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LQR-BASED NONLINEAR TUNING RELAY CONTROL DESIGN WITH FAST CONVERGENCE

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In this article, we present a relay control scheme based on LQR design with fast convergence. This scheme provides a practical and simple way to achieve fast convergence based on the well-known LQR design principle. The controller is a global stabiliser in the sense that for any given initial condition, we can always initialize the controller to drive the system to reach the origin. This controller is tunable in accordance with the position of the system state: the closer to the origin, the larger the control gains, which results in a fast control that maintains bounded control magnitude. It has also been shown that setting matrix Q can significantly influence the tendency of eigenvalues to switch the hyperplane. The relation between matrix Q and the tendency of those eigenvalues has been identified. Simulation results are presented to demonstrate the effectiveness of the scheme.

Keywords: switching control, LQR, relay control, sliding mode.

1. Introduction

Power electronics and relay systems are limited to a finite set of control values. The control techniques for these systems are often based on a conventional controller followed by a saturation function, which is often sufficient (Ledwich, 1995). A fast control is required in a real control situation as the accuracy and time to reach targets constitute performance indicators of a good control system. Commonly used linear control, due to its asymptotic stability nature, does not usually deliver a fast convergence performance. Appropriate nonlinear control can enable finite-time control, but the design of a fast control is difficult and there is no general theory available.

In (Ledwich, 1995) a novel method was proposed for the design of linear switching hyperplanes based on the sequential linear optimal relay principle. The LQR design method was used in a relay controller to yield a sequence of feedback gains k by

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letting a control parameter decrease from a large value to zero such that $kx = 0$ becomes an adaptively adjusted switching hyperplane. Fast convergence performance was reported. Based on the idea proposed in (Ledwich, 1995), in this paper, we will develop a relay controller based on the LQR design for SISO systems that enables a fast control. We will show that under a mild condition, the relay controller is a global stabiliser. This controller is simple to implement because it makes use of the existing LQR design technique which is a standard component of many control softwares. We will use a complexity function approach (Fisher and Reges, 1992) to analyse the fast control performance applicable to general single-input single-output linear systems. The selection of matrix Q in the LQR design will be examined in terms of convergence performance. Simulation results will be presented to show the effectiveness of the control design.

This paper is organized as follows. In Section 2, we present a description of the problem to be dealt with. The stability analysis is given in Section 3. The main results are derived in Section 4. Numerical simulations are shown in Section 5. Conclusions are drawn in Section 6.

2. Problem Statement

Consider the single input single output linear system (SISO) in the canonical form

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^1$ is the control and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix}^T$$

The system (A, B) is assumed to be stabilisable and A is stable. The control law u is a relay control defined as

$$u = \operatorname{sgn}(kx) \tag{2}$$

where $k \in \mathbb{R}^n$ is a control gain vector to be determined.

In this paper, we investigate the design of a fast convergent controller by using the LQR design principle.

Most nonlinear systems are assumed to satisfy the Lipschitz condition. This condition guarantees the existence of a unique solution for each initial condition, and results in asymptotic convergence. The dynamics with finite-time convergence does not satisfy the Lipschitz condition. For example, the dynamics $\dot{z} = -\beta z^{1/3}$

(Zak, 1989) does not satisfy the Lipschitz condition, because the Jacobian $\mathrm{d}\dot{z}/\mathrm{d}z = -(1/2)z^{-2/3} \to -\infty$ as $z \to 0$, which is actually the tangent to the curve (or an eigenvalue), and this means that the closer the trajectory to the origin, the faster the convergence speed, which results in fast (finite time) convergence which yields a near time-optimal response. Hence an important characteristic of fast convergence is that all eigenvalues tend to infinity when the trajectory tends to the origin. Note that a time-optimal control exhibits fast (finite-time) convergence.

The standard LQR problem is as follows: For the system (1) and a given quadratic performance index

$$J = \int_0^\infty \left(x^T Q x + r u^2\right) \mathrm{d}t \tag{3}$$

find an optimal linear state feedback control gain k

$$k = -r^{-1} B^T P \tag{4}$$

such that (3) is minimized. Here P is a positive-definite symmetric matrix which satisfies the algebraic Riccati equation

$$PA + A^T P - PBr^{-1}B^T P + Q = 0 \tag{5}$$

The problem is how to use kx as a relay input to form a switching hyperplane such that the closed-loop system is asymptotically stable and the system exhibit fast convergence. A preliminary study (Ledwich, 1995) showed that decreasing r would improve the convergence speed. A modified fast convergence algorithm (MFCA) based on (Ledwich, 1995) is given as follows:

1) Select a semi-positive definite symmetric matrix Q with rank $n - l$ $(l < n)$ and a large value r.

