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SIMPLIFIED APPROACH TO THE SHAKEDOWN PROBLEM OF I-CROSS-SECTIONS UNDER TWO-DIRECTIONAL BENDING

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The report presents an analytical model suitable to carry out a shakedown and limit analysis of steel thin-walled beam cross-sections under low-cyclic loadings. The loads and actions are quasi-static, any dynamic effects and fatigue failure are not considered. Initial stresses (due to prestressing) as well as residual stresses are allowed. The vector of variable repeated forces contains the axial force, the bending moments about two principal axes of cross-section and warping moment. Shear forces, moments of pure torsion and bending torsion are also taken into account but their influences are assumed to be minor. Prestressing forces and thermal actions are considered herein as one of the load types with zero vector of resultant internal forces. Cross-sections of beams may have any geometrical form, but analysis are performed for the I-type beam under bending about two principal axes, without warping moment. The numerical results for cyclic load are compared with the analogous data for one-path load on the cross-sections.

Keywords: steel thin-walled beams; I cross-section; shakedown; limit analysis; low cyclic and one-path load

1. INTRODUCTION

The numerous literature ([4-6]) is devoted to analysis and design of metal or steel structures from elastic-plastic thin-walled elements. However basically was studied only behavior of elements and cross-sections subjected to monotonic one-path load or to the load with certain known history.

The behavior of steel thin-walled cross-sections with restricted plastic deformations under repeated loads of certain history was investigated in study [6].

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However the load-carrying structures are exposed to the actions (static, thermal, kinematic, etc.), which may vary in a random manner. As a result, there are repeated alternating cross-section forces changed arbitrarily within the specified area ([1, 7-9]). At present, only separate design combinations of loads and influences are usually taken into account in analysis and design procedures. In fact, the strength conditions of elements essentially depend on the interaction of variable repeated loads. The strength conditions in terms of generalized forces (variables) for simple sections from homogeneous ideal plastic material for different types of load cycles have been obtained in the studies [1, 8, 9]. In the work [2] an analytical model to the shakedown analysis of steel thin-walled I-cross-sections for the cycles "from zero" and for the fixed ratio of two-direction bending moments was suggested.

In this paper, an analytical model [2] is evolved to analyze the beam element cross-sections under more general case of the low-cyclic loadings. The loads and actions are quasi-static, no any dynamic effects and fatigue failure are considered. The constitutive model of steel is bilinear elastic-perfectly plastic without strain hardening. The cross-section may have any geometrical form, the prestressing is allowed. The vector of variable repeated forces contains axial force, two bending moments about principal axes of cross-section, and warping moment. The torsion and the shear forces are also taken into account but their influences are assumed to be minor. Prestressing forces and thermal actions are considered herein as one of the load types with zero vector of resultant internal forces. The numerical analysis are performed for the I-type beam under bending about two principal axes (unsymmetrical bending), without warping moment.

2. GENERAL RELATIONS OF SHAKEDOWN PROBLEM

Let the cross-section of metal (steel) element be subjected to the vector of variable repeated forces $\boldsymbol{S} = (N, M_x, M_y, B_\omega, T_0, M_\omega, V_x, V_y) \in \boldsymbol{R}^8$, which are changed arbitrarily within the given domain $\Omega_{\boldsymbol{S}}$. This domain can be simulated by the rectangular parallelepiped

$$\Omega_S = (\boldsymbol{S} \in \boldsymbol{R}^8 : \mu \boldsymbol{S}^- \leq \boldsymbol{S} \leq \mu \boldsymbol{S}^+), \tag{1}$$

where N – normal force (tension or compression); M_x, M_y – bending moments, B_ω – warping moment, T_0, M_ω – moments of pure torsion and bending torsion accordingly, V_x, V_y – shear forces; $\boldsymbol{S}^- = (N^-, M_x^-, \ldots, V_y^-)$, $\boldsymbol{S}^+ = (N^+, M_x^+, \ldots, V_y^+)$ are the vectors of design combinations of cross-section forces due to external loads (static, thermal and kinematic [3]); μ - the parameter of load. Note that the thermal and prestressing action components distributed on the section area may be added to the vector $\boldsymbol{S}$.

