherical Gegenbauer polynomials are utilized to obtain a very accurate numerical estimate for reflection and transmission coefficients which are depicted graphically. From these figures various interesting results are discussed.

**Key words:** finite step, Havelock expansion, Galerkin approximation, Gegenbauer polynomial, reflection and transmission coefficients.

### 1. Introduction

Mathematical study of water wave diffraction problems involving fixed vertical thin barriers of various configurations was initiated long back using a variety of mathematical methods (cf. Dean [1], Ursell [2], Evans [3], Porter [4], Mandal and Dolai [5], etc.). The problems of water wave scattering by an irregular bottom have received some considerable interest in the literature on linearised theory of water waves due to their importance in finding the effects of naturally occurring bottom obstacles such as sand ripples on the wave motion (cf. Roseau [6], Kreisel [7], Fitz Gerald [8], Hamilton [9], Newman [10], Miles [11], Mandal and Gayen [12], Dolai and Dolai [13]).

Problems involving the propagation of water waves in a fluid of variable depth can be divided into three categories: 'beach' problems, where the depth tends to zero, 'obstacle' problems, where the depth is a constant except for variations extending over a finite interval in space, and 'changing-depth' problems, where the depth changes from one limiting value to a second limiting value. There have been many investigations of the beach and obstacle problems (cf. Stoker [14], Wehausen and Laitone [15]), but comparatively few studies have been made on the 'changing depth' case (cf. Bartholomeusz [16], Evans and McIver [17], Newman [10]). The importance of wave propagation in the case of changing depth is obvious in many coastal situations such as the passage of waves over a continental shelf. As an idealization of such a problem, we consider here the case of wave propagation over a finite step, from one constant finite depth to another.

In the present paper we consider the problem of diffraction of an obliquely incident surface wave train on an obstacle in the form of a finite step, from one constant finite depth to another. By a suitable application of Havelock's expansions of water wave potentials, we obtain the reflection and transmission coefficients in terms of integrals. Appropriate multi-term Galerkin approximations involving ultraspherical Gegenbauer polynomials are utilized to obtain a very accurate numerical estimate for reflection and transmission coefficients which are depicted graphically. From these figures various interesting results are discussed.

## 2. Formulation of the problem

We consider the motion in an inviscid, homogeneous, incompressible liquid which is supposed to be confined between two vertical planes perpendicular to the edges of the step. Cartesian axes are chosen with the mean free surface, the  $(x, z)$  plane with the origin directly above the step, while the axis of  $y$  is directed down wards into the liquid. The shallower water is of depth  $h_1$  above the horizontal shelf, the deeper water is of depth  $h_2$ . A simple sketch of the problem is given in Fig.1.

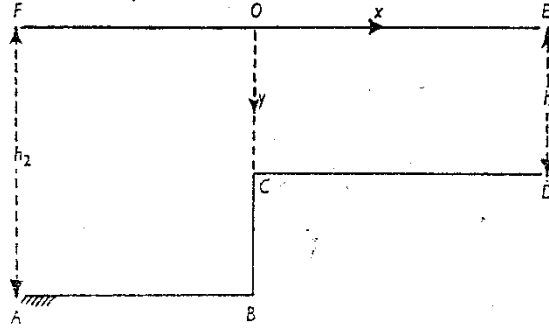


Fig.1. Geometry of the problem.

A simple harmonic progressive oblique wave train originating at  $x \rightarrow -\infty$  is incident on the step, and is partially reflected and partially transmitted. Assuming linear theory, the time harmonic progressive waves from negative infinity can be represented by the velocity potentials  $\text{Re}\{\varphi_-^{\text{inc}}(x, y)\exp(i\vartheta z - i\sigma t)\}$  where

$$\varphi_-^{\text{inc}}(x, y) = \frac{\cosh k_0(h_2 - y)}{\cosh k_0 h_2} \exp(i\mu x), \quad (2.1)$$

