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ESTIMATION OF THE OUTPUT DEVIATION NORM FOR UNCERTAIN, DISCRETE-TIME NONLINEAR SYSTEMS IN A STATE DEPENDENT FORM

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Numerical evaluation of the optimal nonlinear robust control requires estimating the impact of parameter uncertainties on the system output. The main goal of the paper is to propose a method for estimating the norm of an output trajectory deviation from the nominal trajectory for nonlinear uncertain, discrete-time systems. The measure of the deviation allows us to evaluate the robustness of any designed controller. The first part of the paper concerns uncertainty modelling for nonlinear systems given in the state space dependent form. The method for numerical estimation of the maximal norm of the output trajectory deviation with applications to robust control synthesis is proposed based on the introduced three-term additive uncertainty model. Theoretical deliberations are complemented with a numerical, water-tank system example.

Keywords: uncertain systems, uncertain estimates, discrete-time systems, nonlinear systems.

1. Introduction

Analysis and control synthesis for nonlinear uncertain systems or systems with limited information constitutes a wide area of science and engineering. In recent years a lot of research results (Dai *et al.,* 2002) have been published on robust control design. The literature can be classified into two categories: the eigenstructure assignment and Riccati-based methods such as H_2, H_∞ and μ syntheses (Zhou *et al.,* 1996). Other papers focus on simplifications of the nonlinear system, e.g. using a describing function analysis (Impram *et al.,* 2001) and linearization.

Among these multivariable control methods, the H_∞ technique has a broad base because of its robustness to uncertainties and reliable design algorithms. A very important problem is the selection of appropriate weighting matrices reflecting system stability and performance (Postlethwaite *et al.*, 1990; Yang *et al.*, 1997). When the weighting matrices are regarded as variables, the H_∞ robust design problem can be formulated as a multi-objective optimization problem which needs to simultaneously satisfy design specifications in both the time domain and the frequency domain. This optimization problem is usually very complicated with many constraints (Tang *et al.,* 1996; Whidborne *et al.,* 1994). The implementation of the nonlinear optimal control requires solving the Hamilton-Jacobi-Bellman equation (Lewis, 1986). The implementation of nonlinear H_∞ control requires solving the Hamilton-Jacobi-Isaacs equation (Van der Schaft, 1992; Basar, 1995).

The most successful applications of robust control techniques such as μ analysis and synthesis have occurred in problem domains (flexible structures, flight control, distillation) where there may be substantial uncertainty in the available models, the degree of freedom and the dimensions of the input, the output and the state may be high-dimensional, but the basic structure of the system is understood and the uncertainty can be quantified. Nonlinearities are bounded and treated as perturbations on a nominal model or handled by gain scheduling linear point designs.

The concept of state space partitioning called the piecewise affine (linear) (PWA, PWL) decomposition with respect to control synthesis is extensively studied, e.g. in the model predictive control (Bacic *et al.,* 2003; Grancharova *et al.,* 2005).

The main objective of the paper is to develop a new method for estimating the maximal norm of the time-domain output trajectory deviation of the uncertain nonlinear discrete-time system with respect to the nonlinear system with nominal parameters. It extends the previous paper (Orlowski, 2003).

The paper concerns the following aspects:

- Transformation of the uncertain, nonlinear discrete-time system in a general form into a linear time-varying uncertain system in a state dependent form.
- Modelling an additive uncertainty for nonlinear systems—a three-term additive perturbation model for the state dependent form is proposed in Section 3.

The method for bound estimation for system matrices based on the concept of PWL decomposition is stated in Section 4. Furthermore, a nonlinear feedback control is applied to the model with an optimal cost functional. Estimates of the maximal output trajectory deviation norm are given as two theorems with proofs, defined in terms of evolutionary operators taken from linear time-varying systems theory. Theoretical deliberations are complemented by a numerical example of the water tank system.

The approach used in the paper is based on a known approximation of the nonlinear system by a linear time varying system. Such an approach is a very effective method for the synthesis of optimal nonlinear control systems. The nonlinear model predictive control (Kouvaritakis *et al.,* 1999) is a very efficient iterative method which employs the optimal control trajectory calculated in the previous time instant. The system is linearized around the trajectory and can be treated as a linear one. The optimal control can be computed in an iterative way by updating the time-varying approximation of the nonlinear model, calculating a new control and checking whether the convergence condition is satisfied (Ordys *et al.,* 1993; Dutka *et al.,* 2004; Lee *et al.,* 2002).