The domain Ω_S contains the coordinate origin or "zero load" $\boldsymbol{S=0}$ corresponding to initial non-stress state of section with non-prestressing or initial stress state of section with prestressing. The latter state is considered like a thermal action.

In surfents dA of steel area A which have coordinates $\boldsymbol{x} = (x, y)$ the stresses $\boldsymbol{\sigma} = (\sigma_z, \tau_{zx}, \tau_{zy})$, $\boldsymbol{\sigma} \in \boldsymbol{R}^3$, appear; the stresses σ_x, σ_y, τ_{xy} are neglected; normal stresses σ_z in steel of area A_s are only considered. Subscript "z" for stresses σ_z is omitted.

To check the plasticity of steel in compression and in tension a Huber-von Mises criterion in terms of principal stresses for three-dimensional stress state is adopted. It can be written as

$$\sigma^2_1 + \sigma^2_2 + \sigma^2_3 - (\sigma_1 \sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) - f_y^2 \leq 0, \qquad \boldsymbol{x} \in A, \tag{2}$$

where f_y is the steel stress at yield.

For steel cross-section in a state of plane stress, inequality (2) is rewritten as

$$\sigma^2 + 3((\tau_{zx})^2 + (\tau_{zy})^2) - f_y^2 \leq 0, \qquad \boldsymbol{x} \in A. \tag{3}$$

The quadratic inequality (3) may be substituted for linear inequalities for steel of area A^c in compression and for steel of area A^t in tension respectively:

$$-\sigma + f_y^* \leq 0, \qquad \boldsymbol{x} \in A^c, \tag{4}$$

$$\sigma - f_y^* \leq 0, \qquad \boldsymbol{x} \in A^t, \tag{5}$$

where f_y^* is the radical of function located in the left side of (3), which depend on shear stresses $\boldsymbol{\tau} = (\tau_{zx}, \tau_{zy})$. It is given by

$$f_y^* = \sqrt{f_y^2 - 3(\tau_{zx}^2 + \tau_{zy}^2)}\,; \tag{6}$$

its absolute value is the equivalent strength of steel.

Furthermore, residual shear stresses are neglected, i.e.

$$\boldsymbol{\tau}^{\mathrm{r}} = \mathbf{0}, \quad \text{or} \quad \tau^{\mathrm{r}}_{\mathrm{zx}} = \tau^{\mathrm{r}}_{\mathrm{zy}} = 0, \tag{7}$$

then

$$\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{e}}(\boldsymbol{S}_{\tau}) \tag{8}$$

where $\boldsymbol{\tau}^{\mathrm{e}}(\boldsymbol{S}_{\tau})$ – is known vector-function, $\boldsymbol{S}_{\tau} = (T_0, M_{\omega}, V_x, V_y) \in \boldsymbol{R}^4$.

The total stresses in the cross-section of area A are presented as a sum of elastic σ^e and residual σ^r components:

$$\sigma = \sigma^{\mathrm{e}}(\boldsymbol{S}_{\sigma}) + \sigma^{\mathrm{r}}, \qquad \boldsymbol{x} \in A, \tag{9}$$

where $\boldsymbol{S}_{\sigma} = (N, M_x, M_y, B_{\omega}) \in \boldsymbol{R}^4$, so $\boldsymbol{S} = (\boldsymbol{S}_{\sigma}, \boldsymbol{S}_{\tau}) \in \boldsymbol{R}^8$.

The dependence $\sigma^e(\boldsymbol{S}_{\sigma})$ of elastic stresses upon external forces at one-pass loading is defined by

$$\sigma^e = \frac{N}{A} + \frac{M_x}{I_x} y + \frac{M_y}{I_y} x + \frac{B_{\omega}}{I_{\omega}} \omega. \tag{10}$$

