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DYNAMICAL IDENTIFICATION OF CONTROLS FOR PARABOLIC VARIATIONAL INEQUALITIES

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The reconstruction problems of the unknown parameters based on the available information is one of the fundamental problems in the field of optimal control. This paper presents a new method of the positional modelling of distributed controls for the system described by the parabolic variational inequalities.

1. Introduction

The problems of reconstruction of unknown parameters on the basis of the available information about a considered object are well known in engineering and scientific research. For certain modern problems there is the necessity to reconstruct parameters dynamically (in the real time). In such cases the available information can vary in time and depend on the past of the process of reconstruction.

2. Problem Formulation

The goal of the present paper is to demonstrate the method of positional modelling of distributed controls (working on the basis of the feedback principle) for parabolic variational inequalities of the form

$$(\dot{x}(t) - B(t,x(t))u(t) - f(t),\ x(t) - z) + (Ax(t),\ x(t) - z)_{V^* \times V}$$

$$+\ \phi(x(t)) - \phi(z) \le 0 \ \text{ a.e. } \ t \in T = [t_0, \theta] \ \ \forall\, z \in V. \tag{1}$$

Here H and V are real Hilbert spaces with norms $|\cdot|_H$ and $|\cdot|_V$ resp., $H = H^*, V \subset H, V$ is densely and continuously imbedded in $H, (\cdot,\cdot)$ is the scalar product in $H, (\cdot,\cdot)_{V^* \times V}$ is the duality between V and V^*, $f(\cdot) \in L_2(T; H)$ is

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a given function, $A : V \to V^*$ is a linear, continuous and symmetric operator satysfying the condition

$$(Ay, y)_{V^* \times V} + \alpha |y|_H^2 \geq \omega |y|_V^2 \ \forall \ y \in V$$

for certain $\omega > 0$ and $\alpha, (U, |\cdot|_V)$ is a uniform by convex Banach space, $\phi : H \to \overline{R} = RU\{+\infty\}$ is a convex and lower semicontinuous proper function. A family $B(t,x) \in \mathcal{L}(U;H)(t \in T,\ x \in H)$ of operators Lipschitz continuous in x is such that for every function $\psi(\cdot) \in L_2(T;H)$ the operator $B_{\psi(\cdot)}$ defined by $\{B_{\psi(\cdot)}u(\cdot)\}(t) = B(t,\psi(t))u(t)$ a.e. $t \in T$, maps $L_2(T;U)$ into $L_2(T;H)$ – demicontinuously.

The problem in question can be explained as follows. The time interval T is put into parts by intervals $[\tau_i, \tau_{i+1})$, $\tau_{i+1} = \tau_i + \delta$, $\delta > 0$, $i \in [1 : m_\delta]$. An unknown control $u(\cdot) \in L_2(T;U)$ acting upon the system (1) generates a motion $x(\cdot) = x(\cdot; x_0, u(\cdot))$ being a solution of (1). At time instants τ_i the history $x_{\tau_{i-1},\tau_i}(\cdot)$ of the motion is measured approximately, i.e. a function $\psi_{\tau_{i-1},\tau_i}(\cdot)$ being an approximation to $x_{\tau_{i-1},\tau_i}(\cdot)$ is memorized. Let U_* be the set of all controls from $L_2(T;U)$ generating $x(\cdot)$. An algorithm calculating an approximation to a certain $u^*(\cdot)$ from U_* is to be found.

3. Problem Solution

Below, an algorithm solving the above problem and stability with respect to informational and computational hindrances is constructed. The algorithm is based on the approach to the inverse problems of dynamics suggested by Kryazhimski and Osipov (1983) and (1987). According to this approach, calculation of an unknown control $u^*(\cdot)$ on the basis of hindered measurement results $\psi(\cdot)$ is carried out as follows. An auxiliary system M (a model) functioning together with the real system is introduced. Let $y(\cdot)$ be an output and $v^\varepsilon(\cdot)$ be a control for the model M. Then an *algorithm forming* $v^\varepsilon(\cdot)$ *by feedback* $v(\cdot) = v^\varepsilon(\cdot; \psi(\cdot), y(\cdot))$ (see: Krasovski, 1985) is formed $v^\varepsilon(\cdot)$ approximates, in an appropriate sense, the unknown control $u^*(\cdot)$. Thus the problem of calculation of the unknown control is *replaced* by problem of finding an algorithm forming a control for the model. This algorithm provides a desired approximation to the unknown control.