2. Model of the Nonlinear System

Consider a system described by the following, general nonlinear model:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right), \\ \mathbf{y}_k &= \mathbf{g}\left(\mathbf{x}_k\right). \end{aligned} \tag{1}$$

Assume that the system can be transformed into the following nonlinear state space dependent model:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}(\mathbf{x}_k)\mathbf{x}_k + \mathbf{B}(\mathbf{u}_k)\mathbf{u}_k, \\ \mathbf{y}_k &= \mathbf{C}(\mathbf{x}_k)\mathbf{x}_k. \end{aligned} \tag{2}$$

The above description is analogous to the classical linear state space model. Matrix coefficients $\{a_{ij}\}, \{b_{ij}\}$ and $\{c_{ij}\}$ can be arbitrary functions of the state, i.e. $a_{ij} = f_{ij}(\mathbf{x}), c_{ij} = h_{ij}(\mathbf{x})$, and the input $b_{ij} = g_{ij}(\mathbf{u})$. The model given by (2) covers a class of nonlinear systems for which input and state functions can be independently defined. The input-dependent matrix $\mathbf{B}(\mathbf{u}_k)$ can be used to represent input nonlinearities often modelled by the describing function. State-dependent matrices $\mathbf{C}(\mathbf{x}_k)$ and $\mathbf{A}(\mathbf{x}_k)$ cover respectively output nonlinearities and internal system nonlinearities. The model can successfully describe well known nonlinear systems such as a ball and a beam, a water tank system, etc. The model cannot accurately represent a mixed input-state function, neither implicit nor explicit. For example, such a dependence occurs in the inverted pendulum model, e.g. $F\cos(\phi)$, where F is input force and ϕ is the pendulum angle.

3. Model of Uncertainty

Consider the following uncertain, nonlinear model of the system:

$$\begin{aligned} \mathbf{x}^{\Delta}_{k+1} &= \mathbf{A}_{\Delta}(\mathbf{x}^{\Delta}_k)\mathbf{x}^{\Delta}_k + \mathbf{B}_{\Delta}(\mathbf{u}_k)\mathbf{u}_k, \\ \mathbf{y}^{\Delta}_k &= \mathbf{C}_{\Delta}(\mathbf{x}^{\Delta}_k)\mathbf{x}^{\Delta}_k. \end{aligned} \tag{3}$$

The uncertain system produces the uncertain state $\mathbf{x}^{\Delta}_k$ and the uncertain output $\mathbf{y}^{\Delta}_k$. Generally, they are different from the nominal state $\mathbf{x}^{p}_k$ and the nominal output $\mathbf{y}^{p}_k$, and thus $\mathbf{y}^{\Delta}_k \neq \mathbf{y}^{p}_k$, $\mathbf{x}^{\Delta}_k \neq \mathbf{x}^{p}_k$ at least for some k. Of course, they are in general different for the uncertain and nominal systems, $\mathbf{A}(\mathbf{x}^p) \neq \mathbf{A}(\mathbf{x}^{\Delta}), \mathbf{A}_{\Delta}(\mathbf{x}^p) \neq \mathbf{A}_{\Delta}(\mathbf{x}^{\Delta})$, where $\mathbf{x}^p$ and $\mathbf{x}^{\Delta}$ are arbitrary nominal and uncertain states, respectively. Since the nonlinear system (1) is time-invariant, the system matrices are time-independent and they depend on the state/input only. Therefore the index k can be removed from the state and the output. The state- and input-dependent matrices can be expanded in a multivariable Taylor series, e.g. for a matrix $\mathbf{A}$ it is

$$\begin{aligned} \mathbf{A}_{\Delta}(\mathbf{x}^{\boldsymbol{\Delta}}) = \mathbf{A}_{\Delta}(\mathbf{x}^p) + \frac{\mathbf{A}'_{\Delta}(\mathbf{x}^p)}{1!}\left(\mathbf{x}^{\Delta} - \mathbf{x}^p\right) \\ + \frac{\mathbf{A}''_{\Delta}(\mathbf{x}^p)}{2!}\left(\mathbf{x}^{\Delta} - \mathbf{x}^{\mathbf{p}}\right)^2 + \dots . \end{aligned} \tag{4}$$