2) Solve eqn. (5) to obtain P and hence k. Record them in a table.

3) Based on P and k derived in Step 2, find a corresponding maximum Lyapunov function $V = x^T P x$ such that $|kx| < 2$.

4) Decrease r according to $r = r - \Delta r$, where Δr is a small value such that the new vector k derived from the Riccati equation satisfies $|kx| < 2$.

5) Then repeat Step 2 until r reaches an arbitrarily small positive value.

Note the Δr should be selected such that $|kx| < 2$ is always satisfied to maintain stability. The lower bound on r should be zero. As r tends to zero, the control becomes a high-gain feedback control which exhibits robustness. For effective control, ideally when $kx = 0$ under r is reached, Δr should be chosen such that the system state just reach a new $kx = 0$ under $r + \Delta r$. How to choose such a parameter is being studied.

The stability of the system under the relay control can be analyzed (Anderson and Moore, 1971). It can be shown that, as long as the open-loop system (A, B) is stabilisable and (A, D) (here $D^T D = Q$) is detectable, the closed-loop system with the LQR design is globally asymptotically stable. Also with the relay control using the LQR design, the asymptotic stability is guaranteed in the neighborhood of the switching hyperplane $kx = 0$. For states not close to the switching hyperplane, the stability of system (1) can be shown as follows.

Consider the Lyapunov function candidate

$$V = x^T P x \tag{6}$$

Differentiating (6) along the dynamics (1) yields

$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x + 2x^T P B \mathrm{sgn}(kx) \end{aligned}$$

Using eqn. (5), we get

$$\begin{aligned} \dot{V} &= -x^T Q x + x^T P B r^{-1} r r^{-1} B^T P x + 2x^T r r^{-1} P B \mathrm{sgn}(kx) \\ &= -x^T Q x + r(kx)^T (kx) - 2r(kx)^T \mathrm{sgn}(kx) \end{aligned}$$

For $\dot{V} < 0$ to hold, it is necessary and sufficient that

$$kx - 2\mathrm{sgn}(kx) < 0 \quad (\text{except at } kx = 0)$$

A sufficient condition for the asymptotic stability is

$$|kx| < 2 \tag{7}$$

Inequality (7) indicates that, to maintain the asymptotic stability, the larger value the state, the smaller the feedback gain k.

To guarantee the global stability, it is first necessary to know a range of possible initial conditions. For any given initial condition, we can always find a constant r such that (7) is satisfied. So, if at the beginning we derive a constant r such that (7) holds, then the condition (7) holds and hence the global stability is realized. In this sense, the controller is a global stabiliser.

3. Analysis of Fast Convergence Performance

In this section, we shall analyze the fast convergence performance of the system (1) with the MFCA. We assume that ideal adjusting k is done as soon as the switching manifold $kx = 0$ is reached. We shall first look at second-order systems to gain some motivation.

3.1. Second-Order Systems

Consider the second-order system defined by

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Given a positive/semi-positive definite matrix $Q = \mathrm{diag}(q_1, q_2)$ and a positive scale r, using (5) we can obtain a positive definite matrix P. The switching hyperplane $kx = 0$ is expressed through P by virtue of (4):

$$kx = p_{12}x_1 + p_{22}x_2 = 0 \tag{8}$$

The stability of (8) relies on a relation between p_{12}, p_{22} and r. From (5) we have

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} + \begin{bmatrix} 0 & -a_1 \\ 1 & -a_2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$-r^{-1}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} = 0$$

Hence

$$-2a_1p_{12} - r^{-1}p_{12}^2 + q_1 = 0 \tag{9}$$

$$2p_{12} - 2a_2p_{22} - r^{-1}p_{22}^2 + q_2 = 0 \tag{10}$$

As we have shown in the previous section, the LQR-based design for any positive r guarantees the system stability (Anderson and Moore, 1971). It implies when the switching line (8) is reached, it is asymptotically stable as well. Solving (9) and (10) is possible (but hard for higher-order systems), but here we use the complexity function approach, i.e. the O notation to simplify the analysis which can be easily extended to higher-order systems.