$k_0$  being the real positive root of

$$k \tanh kh_2 = K, \quad (2.2)$$

with  $K = \sigma^2 / g$ ,  $\vartheta = k_0 \sin \alpha$ ,  $\mu = k_0 \cos \alpha$ ,  $\sigma$  being the frequency of the incoming waves and  $g$  being the gravity. Due to the presence of the step, the oblique incident wave train is partially reflected by the step and partially transmitted through the gap. If the resulting motion is described by the velocity potential  $\text{Re}\{\varphi(x, y)\exp(i\vartheta z - i\sigma t)\}$ , then  $\varphi$  satisfies

$$\nabla^2 \varphi - \vartheta^2 \varphi = 0 \quad \text{in the fluid region,} \quad (2.3)$$

the free surface condition

$$K\varphi + \frac{\partial \varphi}{\partial y} = 0 \quad \text{on} \quad y = 0, \quad (2.4)$$

the bottom condition

$$\frac{\partial \varphi}{\partial y} = 0 \quad \text{on} \quad y = h_1, \quad x > 0 \quad \text{and} \quad y = h_2, \quad x < 0, \quad (2.5)$$

the condition on the step

$$\frac{\partial \varphi}{\partial x} = 0 \quad \text{on} \quad x = 0-, \quad y \in L(h_1 < y \leq h_2), \quad (2.6)$$

the edge condition

$$r^{1/3} \nabla \varphi \quad \text{is bounded as} \quad r \rightarrow 0, \quad (2.7)$$

$r$  is the distance from the edge, and the infinity condition

$$\varphi(x, y) \rightarrow \begin{cases} T_1 \frac{\cosh \alpha_0 (h_1 - y)}{\cosh \alpha_0 h_1} \exp(i\beta x) & \text{as } x \rightarrow \infty \\ \frac{\cosh k_0 (h_2 - y)}{\cosh k_0 h_2} \{ \exp(i\mu x) + R_1 \exp(-i\mu x) \} & \text{as } x \rightarrow -\infty \end{cases} \quad (2.8)$$

where  $\alpha_0$  satisfies  $\alpha_0 \tanh \alpha_0 h_1 = K$ ,  $\beta^2 = \alpha_0^2 - \mathfrak{G}^2$  and  $T_1, R_1$  are the unknown transmission and reflection coefficients to be determined.

### 3. Method of solution

Since  $\varphi_x(x, y)$  and  $\varphi(x, y)$  are continuous across OC, we can write

$$\left( \frac{\partial \varphi}{\partial x} \right)_{x=0+} = \left( \frac{\partial \varphi}{\partial x} \right)_{x=0-} = f(y), \quad \text{say, for } 0 < y < h_1, \quad (3.1)$$

$$(\varphi)_{x=0+} = (\varphi)_{x=0-} \quad \text{for } 0 < y < h_1. \quad (3.2)$$

A solution for  $\varphi(x, y)$  satisfying Eqs (2.3), (2.4), (2.5) and (2.8) can be represented as

$$\varphi(x, y) \rightarrow \begin{cases} T_1 \frac{\cosh \alpha_0 (h_1 - y)}{\cosh \alpha_0 h_1} \exp(i\beta x) + \sum_I A_n \cos \alpha_n (h_1 - y) \exp(-s_n x), & x \geq 0, \\ \frac{\cosh k_0 (h_2 - y)}{\cosh k_0 h_2} \{ \exp(i\mu x) + R_1 \exp(-i\mu x) \} + \sum_I B_n \cos k_n (h_2 - y) \exp(p_n x), & x \leq 0 \end{cases} \quad (3.3)$$

where  $s_n^2 = \alpha_n^2 + \mathfrak{G}^2$ ,  $p_n^2 = k_n^2 + \mathfrak{G}^2$ ,  $\alpha_n$  satisfy  $\alpha_n \tan \alpha_n h_1 + K = 0$ ,  $k_n$  satisfy  $k_n \tan k_n h_2 + K = 0$ .

Using Eq.(3.3) in Eqs (3.1) and (3.2), we find

$$f(y) = i\beta T_1 \frac{\cosh \alpha_0 (h_1 - y)}{\cosh \alpha_0 h_1} - \sum_I^{\infty} s_n A_n \cos \alpha_n (h_1 - y), \quad 0 < y < h_1, \quad (3.4)$$

$$= i\mu (I - R_1) \frac{\cosh k_0 (h_2 - y)}{\cosh k_0 h_2} + \sum_I^{\infty} p_n B_n \cos k_n (h_2 - y), \quad 0 < y < h_2,$$

and

$$T_1 \frac{\cosh \alpha_0 (h_1 - y)}{\cosh \alpha_0 h_1} + \sum_I^{\infty} A_n \cos \alpha_n (h_1 - y) = (I + R_1) \frac{\cosh k_0 (h_2 - y)}{\cosh k_0 h_2} + \sum_I^{\infty} B_n \cos k_n (h_2 - y), \quad 0 < y < h_1. \quad (3.5)$$