With referring to Eq. (9), conditions (4) and (5) take the forms:

$$-\sigma^e - \sigma^r + f_y^* \le 0, \quad x \in A^c, \tag{11}$$

$$\sigma^e + \sigma^r - f_y^* \le 0, \quad x \in A^t. \tag{12}$$

As function $\sigma^e(\boldsymbol{S}_{\sigma})$ in Eq. (10) is linear, the extremal stresses σ^{e-}, σ^{e+} are induced by the dangerous load combinations:

$$\sigma^{\mathrm{e}-} = \min\left\{ \min \sigma^e(\boldsymbol{S}_{\sigma}^-, \boldsymbol{S}_{\sigma}^+);\ 0 \right\}. \tag{13}$$

$$\sigma^{\mathrm{e}+} = \max\left\{ \max \sigma^e(\boldsymbol{S}_{\sigma}^-, \boldsymbol{S}_{\sigma}^+);\ 0 \right\}. \tag{14}$$

Referring to Eqs. (13), (14), the plasticity conditions (4) and (5) may be written in the following forms:

$$-\sigma^{e-} - \sigma^r + f_y^* \le 0, \quad x \in A^c, \tag{15}$$

$$\sigma^{e+} + \sigma^r - f_y^* \le 0, \quad x \in A^t. \tag{16}$$

Besides, the following equilibrium equations must be satisfied for the residual stresses σ^r in the whole cross-section:

$$\int_A \sigma^r dA = 0\,, \tag{17}$$

$$\int_A \sigma^r x dA = 0\,, \tag{18}$$

$$\int_A \sigma^r y dA = 0\,. \tag{19}$$

If inequality (15) or (16) is actual also in the whole cross-section, the shakedown regime is named “incremental collapse”.

It is obvious that both inequalities (15), (16) may be actual only in a few (one or several) points x of cross-section area. Then we have a shakedown regime of alternating steel yielding.

The strength of steel element cross-section is ensured if there are fields of residual stresses σr(x), x ∈ A provided that inequalities (15), (16) and equalities (17)-(19) hold (shakedown *conditions*).

3. LIMIT ANALYSIS PROBLEMS

The primal limit analysis problem for the ultimate capacity of the element cross-section can be formulated in case when vector S-, S+ of the section force combinations depends only on the one parameter of load μ.

Thus, the following infinite-dimensional nonlinear programming problem A is derived: the parameter of load should be maximized,

$$\mu \rightarrow \max, \tag{20}$$

while constraints (15)–(19) depended on μ are satisfied.

The variables of this problem are the field of optimal control variables $\sigma^r(\boldsymbol{x})$, $\boldsymbol{x} \in A$, and parameter μ.

Similarly, the inverse (design optimization) problem ***B*** can be formulated if the vectors $\boldsymbol{S}^-$, $\boldsymbol{S}^+$ are known, and steel yield stress f_y (unknown) depends on parameter λ; $f_y := \lambda f_y$, where f_y is some positive constant; parameter λ should be minimized,

$$\lambda \rightarrow \min, \tag{21}$$

subject to (15)–(19) depended on λ.

This problem has the same variables as previous problem if substitute μ for λ.

In order to obtain the numerical solutions of these problems they have to be reduced to the finite-dimensional problems by division the cross-section area A into the elementary areas ΔA_i, $i \in I$, where I is the set of indicies of elementary areas. Then the vector of variables (residual stresses σ^r) will have dimensions of value $|I|$, and the problems formulated can be solved by the conventional methods of optimization.

It is possible to use another simple and accurate computer-aided numerical procedures based on the approach [2].

The technique to solving the primal problem A, for the regime of progressive plastic failure and for any geometrical form of cross-section, may be realized by applying the following inverse method:

1. Assume a value to parameter of load μ. It may correspond to the cross-section ultimate capacity derived without considering cyclic load interactions.

2. Determine the extremal elastic stress distributions on the areas of cross-section.

3. Take location of neutral axis.

4. From (15), (16) as from equalities obtain the residual stresses σ^r.

5. Substitute σ^r into Eqs. (17)-(19) and obtain values of parameter μ.

To solve the inverse problem ***B*** the scheme of procedure may be sketched as follows:

1. Determine the extremal elastic stress distributions on the cross-section areas.

2. Assume a value for parameter λ (for example, adopt from results of analyses carried out without considering cyclic load interactions).

3-5. See the same Steps of previous scheme.