Consider the expression (9). Since q_1 and q_2 are constants (without loss of generality we assume $q_1 \neq 0$ but q_2 may be zero), the expression (9) holds only if one of the two terms a_1p_{12} and $r^{-1}p_{12}^2$ converges to a constant. Clearly, p_{12} cannot be a constant, otherwise $r^{-1}p_{12}^2 \to \infty$ as $r \to 0$. Furtheremore, $p_{12} = O(r^\gamma)$ for $\gamma \neq 1/2$ does not hold, otherwise the term $-2a_1p_{12} - r^{-1}p_{12}^2$ tends to either infinity or zero. This will not balance the constant q_1. The only solution is $p_{12} = O(r^{1/2})$.

For the expression (8), we consider two cases:

Case 1 ($q_2 \neq 0$): Similary to the analysis for (9), we can obtain $p_{22} = O(r^{1/2})$. The switching function (8) can now be rewritten as $px_1 + x_2 = 0$ but $p = p_{12}/p_{22} \to$ const. In this case fast control is not achievable because the slope of the switching line tends to a constant, which indicates asymptotic convergence.

Case 2 ($q_2 = 0$): In this case, $2p_{12} - 2a_2p_{22} - r^{-1}p_{22}^2 = 0$. The term $-2a_2p_{22} - r^{-1}p_{22}^2$ must converge to $p_{12} = O(r^{1/2})$. For $p_{22} = O(r^{\gamma})$ ($\gamma \neq 3/4$), it is easy to see that the term $(-2a_2p_{22} - r^{-1}p_{22}^2)/O(r^{1/2})$ will tend to either infinity or zero. Hence the solution is $p_{22} = O(r^{3/4})$. Therefore $p = p_{12}/p_{22} \to \infty$, and finite time convergence is achieved.

Now we look at third-order systems in order to gain a motivation for a possible solution for n-th order systems.

3.2. Third-Order Systems

We now consider the third-order system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For $Q = \mathrm{diag}(q_1, q_2, q_3)$, the corresponding Ricatti equation becomes

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -a_1 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}$$

$$-r^{-1} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} + \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix} = 0 \tag{11}$$

The switching equation is then $p_{13}x_1 + p_{23}x_2 + p_{33}x_3 = 0$. It also can be rewritten as $p_1x_1 + p_2x_2 + x_3 = 0$, where $p_1 = p_{13}/p_{33}$ and $p_2 = p_{23}/p_{33}$. We will split our discussion into three parts according to the three different forms of matrix Q.

Case 1 ($q_1 = 1,\ q_2 = 1,\ q_3 = 1$): The matrix Q has full rank. It follows from (11) that

$$-2a_1p_{13} - r^{-1}p_{13}^2 + 1 = 0 \tag{12}$$

$$2(p_{12} - a_2p_{23}) - r^{-1}p_{23}^2 + 1 = 0 \tag{13}$$

$$2(p_{23} - a_3p_{33}) - r^{-1}p_{33}^2 + 1 = 0 \tag{14}$$

$$-a_1p_{23} + p_{11} - a_2p_{13} - r^{-1}p_{13}p_{23} = 0 \tag{15}$$

$$-a_1p_{33} + p_{12} - a_3p_{13} - r^{-1}p_{13}p_{33} = 0 \tag{16}$$

$$-a_2p_{33} + p_{13} + p_{22} - a_3p_{23} - r^{-1}p_{23}p_{33} = 0 \tag{17}$$

Using the same argument as for the second-order systems, from (12) we can easily derive

$$p_{13} = O(r^{1/2}) \tag{18}$$

Since p_{12} is a function of r, from (13) and (14) we get

$$p_{23} = O(r^{1/2}) \tag{19}$$

$$p_{33} = O(r^{1/2}) \tag{20}$$

With (18), (19) and (20) the switching hyperplane turns out to be $\rho_1 x_1 + \rho_2 x_2 + x_3 = 0$, where $\rho_1 = p_{13}/p_{33} = \text{const}$ and $\rho_2 = p_{23}/p_{33} = \text{const}$. The corresponding characteristic polynomial of the switching function becomes

$$\lambda^2 + \rho_2\lambda + \rho_1 = 0 \tag{21}$$

It is easy to check that two eigenvalues of (21) will have finite values (but all are either negative or have a negative real part) as $r \to 0$. This means that the system trajectory will reach the origin asymptotically, but not in finite time.

