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ALGEBRAIC CONDITION FOR DECOMPOSITION OF LARGE–SCALE LINEAR DYNAMIC SYSTEMS

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The paper concerns the problem of decomposition of a large-scale linear dynamic system into two subsystems. An equivalent problem is to split the characteristic polynomial of the original system into two polynomials of lower degrees. Conditions are found concerning the coefficients of the original polynomial which must be fulfilled for its factorization. It is proved that knowledge of only one of the symmetric functions of those polynomials of lower degrees is sufficient for factorization of the characteristic polynomial of the original large-scale system. An algorithm for finding all the coefficients of the decomposed polynomials and a general condition of factorization are given. Examples of splitting the polynomials of the fifth and sixth degrees are discussed in detail.

Keywords: decomposition, algebraic condition, dynamic systems.

1. Introduction

There are many articles concerning decompositions of polynomials (Alagar and Thanh, 1985; Bartoni and Zippel, 1985; Borodin *et al.*, 1985; Coulter *et al.*, 2001; 1998; Gathen, 1990; Giesbrecht and May, 2005; Kozen *et al.*, 1989; Kozen and Landau, 1989; Watt, 2008).

These methods deal with polynomials in numerical forms. The main idea is to find a decomposition $f(x) = g[h(x)]$. For example, let $f(x) = x^6 + 6x^4 + x^3 + 9x^2 + 3x - 5$. It is possible to decompose this polynomial, where $y = h(x) = x^3 + 3x$ and $g(x) = y^2 + y - 5$. For this type of decomposition, if possible, various algoritms are presented. In this article, a general method for decomposition of polynomials in the symbolic form $f(x) = \sum_{i=0}^{n} a_i s^i$, $a_i \in \mathbb{R}^n$ is presented. Using computer algebra systems it is possible to decompose polynomials of very high degrees.

Decomposition of large-scale linear systems into smaller dynamic systems is always possible if the eigenvalues of the state matrices which characterize the systems are known (Górecki and Popek, 1987). On the other hand, it is well known that, in general, algebraic equations of the degree $n \geq 5$ cannot be solved in radicals when their coefficients belong to the rational field, see the theorem of Ruffini and Abel.

Theorem 1. **(Galois)** *A necessary and sufficient condition for analytical solvability of an algebraic equation of the degree $n > 4$ is that the all of its roots $s_1, s_2, \ldots, s_n$ could be represented in the rational field of its coefficients by a the rational function $f(s_1, s_2)$ of two of its roots s_1 and s_2 (Mostowski and Stark, 1954; Suszkiewicz, 1941; Perron, 1927).*

Our problem is to find algebraic conditions which would enable us to decompose the characteristic polynomial of a large-scale dynamic system into two polynomials which altogether have the same roots as the original characteristic polynomial. Repeating such a decomposition (if possible), we can decompose the original system into more than two parts.

It is shown that such a decomposition is possible if some additional relations are known between the coefficients or the roots of the factorial polynomials.

2. Problem statement

Let us consider the characteristic polynomial

$$F(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0, \qquad (1)$$

where $s_i,\ i = 1, 2, \ldots, n$ are known real coefficients which may depend on some real parameters.

Our goal is the following decomposition of the polynomial (1):

$$F(s) = G(s) \cdot H(s), \tag{2}$$

where

$$G(s) = \sum_{k=0}^{m} b_k s^{m-k}, \tag{3}$$

$$H(s) = \sum_{k=0}^{n-m} c_k s^{n-m-k}, \tag{4}$$

$$a_0 = b_0 = c_0 = 1.$$

3. Problem solution

3.1. Decomposition of the fifth-degree algebraic equation into equations of the second and third degrees. Let

$$f(s) = s^5 + a_1 s^4 + a_2 s^3 + a_3 s^2 + a_4 s + a_5. \tag{5}$$

As for additional information, we assume that we know the sum of the two roots

$$s_1 + s_2 = A_1. \tag{6}$$

Then

$$s_3 + s_4 + s_5 = a_1 - A_1 = B_1, \tag{7}$$

where B_1 is known, as calculated from (7).