Then we have to calculate the safe domain of strength for the alternating yielding from the actual inequalities (15) and (16). The real safe domain of strength is the intersection of first and second domains, corresponding to these two regimes of shakedown.

4. I-CROSS-SECTIONS UNDER TWO-DIRECTIONAL BENDING

Here we analyze the thin-walled I-section cyclically loaded only by the bending moments M_x, M_y; $\boldsymbol{S} = (M_x, M_y) \in \boldsymbol{R}^2$. At first, consider "from zero"-type load cycles, $\boldsymbol{S}^- = \boldsymbol{0}$. In this case of load domain Ω_S (1) a regime of alternating plasticity can not exist, so only incremental collapse of cross-section will be realized.

The diagrams of extremum elastic σ^{e-}, σ^{e+} and residual σ^r ($\sigma_i^r := r_i$, $i \in$ 1:5) stresses in a limit state of shakedown are shown in Fig. 1. The location of neutral axis is determinated by points with coordinates $\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{x} = (-\alpha b, h/2)$.

In this case owing to symmetry the first equilibrium equation (17) is satisfied identically, and the other equations (18), (19) may be written in the following matrix form:

$$\overline{\mathbf{A}} \quad \vec{\sigma} \quad \leq \quad \vec{\mathbf{b}}_{\;1} , \qquad (22)$$

where

$$\overline{\boldsymbol{A}} = \begin{bmatrix} \overline{\boldsymbol{A}}_1 \\ \overline{\boldsymbol{A}}_2 \\ \overline{\boldsymbol{A}}_3 \end{bmatrix}, \quad \vec{\boldsymbol{b}} = \begin{pmatrix} \vec{\boldsymbol{b}}_1 \\ \vec{\boldsymbol{b}}_2 \\ \vec{\boldsymbol{b}}_3 \end{pmatrix},$$

$$\vec{\sigma} = \left(\overline{\sigma}^x, \quad \overline{\sigma}^y, \quad \overline{r}_1, \quad \overline{r}_2, \quad \overline{r}_3, \quad \overline{r}_4 \right)^T,$$
$$\overline{\sigma}^i = \frac{\sigma^i}{f_d}, \overline{r}_i = \frac{r_i}{f_d}, \qquad (23)$$
$$\chi = \frac{ht_w}{bt_f} = \frac{A_s}{A_p},$$

$$\overline{\boldsymbol{A}}_1 = \begin{bmatrix} 0 & 0 & \frac{1}{12} & \frac{(2\alpha-1)(1+\alpha)}{12} & \frac{(2\alpha-1)}{4} & \frac{12\alpha^2-1}{24} \\ 0 & 0 & \frac{1}{4} & \frac{(1-2\alpha)}{4} & \frac{(1-2\alpha)}{4} & -\frac{4\alpha+1}{4}-\frac{\chi}{6} \end{bmatrix},$$

$$\overline{\boldsymbol{A}}_2 = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2\alpha & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\overline{\boldsymbol{A}}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \tag{24}$$

and

$$\begin{aligned} \overline{\mathbf{b}}_1 &= \begin{pmatrix} 0 \\ \frac{\chi}{12} \end{pmatrix}, \\ \overline{\mathbf{b}}_2 &= \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T, \\ \overline{\mathbf{b}}_{32} &= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^T. \end{aligned} \tag{25}$$