Case 2 ($q_1 = 1$, $q_2 = 1$, $q_3 = 0$): The matrix Q does not have full rank. It follows from (10) that (12)–(17) will be the same except for (14) due to $q_3 = 0$ which becomes $2(p_{23} - a_3 p_{33}) - r^{-1}p_{33}^2 = 0$. Using the same argument as for Case 2 of second-order systems, since $p_{23} = O(r^{1/2})$, the complexity of p_{33} is

$$p_{33} = (rp_{23})^{1/2} = O(r^{3/4}) \tag{22}$$

With (18), (19) and (22), the switching hyperplane turns out to be $\rho_1 x_1 + \rho_2 x_2 + x_3 = 0$, where $\rho_1 = p_{13}/p_{33} = O(r^{-1/4})$ and $\rho_2 = p_{23}/p_{33} = O(r^{-1/4})$.

The characteristic polynomial of the switching function becomes

$$\lambda^2 + \rho_2\lambda + \rho_1 = 0 \tag{23}$$

and the two eigenvalues of (23) are $\lambda_1 \to -\infty$ and $\lambda_2 \to -1$ as $r \to 0$. This shows that when the system trajectories reach (23) with r approaching zero, one dimension of the trajectories on the two-dimensional hyperplane will approach zero very fast while the other dimension approaches zero asymptotically. Only one of the two dimensions has fast convergence.

Case 3 ($q_1 = 1$, $q_2 = 0$, $q_3 = 0$): The matrix Q does not have full rank. It follows from (11) again that eqns. (12)–(17) are the same except the ones containing $q_2 = 0$ and $q_3 = 0$. We have

$$2(p_{12} - a_2 p_{23}) - r^{-1}p_{23}^2 = 0 \tag{24}$$

$$2(p_{23} - a_3 p_{33}) - r^{-1}p_{33}^2 = 0 \tag{25}$$

First, because $q_1 = 1$, we have $p_{13} = O(r^{1/2})$. Equation (24) can be alternatively expressed as $p_{23}^2 + 2a_2 r p_{23} - 2rp_{12} = 0$, hence the complexity function of p_{23} is $p_{23} = O(\max(r, r^{1/2}p_{12}^{1/2})) = O(r^{1/2}p_{12}^{1/2})$. The same reasoning applies to (25), so $p_{33} = O(\max(r, r^{1/2}p_{23}^{1/2})) = O(\max(r, r^{3/4}p_{12}^{1/4})) = O(r^{3/4}p_{12}^{1/4})$. Since p_{12} is a function of r, from (16) we have $-a_1 p_{33} + p_{12} - a_3 p_{13} - r^{-1}p_{13}p_{33} = 0$, hence $O(\max(r, r^{3/4}p_{12}^{1/4}) +$

$p_{12} + O(r^{1/2}) + O(\max(r^{1/2}, r^{1/4}p_{12}^{1/4})) = 0$. The term that is slowest decreasing is $p_{12} = O(r^{1/4}p_{12}^{1/4})$, which leads to $p_{12} = O(r^{1/3})$. Hence $p_{23} = O(r^{2/3})$ and

$$p_{33} = O(r^{5/6}) \tag{26}$$

With (18) and (26) the switching hyperplane turns out to be $\rho_1 x_1 + \rho_2 x_2 + x_3 = 0$, where $\rho_1 = p_{13}/p_{33} = O(r^{-1/3})$ and $\rho_2 = p_{23}/p_{33} = O(r^{-1/6})$.

The characteristic polynomial of the switching function becomes

$$\lambda^2 + \rho_2\lambda + \rho_1 = 0 \tag{27}$$

and the real parts of the two eigenvalues of (27) $\operatorname{Re}\lambda_1 \to -\infty$ and $\operatorname{Re}\lambda_2 \to -\infty$ as $r \to 0$. This shows that when the system trajectories reach (27) and r approaches zero, all the trajectories will approach zero, thus the system exhibits fast convergence.