Let

$$(s - s_1)(s - s_2) = s^2 - A_1 s + A_2, \tag{8}$$

$$(s - s_3)(s - s_4)(s - s_5) = s^3 - B_1 s^2 + B_2 s + B_3. \tag{9}$$

Knowing the coefficients $a_1, \ldots, a_5$ and the sums A_1 and B_1, we determine A_2 and B_2, B_3. We have

$$\begin{aligned} f(s) &= \left(s^2 - A_1 s + A_2\right)\left(s^3 - B_1 s^2 + B_2 s + B_3\right) \\ &= s^5 - \left(A_1 + B_1\right)s^4 + \left(A_2 + B_2 + A_1 B_1\right)s^3 \\ &\quad + \left(B_3 - A_1 B_2 - A_2 B_1\right)s^2 \\ &\quad + \left(A_2 B_2 - A_1 B_3\right)s + A_2 B_3. \end{aligned} \tag{10}$$

From (5) and (10), we obtain the following relations:

$$A_2 + B_2 = a_2 - A_1 B_1, \tag{11}$$

$$B_3 - A_1 B_2 - A_2 B_1 = a_3, \tag{12}$$

$$A_2 B_2 - A_1 B_3 = a_4, \tag{13}$$

$$A_2 B_3 = a_5. \tag{14}$$

We consider two cases:

Case 1: $A_1 \neq B_1$. From (11) and (12) we express A_2 and B_2 as linear functions of B_3, A_1 and B_1, which are known:

$$A_2 = \frac{A_1(a_2 - A_1 B_1) + a_3 + B_3}{A_1 - B_1}, \tag{15}$$

$$B_2 = -\frac{B_1(a_2 - A_1 B_1) + a_3 + B_3}{A_1 - B_1}. \tag{16}$$

Substituting (15) into (14), we obtain the equation of the second degree for calculation of B_3:

$$B_3^2 + \left[A_1(a_2 - A_1 B_1) + a_3\right]B_3 - \left(A_1 - B_1\right)a_5 = 0. \tag{17}$$

After solving this equation, from (15) and (16) we calculate the coefficients A_2 and B_2. In this way, decomposition of the polynomial $f(s)$ into the factors of the second and third degrees is completed, and the solution of the equation of the fifth degree is reduced to solving equations of the second and third degrees. Substituting the coefficients A_2 and B_3 into (14), we obtain the relation between the coefficients $a_1, \ldots, a_5$ which must be fulfilled as a consequence of the assumption (6) that $s_1 + s_2 = A_1$.

Case 2: $A_1 = B_1$. Equations (11) and (12) are

$$A_2 + B_2 = a_2 - A_1^2, \tag{18}$$

$$B_3 - A_1(A_2 + B_2) = a_3. \tag{19}$$

From (18) and (19) we obtain

$$B_3 = a_3 + A_1(a_2 - A_1^2). \tag{20}$$

The coefficient B_3 is known.

In this case, from (14) we calculate the coefficient A_2 and from (18) the coefficient B_2.

All the coefficients are computed and the free equation (13) yields a relation between the coefficients $a_1, \ldots, a_5$. In this way, we decomposed the equation of the fifth degree into two equations which can be solved analytically.

3.2. Decomposition of the equation of the sixth degree into equations of the fourth and second degrees. Let

$$f(s) = s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s + a_6 = 0. \tag{21}$$

We assume that

$$s_1 + s_2 = A_1, \tag{22}$$

where A_1 is known. In this case, we have

$$s_3 + s_4 + s_5 + s_6 = a_1 - A_1 = B_1, \tag{23}$$

so that B_1 is calculated.

We can write

$$(s - s_1)(s - s_2) = s^2 - A_1 s + A_2 \tag{24}$$

and

$$(s-s_3)(s-s_4)(s-s_5)(s-s_6) = s^4 - B_1 s^3 + B_2 s^2 - B_3 s + B_4. \quad (25)$$

Knowing the coefficients $a_1, \dots, a_6$ and the sums A_1 and B_1, we determine A_2, B_2 and B_3, B_4.