When the state trajectory error is $\left\|\mathbf{x}^{\Delta}(\cdot) - \mathbf{x}^p(\cdot)\right\| \ll 1$, the series is convergent and it is possible to rewrite it in the following form:

$$\mathbf{A}_{\Delta}(\mathbf{x}^{\Delta}) = \mathbf{A}_{\Delta}(\mathbf{x}^p) + \Delta_{\mathbf{A}r}\left(\mathbf{x}^{\Delta} - \mathbf{x}^p\right), \tag{5}$$

where Δ_{Ar} satisfies the conditions

$$\Delta_{\mathbf{A}r} = \frac{\mathbf{A}'_{\boldsymbol{\Delta}}(\mathbf{x}^{\mathbf{p}})}{1!} + \frac{\mathbf{A}''_{\boldsymbol{\Delta}}(\mathbf{x}^{\mathbf{p}})}{2!}\left(\mathbf{x}^{\Delta} - \mathbf{x}^p\right) + \dots, \tag{6}$$

$$\mathbf{A}_{\Delta}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \Delta_{\mathbf{A}}(\mathbf{x}). \tag{7}$$

Finally, all system matrices have the following additive form:

$$\begin{aligned} &\mathbf{A}_{\Delta}(\mathbf{x}^{\Delta}) \\ &\quad = \mathbf{A}(\mathbf{x}^p) + \Delta_{\mathbf{A}}(\mathbf{x}^p) + \Delta_{\mathbf{A}r}(\mathbf{x}^p)\left(\mathbf{x}^{\Delta} - \mathbf{x}^p\right), \end{aligned} \tag{8}$$

$$\mathbf{B}_{\Delta}(\mathbf{u}^{\Delta}) = \mathbf{B}(\mathbf{u}^{p}) + \Delta_{\mathbf{B}}(\mathbf{u}^{p}) + \Delta_{\mathbf{B}r}(\mathbf{u}^{p}) \left(\mathbf{u}^{\Delta} - \mathbf{u}^{p}\right), \quad (9)$$

$$\mathbf{C}_{\Delta}(\mathbf{x}^{\Delta}) = \mathbf{C}(\mathbf{x}^{p}) + \Delta_{\mathbf{C}}(\mathbf{x}^{p}) + \Delta_{\mathbf{C}r}(\mathbf{x}^{p}) \left(\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right). \quad (10)$$

The model of uncertainty for any perturbed system matrices $\mathbf{A}, \mathbf{B}$ or $\mathbf{C}$ consists of three components:

- the one corresponding to the nominal matrix in the nominal state (or the input), e.g. $\mathbf{A}(\mathbf{x}^p)$,
- an additive perturbation in the nominal state (or the input), e.g. $\Delta_{\mathbf{A}}(\mathbf{x}^p)$, which does not depend on the deviation from the nominal state (or the input),
- a differential perturbation in the nominal state (or the input), e.g. $\Delta_{\mathbf{A}\mathrm{r}}(\mathbf{x}^p)$, which represents an uncertainty increase in connection with the state (or input) deviation.

Generally, one does not need to know the additive perturbation matrices $\Delta_A, \Delta_B, \Delta_C$, and the differential perturbation matrices, $\Delta_{Ar}, \Delta_{Br}, \Delta_{Cr}$, but only has to find their estimates $\delta_A, \delta_B, \delta_C, \delta_{Ar}, \delta_{Br}, \delta_{Cr}$. In such a case the following conditions are held for the matrix $\mathbf{A}$:

$$\|\Delta_{\mathbf{A}}(\mathbf{x}_k^p)\| \le \delta_A < \infty, \quad (11)$$

$$\|\Delta_{\mathbf{A}r}(\mathbf{x}_k^p)\| \le \delta_{Ar} < \infty, \quad (12)$$

where $\Delta_A(x_k^p) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $\Delta_{Ar}(x_k^p) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and for the matrices $\mathbf{B}$ and $\mathbf{C}$ we have