In summary, by varying the form of matrix Q, the system convergence can change from asymptotic convergence to partial and further complete finite-time convergence.

3.3. Higher-Order Systems

From the analysis for second-order and third-order systems, we can see that by keeping one entry in the Q matrix non-zero and the others zero, fast convergence can be achieved. Now we study higher-order systems. The matrix Q is of the form $Q = \operatorname{diag}(q_1, q_2, \ldots, q_n)$. Without loss of generality, we set $q_i = 1, \ i = 1, \ldots, m$ and $q_j = 0, \ j = m+1, \ldots, n$. The Ricatti equation yields the set of scalar equations

$$-2a_1 p_{1n} - r^{-1}p_{1n}^2 + 1 = 0 \tag{28}$$

$$2(p_{i,i-1} - a_i p_{i,n}) - r^{-1}p_{i,n}^2 + 1 = 0, \quad i = 1, \ldots, m \tag{29}$$

$$2(p_{j,j-1} - a_j p_{j,n}) - r^{-1}p_{j,n}^2 = 0, \quad j = m+1, \ldots, n \tag{30}$$

$$p_{i,j} - a_{j+1}p_{i,n} - a_i p_{j+1,n} + p_{i-1,j+1} - r^{-1}p_{i,n}p_{j+1,n} = 0$$
$$\text{for} \quad i = 1, \ldots, n-1 \quad \text{and} \quad j = i+1, \ldots, n-1 \tag{31}$$

$$p_{0,j+1} = 0, \quad j = i, \ldots, n-1$$

The formats of (28)–(31) are similar to the ones for the third-order systems. We omit tedious computations here. For $r \to 0$, similary to the third-order systems, we can get

$$p_{i,n} = O(r^{1/2}), \quad i = 1, \ldots, m \tag{32}$$

$$p_{i,n} = O(r^{1/2}p_{i-1,i}^{1/2}), \quad i = m+1, \ldots, n \tag{33}$$

$$p_{i,i+1} = O(r^{-1}p_{i,n}p_{i+2,n}), \quad i = 1, \ldots, n-2 \tag{34}$$

As in the case for the third-order systems, by solving (32)–(34), the complexity of $p_{i,n}$ is

$$p_{i,n} = \begin{cases} O(r^{1/2}) & \text{if } \ 1 \le i \le m \\ O(r^{\frac{n-2m+1+i}{2(n+1-m)}}) & \text{if } \ m < i \le n \end{cases}$$

The switching hyperplane is then written as

$$\rho_1 x_1 + \rho_2 x_2 + \cdots + \rho_{n-1} x_{n-1} + x_n = 0$$

where

$$\rho_i = \frac{p_{i,n}}{p_{nn}} = \begin{cases} O(r^{-\frac{n-m}{2(n+1-m)}}) & \text{if } \ 1 \le i \le m \\ O(r^{-\frac{n-i}{2(n+1-m)}}) & \text{if } \ m < i \le n-1 \end{cases}$$

The corresponding characteristic polynomial is

$$\lambda^{n-1} + \rho_{n-1}\lambda^{n-2} + \cdots + \rho_2\lambda + \rho_1 = 0 \tag{35}$$

As is demonstrated in (Anderson and Moore, 1971), when the system trajectories are near the switching hyperplane, the stability of the system is guaranteed. That implies that all the eigenvalues of (35) are located on the left half of the complex plane as $r \to 0$. What we are concerned with, is the tendency of each eigenvalue of (35).

Set $\theta = O(r^{-1/2(n+1-m)})$. The coefficients become

$$\rho_i = \begin{cases} O(\theta^{n-m}) & \text{if } \ 1 \le i \le m \\ O(\theta^{n-i}) & \text{if } \ m < i \le n-1 \end{cases}$$

The polynomial (35) can be rewritten as

$$\lambda^{n-1} + O(\theta)\lambda^{n-2} + O(\theta^2)\lambda^{n-3} + \cdots + O(\theta^{n-m})\lambda^{n-m-1} + \cdots + O(\theta^{n-m})\lambda + O(\theta^{n-m}) = 0 \tag{36}$$