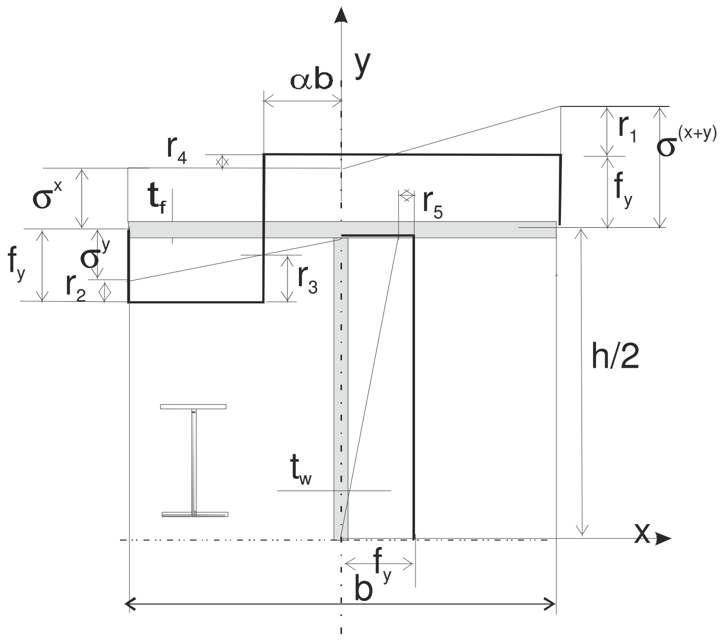

Fig. 1. Thin-walled I-section under "from zero" cyclic load; diagrams of extremum elastic σ^{e-}, σ^{e+} and residual r_i ($i \in$ 1:5) stresses

If the value αb is fixed, the unknowns M_x, M_y for the limit state of shakedown may be found from the Eq. (22) taking into account Egs. (23) - (25),

$$M_x = \overline{\sigma}^x f_d W_{x,}\,, \qquad M_y = \overline{\sigma}^y f_d W_{y,}\,, \tag{26}$$

Note, that this solution is true only for the meaning of the parameter $t_w/b > \alpha < 0,5$. For the domain $0 < \alpha < t_w/b$ it is necessary to accept another (thick-walled) model of cross-section. For such case we propose a simple Eq. to approximate the relations between nondimensional bending moments m_x and m_y in the following form:

$$m_y = 1 - a m_x^2\,, \tag{27}$$

where $m_x = \dfrac{M_x}{M_{x,pl}}$, $m_y = \dfrac{M_y}{M_{y,pl}}$, $a = (1 - m_{yw}) / m_{xw}^2$, m_{xw}, m_{yw} are the meanings of moments m_x and m_y respectively for the parameter $\alpha = t_w/b$.

Then let us consider sign-varying cyclic load of the cross-section, $\boldsymbol{S^- < 0}$. The calculations in this case will be analogous, but the strength for the alternating plasticity has to be analyzed (see parts 3, 5).

5. NUMERICAL RESULTS

On the basis of described analytical models the computer program has been developed, and some numerical results have been obtained. Thus, it was found that traditional analysis may overestimate the ultimate carrying capacity of cross-section if the influences of repeated forces interaction are neglected.

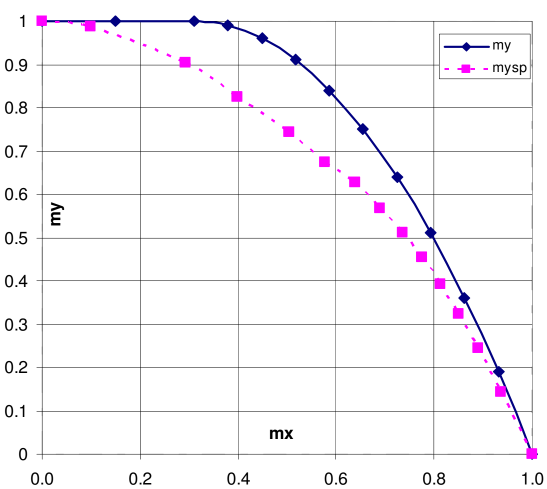

Fig. 2. Domain of strength for thin-walled section I 500 under one-pass (my) and under "from zero" cyclic (mysp) load; $0 \le m_x \le m_x^+$; $0 \le m_y \le m_y^+$

5.1. There are some numerical results of limit and shakedown analyses of steel beam I 120 with thin-walled cross-section under "from zero"-type cycles of bending moments M_x, M_y. Domains of strength for this section under one-pass (my) and cyclic (mysp) loading are shown in Fig. 2, where m_x^+ and m_y^+ are relative (dimensionless) sizes. The difference between these solutions is reached to 28 %.

5.2. The results of numerical analysis for the I-type beam under sign-varying cyclic bending moment; -0,1≤ $m_x \le m_x^+$; -0,2≤ $m_y \le m_y^+$ are presented in Fig. 3. Here the curves 1 and 2 correspond to the incremental collapse and to the alternating plasticity of cross-section respectively.