The use of Havelock's [18] inversion theorem in Eqs (3.4) produces

$$i\beta T_1 = \frac{4\alpha_0 \cosh \alpha_0 h_1}{2\alpha_0 h_1 + \sinh 2\alpha_0 h_1} \int_0^{h_1} f(y) \cos \alpha_0 (h_1 - y) dy, \quad (3.6)$$

$$-s_n A_n = \frac{4\alpha_n}{2\alpha_n h_1 + \sin 2\alpha_n h_1} \int_0^{h_1} f(y) \cos \alpha_n (h_1 - y) dy, \quad (3.7)$$

$$i\mu (I - R_1) = \frac{4k_0 \cosh k_0 h_2}{2k_0 h_2 + \sinh 2k_0 h_2} \int_0^{h_1} f(y) \cos k_0 (h_2 - y) dy, \quad (3.8)$$

$$p_n B_n = \frac{4k_n}{2k_n h_2 + \sin 2k_n h_2} \int_0^{h_1} f(y) \cos k_n (h_2 - y) dy. \quad (3.9)$$

Using Eqs (3.6), (3.7), (3.9) in Eq.(3.5) and from Eq.(3.8), we find

$$\int_0^{h_1} F_1(u) M_1(y, u) du = \frac{\cosh k_0 (h_2 - y)}{\cosh k_0 h_2}, \quad 0 < y < h_1, \quad (3.10)$$

$$\int_0^{h_1} F_1(y) \frac{\cosh k_0 (h_2 - y)}{\cosh k_0 h_2} dy = C_1 \quad (3.11)$$

where

$$F_1(u) = \frac{-f(u)}{I + R_1} \frac{4k_0 \cosh^2 k_0 h_2}{\mu (2k_0 h_2 + \sinh 2k_0 h_2)},$$

$$\begin{aligned}
 M_I(y, u) &= \frac{\mu(2k_0h_2 + \sinh 2k_0h_2)}{k_0 \cosh^2 k_0h_2} \left[ \frac{i\alpha_0 \cosh \alpha_0(h_1 - y) \cosh \alpha_0(h_1 - u)}{\beta(2\alpha_0h_1 + \sinh 2\alpha_0h_1)} + \right. \\
 &+ \left. \sum_I^{\infty} \left\{ \frac{\alpha_n \cos \alpha_n(h_1 - y) \cos \alpha_n(h_1 - u)}{s_n(2\alpha_nh_1 + \sin 2\alpha_nh_1)} + \frac{k_n \cos k_n(h_2 - y) \cos k_n(h_2 - u)}{p_n(2k_nh_2 + \sin 2k_nh_2)} \right\} \right], \\
 C_I &= \frac{-i(I - R_I)}{I + R_I}.
 \end{aligned} \tag{3.12}$$

It may be noted that the function  $F_I(y)$  and the constant  $C_I$  are real. The integral Eq.(3.10) is to be solved by  $(N + I)$  multi-term Galerkin approximations of  $F_I(y)$  in terms of ultraspherical Gegenbauer polynomials  $C_{2n}^{I/6}(y/h_1)$  by noting the behavior of  $F_I(y) \sim (h_1 - y)^{-1/3}$  as  $y \rightarrow h_1 - 0$  given by (cf. Kanoria *et al.* [19])

$$F_I(y) = \sum_{n=0}^N a_n f_n(y), \quad 0 < y < h_1 \tag{3.13}$$

where

$$f_n(y) = -\frac{d}{dy} \exp(-Ky) \int_y^{h_1} \exp(Kt) \hat{f}_n(t) dt, \quad 0 < y < h_1,$$

with

$$\hat{f}_n(y) = \frac{2^{7/6} \Gamma(I/6) (2n)!}{\pi \Gamma(2n + I/3) h_1^{1/3} (h_1^2 - y^2)^{1/3}} C_{2n}^{I/6}(y/h_1).$$

The unknown coefficients  $a_n$  ( $n = 0, 1, 2, \dots, N$ ) are obtained by solving the system of linear equations

$$\sum_{n=0}^N a_n \mathfrak{R}_{nm} = d_m, \quad m = 0, 1, 2, \dots, N \tag{3.14}$$