We have

$$\begin{aligned} f(s) &= \big(s^2 - A_1 s + A_2\big) \\ &\quad \cdot \big(s^4 - B_1 s^3 + B_2 s^2 - B_3 s + B_4\big) \\ &= s^6 + a_1 s^5 + \big(A_2 + A_1 B_1 + B_2\big) s^4 \\ &\quad - \big(B_1 A_2 + A_1 B_2 + B_3\big) s^3 \\ &\quad + \big(A_2 B_2 + A_1 B_3 + B_4\big) s^2 \\ &\quad - \big(A_2 B_3 + A_1 B_4\big) s + A_2 B_4. \end{aligned} \quad (26)$$

From (21) and (26) we obtain

$$A_2 + A_1 B_1 + B_2 = a_2, \quad (27)$$

$$B_1 A_2 + A_1 B_2 + B_3 = -a_3, \quad (28)$$

$$A_2 B_2 + A_1 B_3 + B_4 = a_4, \quad (29)$$

$$A_2 B_3 + A_1 B_4 = -a_5, \quad (30)$$

$$A_2 B_4 = a_6. \quad (31)$$

It is necessary to consider the following two different cases.

Case 1: $A_1 \neq B_1$. From (27) and (28) we express A_2 and B_2 as linear functions of B_3:

$$A_2 = \frac{A_1(a_2 - A_1 B_1) + a_3 + B_3}{A_1 - B_1}, \quad (32)$$

$$B_2 = -\frac{B_1(a_2 - A_1 B_1) + a_3 + B_3}{A_1 - B_1}. \quad (33)$$

Substituting (32) and (33) into (29), we obtain B_4 as the following function of the second degree of the unknown B_3:

$$\begin{aligned} B_4 = a_4 &+ \left[\frac{A_1(a_2 - A_1 B_1) + a_3 + B_3}{A_1 - B_1}\right] \\ &\cdot \left[\frac{B_1(a_2 - A_1 B_1) + a_3 + B_3}{A_1 - B_1}\right] - A_1 B_3. \end{aligned} \quad (34)$$

From (32) and (34) substituted into (30), we obtain the following equation of the second degree in the unknown B_3:

$$\begin{aligned} &\frac{A_1(a_2 - A_1 B_1) + a_3 + B_3}{A_1 - B_1} B_3 \\ &\quad + A_1 \left[a_4 - A_1 B_3 + \frac{A_1(a_2 - A_1 B_1) + a_3 + B_3}{A_1 - B_1} \right. \\ &\quad \left. \cdot \frac{B_1(a_2 - A_1 B_1) + a_3 + B_3}{A_1 - B_1} \right] = -a_5. \end{aligned} \quad (35)$$

We can solve this equation and obtain B_3. Then we compute A_2 and B_2 from (32) and (33), and, finally, B_4 from (34). In this way we decomposed $f(s)$ into factors of the second and fourth degrees. The problem is reduced to solving equations of the second and fourth degrees. Substituting the values of A_2 and B_4 into (31), we obtain the relation between the coefficients $a_1, \dots, a_6$, which must be fulfilled as a consequence of the assumption (22) that $s_1 + s_2 = A_1$.

Case 2: $A_1 = B_1$. Equations (27) and (28) are as follows:

$$A_2 + B_2 = a_2 - A_1^2, \quad (36)$$

$$B_3 + A_1(A_2 + B_2) = -a_3. \quad (37)$$

From (36) and (37) we obtain

$$B_3 = -a_3 - A_1(a_2 - A_1 B_1), \quad (38)$$

and B_3 is known.

From (30) and (31) we compute A_2 and B_4, whereupon from (27) we determine the coefficient B_2. All the unknown coefficients are calculated, and (29) yields the desired relation between the coefficients $a_1, \dots, a_6$.

3.3. Decomposition of the sixth-degree equation into two equations of the third degree.

We consider the relation (21) in the general form

$$f(s) = a_0 s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s + a_6 = 0. \quad (39)$$

We assume that the following relation holds:

$$s_1 + s_2 + s_3 = s_4 + s_5 + s_6. \quad (40)$$

From this relation and (39), we conclude that

$$s_1 + s_2 + s_3 = s_4 + s_5 + s_6 = -\frac{a_1}{2a_0}. \quad (41)$$

We also have

$$\begin{aligned} \sum_{i \neq j} s_i s_j &= (s_1 + s_2 + s_3)(s_4 + s_5 + s_6) \\ &\quad + (s_1 s_2 + s_1 s_3 + s_2 s_3) \\ &\quad + (s_4 s_5 + s_4 s_6 + s_5 s_6). \end{aligned} \quad (42)$$