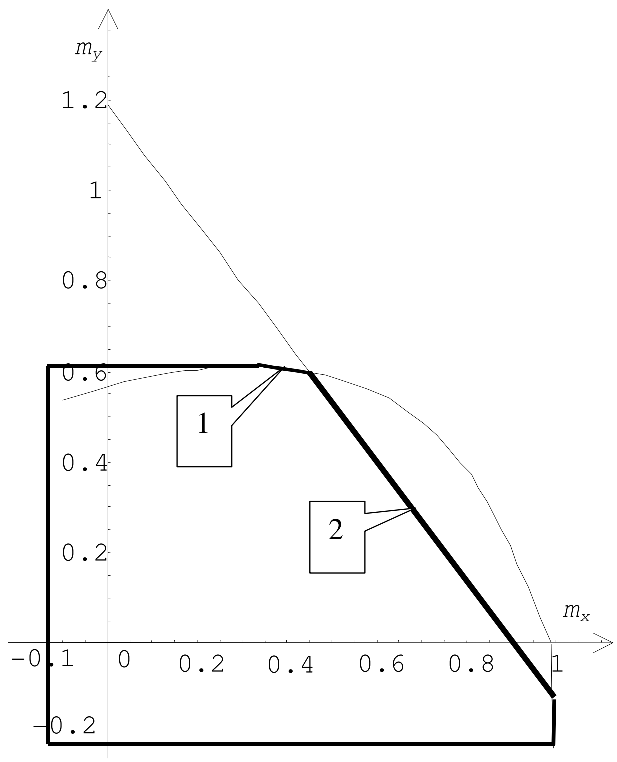

Fig. 3. Domain of strength for thin-walled section I 500 under sign-varying cyclic load; -0,1≤ $m_x \le m_x^+$; -0,2≤ $m_y \le m_y^+$

The analytical equation for the regime 2 of alternating plasticity is the following:

$$(m_x^+ - m_x^-)W_{x,pl} / W_x + (m_y^+ - m_y^-)W_{y,pl} / W_y = 2, \qquad (28)$$

where $W_{x,pl}, W_{y,pl}$ are the plastic moments of resistance for the corresponding axes *x*, *y*.

6. CONCLUSIONS

In this study, an analytical model is formulated and simplified approach methods are proposed to carry out a limit analysis of steel elements and shakedown analysis of beams subjected to the low-cyclic loads.

It is shown that analysis may overestimate the shakedown of the structure elements if the influences of interactions of variable repeated forces are neglected. The possible significant influence of these effects on the cross-section capacity of steel beams is demonstrated by the numerical results for the I-beams under two-direction bending.

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UPROSZCZONA METODA ROZWIĄZANIA PROBLEMU PRZYSTOSOWANIA PRZEKROJÓW DWUTEOWYCH NA ZGINANIE DWUKIERUNKOWE

Streszczenie

W referacie zaprezentowano dogodny, odwrotny sposób rozwiązania problemu przystosowania i nośności granicznej stalowych, cienkościennych przekrojów poddanych obciążeniu niskocyklicznemu. W rozważaniach przyjęto kwazistatyczny przyrost obciążeń, oraz pominięto efekty dynamiczne i zmęczeniowe. Naprężenia początkowe (odpowiadające np. sprężeniu wstępnemu) dopuszczono na równi z naprężeniami rezydualnymi. Wektor sił zmiennych w ogólności zawiera obciążenia osiowe, momenty zginające w obydwu głównych osiach i bimoment. Przyjęto, że wpływ sil ścinających i momentu czystego skręcania jest nieznaczny. Siły sprężające i oddziaływania termiczne zawarto jako zerowy wektor sił wewnętrznych. Przekrój poprzeczny belki może mieć różny kształt, jednak w konkretnym przykładzie obliczeniowym rozważono bisymetryczny przekrój dwuteowy przy zginaniu dwukierunkowym. Wyniki obliczeń numerycznych przystosowania porównano z nośnością graniczną przekroju przy obciążeniu jednokrotnym.