where

$$\begin{aligned}
 \mathfrak{R}_{nm} &= \frac{\mu(2k_0h_2 + \sinh 2k_0h_2)}{k_0 \cosh^2 k_0h_2} \left[ 4(-I)^{n+m} \sum_{r=1}^{\infty} \left\{ \frac{k_r \cos^2 k_r h_2}{p_r(2k_r h_2 + \sin 2k_r h_2)} \frac{J_{2n+1/6}(k_r h_2) J_{2m+1/6}(k_r h_2)}{(k_r h_2)^{1/3}} + \right. \right. \\
 &+ \left. \frac{\alpha_r \cos^2 \alpha_r h_1}{s_r(2\alpha_r h_1 + \sin 2\alpha_r h_1)} \frac{J_{2n+1/6}(\alpha_r h_1) J_{2m+1/6}(\alpha_r h_1)}{(\alpha_r h_1)^{1/3}} \right\} + \\
 &+ \left. \frac{i\alpha_0 \cosh^2 \alpha_0 h_1}{\beta(2\alpha_0 h_1 + \sinh 2\alpha_0 h_1)} \frac{I_{2n+1/6}(\alpha_0 h_1) I_{2m+1/6}(\alpha_0 h_1)}{(\alpha_0 h_1)^{1/3}} \right],
 \end{aligned}$$

$$d_m = \frac{I_{2m+1/6}(k_0 h_2)}{(k_0 h_2)^{1/6}}$$

Once  $a_n$  ( $n = 0, 1, 2, \dots, N$ ) are solved, the real constant  $C_I$  can be determined from Eq.(3.11)

$$C_I = \sum_{n=0}^N a_n d_n. \quad (3.15)$$

Then  $R_I$  can be found using Eq.(3.12) and  $T_I$  can be found from Eqs (3.6) using (3.13) as

$$R_I = \frac{I - iC_I}{I + iC_I}, \quad (3.16)$$

$$T_I = \frac{iA \cosh \alpha_0 h_1}{(\alpha_0 h_1)^{1/6}} \sum_{n=0}^N a_n I_{2n+1/6}(\alpha_0 h_1) \quad (3.17)$$

where

$$A = \frac{\alpha_0 \mu (I + R_I) \cosh \alpha_0 h_1 (2k_0 h_2 + \sinh 2k_0 h_2)}{\beta k_0 \cosh^2 k_0 h_2 (2\alpha_0 h_1 + \sinh 2\alpha_0 h_1)}.$$

If a simple harmonic progressive oblique wave train originating at  $x \rightarrow +\infty$  is incident on the step, and is partially reflected and partially transmitted, the time harmonic progressive waves from positive infinity can be represented by the velocity potentials  $\text{Re} \left\{ \varphi_+^{\text{inc}}(x, y) \exp(i\vartheta_I z - i\sigma t) \right\}$  where

$$\varphi_+^{\text{inc}}(x, y) = \frac{\cosh \alpha_0 (h_1 - y)}{\cosh \alpha_0 h_1} \exp(-i\beta_I x), \quad (3.18)$$

with  $\vartheta_I = \alpha_0 \sin \alpha$ ,  $\beta_I = \alpha_0 \cos \alpha$ .

If the resulting motion is described by the velocity potential  $\text{Re} \left\{ \varphi(x, y) \exp(i\vartheta_I z - i\sigma t) \right\}$ , then  $\varphi$  satisfies

$$\nabla^2 \varphi - \vartheta_I^2 \varphi = 0 \quad \text{in the fluid region,} \quad (3.19)$$

and the conditions from Eqs (2.4) to (2.7).

The behavior of  $\varphi(x, y)$  at infinity gives

$$\varphi(x, y) \rightarrow \begin{cases} T_2 \frac{\cosh k_0 (h_2 - y)}{\cosh k_0 h_2} \exp(-i\mu_I x) & \text{as } x \rightarrow -\infty \\ \frac{\cosh \alpha_0 (h_1 - y)}{\cosh \alpha_0 h_1} \{ \exp(-i\beta_I x) + R_2 \exp(i\beta_I x) \} & \text{as } x \rightarrow +\infty \end{cases} \quad (3.20)$$

where  $\mu_1^2 = k_0^2 - \mathfrak{G}_1^2$  and  $T_2, R_2$  are the unknown transmission and reflection coefficients to be determined.

A solution for  $\varphi(x, y)$  satisfying Eqs (3.19), (2.4), (2.5) and (3.20) can be represented as

$$\varphi(x, y) \rightarrow \begin{cases} T_2 \frac{\cosh k_0 (h_2 - y)}{\cosh k_0 h_2} \exp(-i\mu_1 x) + \sum_I^\infty D_n \cos k_n (h_2 - y) \exp(p_n' x), & x \leq 0, \\ \frac{\cosh \alpha_0 (h_1 - y)}{\cosh \alpha_0 h_1} \{ \exp(-i\beta_1 x) + R_2 \exp(i\beta_1 x) \} + \sum_I^\infty C_n \cos \alpha_n (h_1 - y) \exp(-s_n' x), & x \geq 0 \end{cases} \quad (3.21)$$

where  $s_n' = \sqrt{\alpha_n^2 + \mathfrak{G}_1^2}$ ,  $p_n' = \sqrt{k_n^2 + \mathfrak{G}_1^2}$ .