We obtain

$$s_1 s_2 + s_1 s_3 + s_2 s_3 = A_2, \quad (43)$$

$$s_4 s_5 + s_4 s_6 + s_5 s_6 = B_2, \quad (44)$$

$$s_1 s_2 s_3 = A_3, \quad (45)$$

$$s_4 s_5 s_6 = B_3. \quad (46)$$

Taking into account (41)–(44), we have

$$A_2 + B_2 + \frac{a_1^2}{4a_0^2} = \frac{a_2}{a_0}.$$

This means that

$$A_2 + B_2 = \frac{a_2}{a_0} - \frac{a_1^2}{4a_0^2} \tag{47}$$

and

$$\sum_{\substack{i\neq j\neq k\\ i\neq k}} s_i s_j s_k$$
$$= (s_1+s_2+s_3)(s_4s_5+s_4s_6+s_5s_6) + (s_4+s_5+s_6)(s_1s_2+s_1s_3+s_2s_3) + s_1s_2s_3 + s_4s_5s_6,$$

or

$$-\frac{a_1}{2a_0}\big(A_2+B_2\big) + A_3 + B_3 = -\frac{a_3}{a_0}.$$

From this and (47), we obtain

$$B_3 + A_3 = -\frac{a_3}{a_0} + \frac{a_1a_2}{2a_0^2} - \frac{a_1^3}{8a_0^3}. \tag{48}$$

Similarly,

$$\sum_{\substack{i\neq j\neq k\\ i\neq k\neq l\ i\neq l\neq j}} s_i s_j s_k s_l$$
$$= (s_1s_2+s_1s_3+s_2s_3)(s_4s_5+s_4s_6+s_5s_6) + (s_1+s_2+s_3)s_4s_5s_6 + (s_4+s_5+s_6)s_1s_2s_3. \tag{49}$$

Taking account (43) and (44), from (49) we have

$$A_2B_2 - \frac{a_1}{2a_0}\big(A_3+B_3\big) = \frac{a_4}{a_0}. \tag{50}$$

Substitution of (48) into (50) gives

$$A_2B_2 = \frac{a_4}{a_0} - \frac{a_1a_3}{2a_0^2} + \frac{a_1^2a_2}{4a_0^3} - \frac{a_1^4}{16a_0^4}. \tag{51}$$

Now, from (47) and (51), we can determine A_2 and B_2.

Apart from that,

$$(s_1s_2s_3)(s_4s_5s_6) = \frac{a_6}{a_0}. \tag{52}$$

This means that

$$A_3B_3 = \frac{a_6}{a_0}. \tag{53}$$

From (48) and (53), we can compute A_3 and B_3.

The roots s_1, s_2, s_3 are obtained as the solutions of the equation

$$s^3 + \frac{a_1}{2a_0}s^2 + A_2s - A_3 = 0, \tag{54}$$

and the roots s_4, s_5, s_6 result from the equation

$$s^3 + \frac{a_1}{2a_0}s^2 + B_2s - B_3 = 0. \tag{55}$$

Finally, we also have the equality

$$\sum_{i,j,k,l,m} s_is_js_ks_ls_m = s_1s_2s_3(s_4s_5+s_4s_6+s_5s_6) + s_4s_5s_6(s_1s_2+s_1s_3+s_2s_3). \tag{56}$$

From (56) and (43)–(46) we obtain

$$A_2B_3 + B_2A_3 = -\frac{a_5}{a_0}. \tag{57}$$

Substituting the computed coefficients A_2, B_2, A_3 and B_3 into (57), we obtain the relation between the coefficients $a_0, \dots, a_6$ which results from the assumption (40).

Remark 1. The coefficients A_2 and B_2 can be obtained from equations of the second degree. The coefficients A_3 and B_3 can be determined from the relations (48) and (57), which are linear. Then the relation between the coefficients of (39) can be obtained by substitution of A_3 and B_3 into (53). In this way, we decomposed the equation of the sixth degree into two equations which can be solved analytically. We omit the case when we know two relations between couples of roots because it is straightforward and can lead to the first case.

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