In the papers (Kryazhimski and Osipov, 1987, 1983) certain control reconstruction procedures stable with respect to informational hindrances were suggested for systems described by ordinary differential equations. These procedures prove to be special regularizing procedures (see: Tikhonov and Arsenin, 1977) stabilizing Lyapunov type functionals. For certain classes of nonlinear systems with distributed parameters, the constructions analogous to these from (see: Tikhonov and Arsenin, 1977) are given by Korbicz *et al.* (1990) and by Maksimov (1988). The basic difference between results obtained by Korbicz *et al.* (1990) and by Maksimov (1988) and the present paper is that the latter utilizes finite-dimensional models.

Let us pass to the description of a procedure modelling $u^*(\cdot)$. A solution of the inequality (1) corresponding to a control $u(\cdot) \in L_2(T;U)$ and an initial state $x_0 \in D(\phi) \cap V(D(\phi) = \{y \in V : \phi(y) \neq +\infty\})$ is said to be function $x(\cdot) \in L_2(T;V)$, $\dot{x}(\cdot) \in L_2(T;H)$, $x(t_0) = x_0$, satisfying (1) a.e. $t \in T$. We suppose that an unknown input $u(\cdot) \in L_2(T;U)$ and an initial state x_0 generate the single solution of the inequality (1). Besides assume that

$$||B(t,x(t))||_{\mathcal{L}(U;H)} \leq a \quad \text{a.e } t \in T.$$

For $B(t,x) = B$, sufficient existence and uniqueness of conditions for solutions can be found, for example, in (see: Glowinski *et al,* 1976; Barbu, 1984).

Let triples $\{V_h, p_h, r_h\}$ and $\{U_h, q_h, s_h\}$ – ($h \in H_0$, H_0 is a neighbourhood of zero in R^n) form inner approximations (see: Glowinski at al, 1976; Temam, 1979) to the spaces V and U, i.e.

1° V_h, U_h are finite-dimensional spaces whose norms $|\cdot|_h$ and $|||\cdot|||_h$ are generated by the norms $|\cdot|_H$ and $||\cdot||_U$,

2° $p_h : V_h \to V$, $r_h : H \to V_h$, $q_h : U_h \to U$ and $s_h : U \to U_h$ are linear and continuous operators, p_h and q_h are one-to-one,

3° $\forall\, y \in H p_h r_h y \to y$ in H as $h \to 0$,

4° $|p_h r_h|_{\mathcal{L}(H:H)} \leq c_1$,

5° $q_h s_h u \to u$ in U as $h \to 0\; \forall\, u \in P$,

6° $|q_h s_h|_{\mathcal{L}(U;U)} \leq c_2$.

$A_h : V_h \to V_h$ is a discrete elliptic operator: $(A_h y_h, z_h)_h = (A p_h y_h, p_h z_h)_{V^* \times V}$ $\forall\, y_h, z_h \in V_h, f(\cdot) \in L_2(T;V_h) : (f_h(t), y_h)_h = (f(t), p_h y_h)$ a.e. $t \in T\; \forall\, y_h \in V_h$, the family of operators $B_h(t,\psi^h) : U_h \to V_h$ is determined by $(B_h(t,\psi^h)\, u_h, y_h)_h = (B(t, p_h\psi^h) q_h u_h, p_h y_h)$ a.e. $t \in T\; \forall\, y_h \in V_h,\; u_h \in U_h, (\cdot,\cdot)_h$ is the scalar product in $V_h, \phi_h(y_h) = \phi(p_h y_h)\; \forall\, y_h \in V_h$.

For inner approximations to separable normed spaces, the Galerkin approximation or that of finite elements (see: Glowinski and Lions, 1976; Temam, 1979; Aubin, 1972) can be taken.

The measurement results $\psi^h(\cdot) \in L_2(\delta_{i-1}; V_h)$, $\delta_i = [\tau, \tau_{i+1}]$, are available at instants $\tau_i, i \geq 1$. Consider two cases.