Using Eq.(3.21) in Eqs (3.1) and (3.2), we find

$$\begin{aligned} f(y) &= -i\mu_1 T_2 \frac{\cosh k_0 (h_2 - y)}{\cosh k_0 h_2} + \sum_I^\infty p_n' D_n \cos k_n (h_2 - y), \quad 0 < y < h_2, \\ &= i\beta_1 (R_2 - I) \frac{\cosh \alpha_0 (h_1 - y)}{\cosh \alpha_0 h_1} - \sum_I^\infty s_n' C_n \cos \alpha_n (h_1 - y), \quad 0 < y < h_1 \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} T_2 \frac{\cosh k_0 (h_2 - y)}{\cosh k_0 h_2} + \sum_I^\infty D_n \cos k_n (h_2 - y) &= (I + R_2) \frac{\cosh \alpha_0 (h_1 - y)}{\cosh \alpha_0 h_1} + \\ &+ \sum_I^\infty C_n \cos \alpha_n (h_1 - y), \quad 0 < y < h_1. \end{aligned} \quad (3.23)$$

Use of Havelock's [18] inversion theorem in Eq.(3.22) produces

$$-i\mu_1 T_2 = \frac{4k_0 \cosh k_0 h_2}{2k_0 h_2 + \sinh 2k_0 h_2} \int_0^{h_1} f(y) \cos k_0 (h_2 - y) dy, \quad (3.24)$$

$$-s_n' C_n = \frac{4\alpha_n}{2\alpha_n h_1 + \sin 2\alpha_n h_1} \int_0^{h_1} f(y) \cos \alpha_n (h_1 - y) dy, \quad (3.25)$$

$$i\beta_1 (R_2 - I) = \frac{4\alpha_0 \cosh \alpha_0 h_1}{2\alpha_0 h_1 + \sinh 2\alpha_0 h_1} \int_0^{h_1} f(y) \cos \alpha_0 (h_1 - y) dy, \quad (3.26)$$

$$p_n' D_n = \frac{4k_n}{2k_n h_2 + \sin 2k_n h_2} \int_0^{h_1} f(y) \cos k_n (h_2 - y) dy. \quad (3.27)$$

Using Eqs (3.24), (3.25), (3.27) in Eq.(3.23) and from Eq.(3.26), we find

$$\int_0^{h_1} F_2(u) M_2(y, u) du = \frac{\cosh \alpha_0 (h_1 - y)}{\cosh \alpha_0 h_1}, \quad 0 < y < h_1, \quad (3.28)$$

$$\int_0^{h_1} F_2(y) \frac{\cosh \alpha_0 (h_1 - y)}{\cosh \alpha_0 h_1} dy = C_2 \quad (3.29)$$

where

$$F_2(u) = \frac{f(u)}{I + R_2} \frac{4\alpha_0 \cosh^2 \alpha_0 h_1}{\beta_1 (2\alpha_0 h_1 + \sinh 2\alpha_0 h_1)},$$

$$M_2(y, u) = \frac{\beta_1 (2\alpha_0 h_1 + \sinh 2\alpha_0 h_1)}{\alpha_0 \cosh^2 \alpha_0 h_1} \left[ \frac{ik_0 \cosh k_0 (h_2 - y) \cosh k_0 (h_2 - u)}{\mu_1 (2k_0 h_2 + \sinh 2k_0 h_2)} + \right. \\ \left. + \sum_1^{\infty} \left\{ \frac{\alpha_n \cos \alpha_n (h_1 - y) \cos \alpha_n (h_1 - u)}{s_n' (2\alpha_n h_1 + \sin 2\alpha_n h_1)} + \frac{k_n \cos k_n (h_2 - y) \cos k_n (h_2 - u)}{p_n' (2k_n h_2 + \sin 2k_n h_2)} \right\} \right]. \quad (3.30)$$

$$C_2 = \frac{i(R_2 - I)}{R_2 + I}.$$

It may be noted that the function  $F_2(y)$  and the constant  $C_2$  are real. The integral Eq.(3.28) is to be solved by  $(N+1)$  multi-term Galerkin approximations of  $F_2(y)$  in terms of ultraspherical Gegenbauer polynomials  $C_{2n}^{1/6}(y/h_1)$  by noting the behavior of  $F_2(y) \sim (h_1 - y)^{-1/3}$  as  $y \rightarrow h_1 - 0$  given by (cf. Kanoria *et al.* [19])