It is easy to see that among $n-1$ eigenvalues of the polynomial, there are $n-m$ eigenvalues which are of $O(\theta)$ and the other $m-1$ eigenvalues are constant. Indeed, from the relations between the coefficients and eigenvalues, assuming that the $n-1$ eigenvalues are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$, it is well known that

$$\sum_{i=1}^{n-1} \lambda_i = -O(\theta) \tag{37}$$

$$\sum_{i,j=1}^{n-1} \lambda_i \lambda_j = O(\theta^2) \tag{38}$$

$$\vdots$$

$$\lambda_1 \lambda_2 \cdots \lambda_{n-1} = O(\theta^{n-m}) \tag{39}$$

From (36), one can see that there exists at least one eigenvalue which is $O(\theta)$. Without loss of generality, assume that λ_1 is such an eigenvalue. From (37) one can see that, since $\lambda_1 = O(\theta)$, there exists at least another eigenvalue which is $O(\theta)$ such that (37) holds. Continuing this deduction, we can draw the conclusion that there are $n-m$ eigenvalues which are $O(\theta)$ and the other $m-1$ eigenvalues are constant.

In conclusion, for higher-order systems with Q of rank m, as r tends to zero, $n-m$ eigenvalues will approach $-\infty$, while the others will converge to fixed values on the left side of the complex plane. In order to achieve fast convergence, the rank of Q should be set to one.

3.4. Occurrence of Sliding Modes

Since the system (1) under the control (2) with MFCA is asymptotically stable, the system trajectory will approach the equilibrium. It should be noted that the sliding mode will occur when the trajectory is close to the equilibrium. Indeed, take the Lyapunov function $V(x) = (kx)^2/2$. Differentiating it along the system dynamics (1) with (2) yields

$$\begin{aligned} \dot{V} &= (kx)k\dot{x} \\ &= (kx)k\big(Ax + B\text{sgn}(kx)\big) \\ &= (kx)\big(kAx + kB\text{sgn}(kx)\big) \\ &= (kx)kAx + kB|kx| \end{aligned}$$

Since $k = -r^{-1}B^TP$ and the system is asymptotically stable, there exists a moment t_0 such that for $t > t_0$ we have $|kB| > |kAx|$, hence a sliding mode on $s = kx = 0$ occurs. The system then exhibits the invariance properties of the sliding mode control (Utkin, 1992).

4. Numerical Simulation

For simulation studies, a second-order system and a fourth-order system were used and the simulation software was Matlab.

4.1. Second-Order System

The parameters were chosen as

$$A = \begin{bmatrix} 0 & 1 \\ -8 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and the value of r was decreased from 10^2 to 10^{-8}. The initial condition was $x(0) = (0.3, 0)$.

Figure 1(a) shows the response of the system states versus time. Figure 1(b) depicts the phase-plane trajectory. It can be seen that when close to the equilibrium, the slope of the trajectory tends to $-\infty$, which is an indication of finite time convergence. Figure 1(c) presents the corresponding switching function. The ripple is due to the sampling period which was set to 0.01 sec. The zigzagging was due to changing k. The tendency of finite time convergence was observed. However, the power of this scheme is better shown in the following higher-order system.

4.2. Fourth-Order System

The parameters were chosen as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

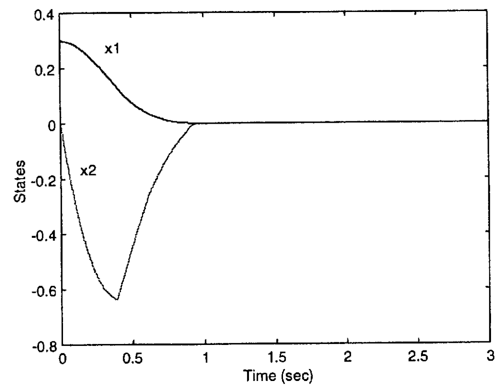
and the value of r was decreased from 10^2 to 10^{-8}. The initial condition was $x(0) = (0.3, 0, 0, 0)$.

Figure 2(a) shows the system response as a function of time. Figure 2(b) presents the corresponding switching function. It is seen that the switching hyperplane is reached before the state reaches zero. The ripples are due to the iterative changes in k. Along the switching hyperplane, the system trajectory is guided to reach zero very quickly.