Case I: $\psi^h(\cdot)$ satisfies conditions

$$|p_h\psi^h(t) - x(t)|_H \leq \varepsilon_2, \quad t \in \delta_{i-1}, \tag{2}$$

$$\int_{\tau_{i-1}}^{\tau_i} |Ax(t) - p_h A_h \psi^h(t)|_H dt \leq \varepsilon_1, \; Ax(t) \in L_2(T; H).$$

Case II: only the inequalities (2) are true.

Let the model be the system described by the finite dimensional ordinary differential equation

$$\dot{z}_h(t) = \begin{cases} 0, \; t \in [t_0, \tau_1) \\ \phi_{\varepsilon,\lambda,h}(t-\delta) + B_h(t-\delta, \psi^h(t-\delta))v_h^\varepsilon(t), \end{cases} \tag{3}$$

$$t \in [\tau_1, \theta], \quad z_h(t_0) = \psi^h(t_0),$$

where $\phi_{\varepsilon,\lambda,h}(\xi) = \overline{A}_h\psi^h(\xi) - \partial\phi^*_{\lambda h}(\psi^h(\xi)) + f_h(\xi)$, $\phi^*_{\lambda h}(y_h) = \phi^*(p_h y_h) \; \forall \; y_h \in V_h$, $\overline{A}_h\psi^h = -A_h\psi^h, \phi^*(y) = \phi(y)$ in Case I, $\overline{A}_h\psi^h = \alpha\psi^h$, $\phi^*(y) = \phi(y) + 1/2(Ay, y)_{V^* \times V} + \alpha/2|y|_H^2$ in Case II, $\phi^*_\lambda : H \to R$, $\phi^*_\lambda(y) = \inf\{\phi^*(z) + |z - y|_H/2\lambda : z \in H\}$.

Define the control $v_h^\varepsilon(t) = v_{hi}^\varepsilon(t) \in U_h$, $t \in [\tau_i, \tau_{i+1})$ for the model by

$$v_{hi}^\varepsilon(t) = \begin{cases} 0, \text{ if } \chi_i^h \leq 0 \text{ or } a_i^h \leq \nu\delta^{1/2} \\ \chi_i^h (a_i^h)^{-2} d_{i,\psi,h}(t-\delta), \text{ in the contrary.} \end{cases}$$

Here $\chi_i^h = (\pi^*_{h,i}, \int_{\tau_{i-1}}^{\tau_i} \phi_{\varepsilon,\lambda,h}(\xi)d\xi - x_i)_h$, $\varepsilon = (\varepsilon_1, \varepsilon_2)$, $\pi^*_{h,i} = z_h(\tau_i) - \psi^h(\tau_{i-1})$, $x_i = \psi^h(\tau_i) - \psi^h(\tau_{i-1})$, $a_i^h = \|d_{i,\psi,h}(\cdot)\|_{L_2(\delta_{i-1};U_h)}$, $d_{i,\psi,h}(t) = -B_h^*(t, \psi^h(t))\pi^*_{h,i}$.

Assume in the sequel that the 4^o from (Maksimov, 1988) is satisfied, i.e. there exists the single minimal norm element $u_*(\cdot) \in L_2(T; U)$ with the property

$$\dot{x}(t) - A_* x(t) - f(t) + \partial\phi^{*,0}(x(t)) = \{B_{x(\cdot)} u_*(\cdot)\}(t) \text{ a.e. } t \in T.$$

It is easily seen that $t \to \partial\phi^{*,0}(x(t)) \in L_2(T;H)$. Here $\partial\phi^*$ is the subdifferential for the function $\phi^* : H \to \overline{R}$, $\partial\phi^{*,0}(y) = \{z \in H : |z|_H = \inf |w|_H, w \in \partial\phi^*(y)\}$.