$$F_2(y) = \sum_{n=0}^N a_n' f_n(y), \quad 0 < y < h_1 \quad (3.31)$$

where

$$f_n(y) = -\frac{d}{dy} \exp(-Ky) \int_y^{h_1} \exp(Kt) \hat{f}_n(t) dt, \quad 0 < y < h_1,$$

with

$$\hat{f}_n(y) = \frac{2^{7/6} \Gamma(1/6) (2n)!}{\pi \Gamma(2n + 1/3) h_1^{1/3} (h_1^2 - y^2)^{1/3}} C_{2n}^{1/6}(y/h_1).$$

The unknown coefficients  $a_n'$  ( $n=0, 1, 2, \dots, N$ ) are obtained by solving the system of linear equations



$$\sum_{n=0}^N a_n' \mathfrak{R}_{nm}' = d_m', \quad m = 0, 1, 2, \dots, N \quad (3.32)$$

where

$$\mathfrak{R}_{nm}' = \frac{\beta_l (2\alpha_0 h_1 + \sinh 2\alpha_0 h_1)}{\alpha_0 \cosh^2 \alpha_0 h_{12}} \left[ 4(-1)^{n+m} \sum_{r=1}^{\infty} \left\{ \frac{\alpha_r \cos^2 \alpha_r h_{12}}{s_r' (2\alpha_r h_{12} + \sin 2\alpha_r h_{12})} \frac{J_{2n+1/6}(\alpha_r h_1) J_{2m+1/6}(\alpha_r h_1)}{(\alpha_r h_1)^{1/3}} + \frac{k_r \cos^2 k_r h_2}{p_r' (2k_r h_2 + \sin 2k_r h_2)} \frac{J_{2n+1/6}(k_r h_2) J_{2m+1/6}(k_r h_2)}{(k_r h_2)^{1/3}} \right\} + \frac{ik_0 \cosh^2 k_0 h_2}{\mu_l (2k_0 h_2 + \sinh 2k_0 h_2)} \frac{I_{2n+1/6}(k_0 h_2) I_{2m+1/6}(k_0 h_2)}{(k_0 h_2)^{1/3}} \right],$$

$$d_m' = \frac{I_{2m+1/6}(\alpha_0 h_1)}{(\alpha_0 h_1)^{1/6}}.$$

Once  $a_n'$  ( $n = 0, 1, 2, \dots, N$ ) are solved, the real constant  $C_2$  can be determined from Eq.(3.29)

$$C_2 = \sum_{n=0}^N a_n' d_n'. \quad (3.33)$$

Then  $R_2$  can be found using Eq.(3.30) and  $T_2$  can be found from Eq.(3.24) using (3.31) as

$$R_2 = \frac{I - iC_2}{I + iC_2}, \quad (3.34)$$

$$T_2 = \frac{iB \cosh k_0 h_2}{(k_0 h_2)^{1/6}} \sum_{n=0}^N a_n' I_{2n+1/6}(k_0 h_2) \quad (3.35)$$

where

$$B = \frac{k_0 \beta_l (I + R_2) \cosh k_0 h_2 (2\alpha_0 h_1 + \sinh 2\alpha_0 h_1)}{\mu_l \alpha_0 \cosh^2 \alpha_0 h_1 (2k_0 h_2 + \sinh 2k_0 h_2)}.$$

#### 4. Numerical results

Multi-term Galerkin approximations are used to obtain a numerical estimate for  $|R_I|$ ,  $|T_I|$  and  $|R_2|$ ,  $|T_2|$ . In the numerical computations we take at most six terms to produce a fairly accurate numerical estimate for  $|R_I|$ ,  $|T_I|$  and  $|R_2|$ ,  $|T_2|$ .

We display a representative set of numerical estimates for  $|R_I|$ ,  $|T_I|$  and  $|R_2|$ ,  $|T_2|$  in Tab.1 and Tab.2, taking  $N = 0, 1, 2, 3, 4$  and  $5$  in the  $(N + 1)$ - term of Galerkin approximations and some particular values of the different parameters and the wave numbers.