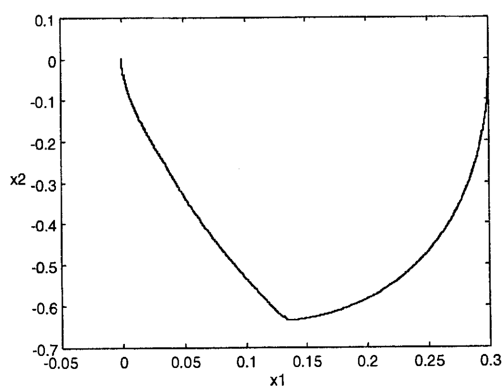
5. Conclusion

In this paper, we have presented a relay control scheme based on the LQR design with fast convergence. This approach provides a practical and simple way to achieve fast convergence based on the well-developed LQR approach. The controller is a kind of global stabiliser which is tunable in accordance with the position of the system state: the closer to the origin, the larger the control gains, which results in a non-linear tuning relay control that maintains a bounded control magnitude as well and exhibits fast convergence. It has also been proved that setting the Q matrix can significantly influence the tendency of eigenvalues to switch the hyperplane. A relation between the Q matrix and the tendency of the eigenvalues have been identified. It should be noted, under ideal switching, that the convergence of the proposed scheme is nearly time-optimal. The advantage of the scheme is that, unlike the situation in time-optimal control where obtaining an analytic solution for linear systems of orders higher than two is very hard if not possible, it provides an alternative analysis and synthesis tool for fast control design.

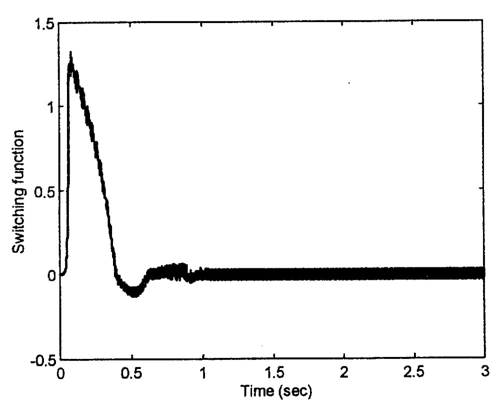
It should be noted that when r tends to zero, the gain k tends to infinity. However, this does not translate to a high gain control, rather the 'slope' of the switching line becomes 'deeper', which results in a fast transient process. The robustness is reflected in the situation that in most cases, when r tends to zero, the magnitudes of the states become small, hence they make the asymptotic stability condition easier to be satisfied. A further study will be conduced for MIMO linear systems and non-minimum phase systems.



(a)

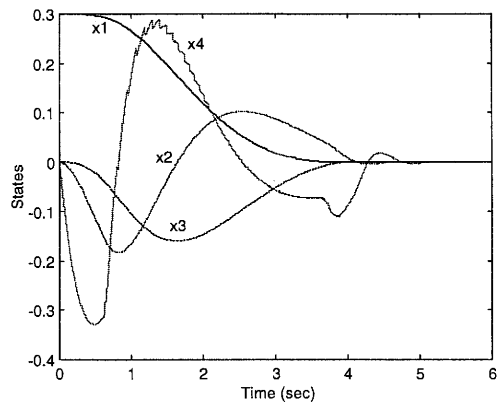


(b)

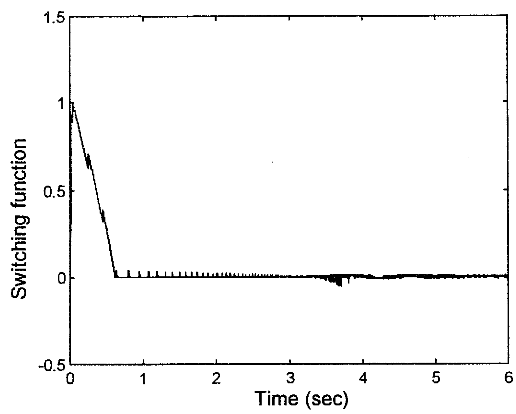


(c)

Fig. 1. The second-order system: the system responses (a), phase trajectory (b) and switching function (c).



(a)



(b)

Fig. 2. The fourth-order system: the system responses (a) and switching function (b).

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