Let mappings $h(\nu)$, $\lambda(\nu)$, $\delta(h,\nu)$, $\varepsilon_1(\nu,\delta,h)$ and $\varepsilon_2(\nu,\delta,h,\lambda)$ satisfy the following conditions

Condition A:

$$\int_{t_0}^{\theta} |\partial\phi^{*,0}(x(t)) - \partial\phi^{*,0}_{\lambda(\nu)}(x(t))|_H^2 \le \nu^2, \ \lambda(\nu) \to +0 \text{ as } \nu \to 0,$$

$$\int_{t_0}^{\theta} \|(I - q_h s_h)u_*(t)\|_U^2 \le \nu^2, \tag{4}$$

$$\delta(h,\nu) \le \nu(\max\{1, |p_h|^4_{\mathcal{L}(V_h,H)}\})^{-1}, \tag{5}$$

$$\varepsilon_2(\nu,\delta,h,\lambda) \le \nu \min\{\lambda,\delta\}, \ \ \varepsilon_2(\nu,\delta,h) \le \nu\delta.$$

In the case II the least correlation should be omitted. We shall say that values $(h,\lambda,\delta,\varepsilon_1,\varepsilon_2)$ and a partition of the interval T with diameter $\delta = \delta(\Delta)$ are ν – compatible, if condition A and the following condition B are fulfilled.

Condition B: $h \in (0,h(\nu))$, $\lambda \in (0,\lambda(\nu))$, $\delta \in (0,\delta(h,\nu))$, $\varepsilon_1 \in (0,\varepsilon_1(\nu,\delta, h))$, $\varepsilon_2 \in (0,\varepsilon_2(\nu,\delta,h,\lambda))$.

Introduce the notations: $v_h^{-\varepsilon}(t) = q_h v_h^{\varepsilon}(t)$ a.e. $t \in [\tau_1,\theta]$, $v_h^{-\varepsilon}(t) = v \in P$ a.e. $t \in [t_0,\tau_1]$.

Theorem. *For every $\alpha_0 > 0$ there exists ν_0 such that for any $\nu \in (0,\nu_0)$ ν–compatible values $(h,\lambda,\delta,\varepsilon_1,\varepsilon_2)$ and a partition $\Delta = \{\tau_i\}_{i=0}^{m_\delta}$ with a diameter $\delta = \delta(\Delta)$ and any measurement results $\psi^h(\cdot)$ with the above described properties, the following inequality is true:*

$$\|v_h^{-\varepsilon}(\cdot) - u_*(\cdot)\|_{L_2(T;U)} \le \alpha_0.$$

The proof of the Theorem is based on the following statement.

Lemma 1. *If values $(h,\lambda,\delta,\varepsilon_1,\varepsilon_2)$ are ν–compatible, then for every $\nu > 0$*

$$|x(t) - p_h z_h(t)|_H^2 \le k_1\nu,$$

$$\|v_h^{-\varepsilon}(\cdot)\|^2_{L_2([\tau_1,\theta];U)} \le (1+\nu^{1/2})\|q_h s_h u_*(\cdot)\|^2_{L_2(T;U)} + k_2\nu^{1/2},$$

where the constans k_1, k_2 do not depend on $\nu, h, \varepsilon_1, \varepsilon_2, \delta$ and λ.

Let the initial information be completed with the following data. It is known that

i) the mapping $x \to \partial\phi^0(x)$ is Lipschitz,

ii) $p_h\psi^h(t) \in D(\partial\phi)$ a.e. $t \in T$.

Then we take the system (3) for the model, where we put $\phi_{\varepsilon,\lambda,h}(\xi) = -A_h\psi^h(\xi) - p_h^*\partial\phi^0(p_h\psi^h(\xi)) + f_h(\xi)$ in Case I, $\phi_{\varepsilon,\lambda,h}(\xi) = \alpha\psi^h(\xi) - p_h^*\partial\phi^h(\xi))$ $+f_h(\xi) - p_h^*\partial\Phi_\lambda(p_h\psi^h(\xi))$ in Case II. Here $\Phi(y) = 0.5\alpha|y|_H^2 + 0.5(Ay,y)_{V^*\times V}$. In Case II we replace ϕ^* in condition A by the function Φ, and in Case I we formulate condition A as follows: mappings $h(\nu)$, $\delta(h,\nu)$, $\varepsilon_1(\nu,\delta,h)$ and $\varepsilon_2(\nu,\delta,h)$ have properties (4), (5) and max $\{\varepsilon_1,\varepsilon_2\} \leq \nu\delta$. Under these conditions the theorem is true.

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