Table 1

$h_1 / h_2 = 0.5, Kh_2 = 0.2, \alpha = 0^0$				
	$ R_1 $	$ R_2 $	$ T_1 $	$ T_2 $
0	0.236756	0.236756	1.126449	0.837985
1	0.234018	0.234018	1.127218	0.838556
2	0.233981	0.233981	1.127228	0.838564
3	0.233979	0.233979	1.127229	0.838565
4	0.233979	0.233979	1.127229	0.838564
5	0.233979	0.233979	1.127229	0.838564
$h_1 / h_2 = 0.5, Kh_2 = 0.2, \alpha = 30^0$				
	$ R_1 $	$ R_2 $	$ T_1 $	$ T_2 $
0	0.201124	0.157099	1.094152	0.934879
1	0.198444	0.154480	1.094762	0.935270
2	0.198408	0.154444	1.094771	0.935275
3	0.198406	0.154442	1.094771	0.935275
4	0.198405	0.154442	1.094771	0.935275
5	0.198405	0.154442	1.094771	0.935275

Table 2

$h_1 / h_2 = 0.5, Kh_2 = 1.8, \alpha = 0^0$				
	$ R_1 $	$ R_2 $	$ T_1 $	$ T_2 $
0	0.259046	0.259046	0.934748	0.998018
1	0.259317	0.259317	0.934678	0.997943
2	0.259138	0.259138	0.934724	0.997992
3	0.259127	0.259127	0.934727	0.997995
4	0.259126	0.259126	0.934727	0.997996
5	0.259126	0.259126	0.934727	0.997996
$h_1 / h_2 = 0.5, Kh_2 = 1.8, \alpha = 30^0$				
	$ R_1 $	$ R_2 $	$ T_1 $	$ T_2 $
0	0.217419	0.201057	0.923292	1.049337
1	0.218222	0.201998	0.923122	1.049129
2	0.218054	0.201835	0.923159	1.049165
3	0.218040	0.201825	0.923161	1.049167
4	0.218039	0.201824	0.923161	1.049167
5	0.218039	0.201824	0.923161	1.049167

It is observed from Tabs 1 and 2 that the computed results for  $|R_1|, |T_1|$  and  $|R_2|, |T_2|$  converge very rapidly with  $N$ , and for  $N \geq 3$  an accuracy of almost six decimal places is observed. It appears that the present numerical procedure for the numerical computations of reflection and transmission coefficients is quite efficient. We also note from these tables that for normal incidence of the wave train ( $\alpha = 0^0$ ),  $|R_1| = |R_2|$  ( $= |R|$ , say) and  $|T_1 T_2| = 1 - |R|^2$ . Similar observations were made by Newman (1965) for the infinite step problem.

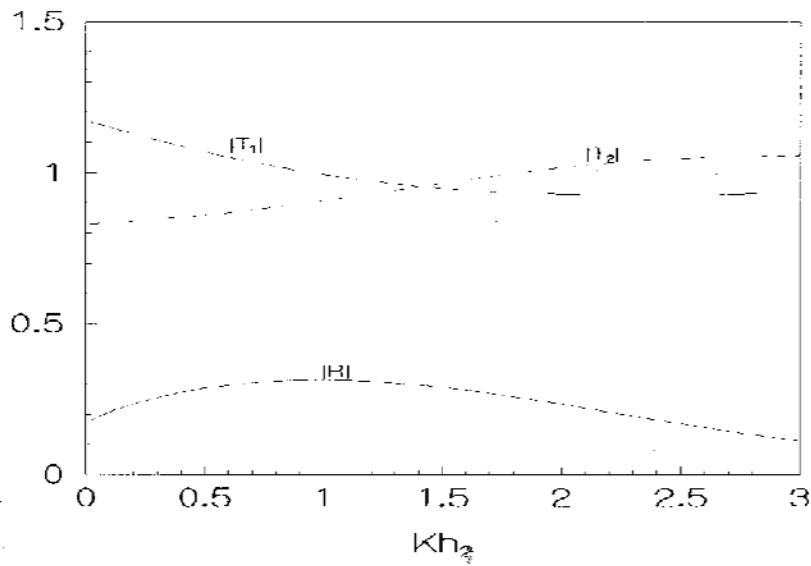


Fig.2. Reflection and transmission coefficients for  $h_1/h_2 = 0.5, \alpha = 0^\circ$ .

The computed results for  $|R|$  and  $|T_1|, |T_2|$  are plotted in Fig.2 against  $Kh_2$  for  $h_1/h_2 = 0.5, \alpha = 0^\circ$ . It is observed that  $|R|$  increases for some moderate values of  $Kh_2$  and then decreases as  $Kh_2$  increases. On the other hand,  $|T_1|$  decreases and  $|T_2|$  increases as  $Kh_2$  increases and  $|T_1| = |T_2| \approx 1.0$  for some particular value of  $Kh_2 \approx 1.5$ . Also, it is observed that  $|R| \rightarrow 0$  and  $|T_1| \rightarrow 1, |T_2| \rightarrow 1$  as  $Kh_2 \rightarrow \infty$ . These type of observations are quite plausible because for small  $Kh_2$ , waves penetrate on the step and for large  $Kh_2$ , waves are mainly confined at the free surface, so that waves are totally transmitted through the gap.

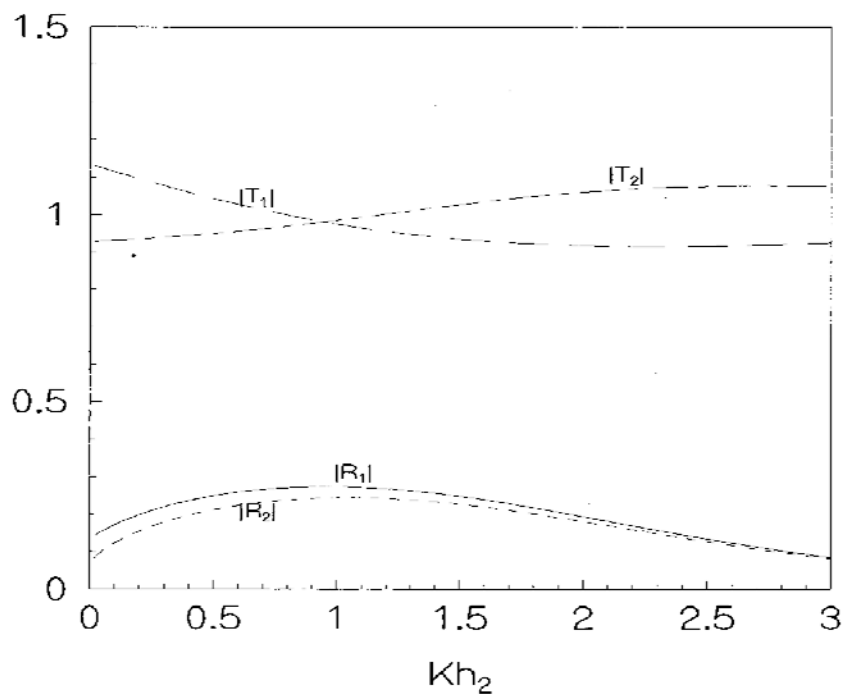


Fig.3. Reflection and transmission coefficients for  $h_1/h_2 = 0.5, \alpha = 30^\circ$ .

In Fig.3, the computed results for  $|R_1|, |R_2|$  and  $|T_1|, |T_2|$  are plotted against  $Kh_2$  for  $h_1/h_2 = 0.5, \alpha = 30^\circ$ . It is observed that  $|R_1| > |R_2|$  for all  $Kh_2$ . It is quite obvious because the reflection coefficient for penetration from the front side of the step will be greater than penetration from the back side of the step.  $|R_1|$  and  $|R_2|$  first increases then decreases as  $Kh_2$  increases and tending to zero as  $Kh_2 \rightarrow \infty$ . Also,  $|T_1|$  decreases and  $|T_2|$  increases as  $Kh_2$  increases and  $|T_1| = |T_2| \approx 1.0$  for some particular value of  $Kh_2 \approx 1.0$ . It is also observed that  $|T_1| \rightarrow 1, |T_2| \rightarrow 1$  as  $Kh_2 \rightarrow \infty$ .

#### 4. Conclusion

The method of multi-term Galerkin approximations in terms of ultra spherical Gegenbauer polynomials has been utilized here to obtain very accurate numerical estimates for the reflection and transmission coefficients in the water wave scattering problem of an obliquely incident surface wave train on an obstacle in the form of a finite step. By choosing only five terms in the Galerkin approximations, we achieve accuracy almost to the sixth digit after the decimal point. In the numerical estimates for the reflection and transmission coefficients. The numerical results are illustrated in tables and figures. For normal incidence of the surface wave train, the results are compared with the known results available in the literature for similar problems and very good agreement is achieved.

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#### Nomenclature

- $g$  – gravity
- $h_1$  – depth of the upper step
- $h_2$  – depth of the lower step
- $K$  – wave number
- $R_1, R_2$  – reflection coefficients
- $T_1, T_2$  – transmission coefficient
- $t$  – time
- $x$  – horizontal distance
- $y$  – vertical distance
- $\sigma$  – wave frequency
- $\varphi$  – velocity potential

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