$$\|\Delta_{\mathbf{B}}(\mathbf{u}_k^p)\| \le \delta_B < \infty, \quad (13)$$
$$\|\Delta_{\mathbf{B}r}(\mathbf{u}_k^p)\| \le \delta_{Br} < \infty, \quad (14)$$
$$\|\Delta_{\mathbf{C}}(\mathbf{x}_k^p)\| \le \delta_C < \infty, \quad (15)$$
$$\|\Delta_{\mathbf{C}r}(\mathbf{x}_k^p)\| \le \delta_{Cr} < \infty, \quad (16)$$

where $\Delta_{\mathbf{B}}(\mathbf{u}_k^p) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $\Delta_{\mathbf{B}r}(\mathbf{u}_k^p) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $\Delta_{\mathbf{C}}(\mathbf{x}_k^p) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, $\Delta_{\mathbf{C}r}(\mathbf{x}_k^p) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, $k = 0, 1, \dots, N-1$.

Perturbation norms can be estimated for the nominal state and input trajectories (11)–(16). It is also possible to estimate perturbation norms for all reachable states and inputs. In such a case the index k must be removed from (11)–(16).

4. Estimation of the Perturbation Norm

The procedure of state space partitioning is known as piecewise affine (linear) and it is borrowed from, e.g. (Bacic *et al.,* 2003; Grancharova *et al.,* 2005). The space of allowed states can be partitioned into a set of small PWL clusters. The median state of each cluster can be interpreted as a working point for the linearization of the cluster. The dynamics of the system, i.e. the matrices $\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{u})$ and $\mathbf{C}(\mathbf{x})$, can be identified locally in the neighbourhood of each working point under the assumption that the system does not change the working point. The matrices are nonlinear but time independent, and therefore the discrete-time index can be omitted. Additive perturbations in the nominal state may be expressed as follows:

$$\delta_{ij}^a = \max\left(\left|a_{ij}^{\Delta}(x_j)\right| - \left|a_{ij}^{n}(x_j)\right|\right), \quad (17)$$
$$\Delta^{\mathbf{a}}(\mathbf{x}) = \left\{\delta_{ij}^a(x_j)\right\}, \quad (18)$$
$$\delta_{ij}^b = \max\left(\left|b_{ij}^{\Delta}(u_j)\right| - \left|b_{ij}^{n}(u_j)\right|\right), \quad (19)$$
$$\Delta^{\mathbf{b}}(\mathbf{u}) = \left\{\delta_{ij}^b(u_j)\right\}, \quad (20)$$
$$\delta_{ij}^c = \max\left(\left|c_{ij}^{\Delta}(x_j)\right| - \left|c_{ij}^{n}(x_j)\right|\right), \quad (21)$$
$$\Delta^{\mathbf{c}}(\mathbf{x}) = \left\{\delta_{ij}^c(x_j)\right\}, \quad (22)$$

where the index n denotes the nominal value of the (i,j)–coefficient. Equations (17), (19) and (21) applied to the matrices (18), (20) and (22) allow us to estimate the additive perturbations δ_A, δ_B and δ_C, given by Eqns. (11), (13) and (15):

$$\delta^{\mathbf{A}}(\mathbf{x}) = \|\Delta^{\mathbf{a}}(\mathbf{x})\| \approx \max_{\delta_{ij}^A}\left(\|\Delta_{\mathbf{A}}(\mathbf{x})\|\right),$$
$$\delta_A = \max_{\mathbf{x}} \delta^{\mathbf{A}}(\mathbf{x}). \quad (23)$$

Similar relations hold for δ_B and δ_C. The estimates for differential perturbations δ_{Ar}, δ_{Br} and δ_{Cr} can be calculated from differences between nominal matrices and estimates of additive perturbations for different working points. The following relations may be used to estimate the norms of differential perturbations:

$$\delta_{Ar} = \max_{i,j,i\neq j} \frac{\left\|\mathbf{A}(\mathbf{x}_i^p) - \mathbf{A}(\mathbf{x}_j^p)\right\| + \left|\delta^{\mathbf{A}}(\mathbf{x}_i) - \delta^{\mathbf{A}}(\mathbf{x}_j)\right|}{\left\|\mathbf{x}_i^p - \mathbf{x}_j^p\right\|}, \quad (24)$$

$$\delta_{Br} = \max_{i,j,i\neq j} \frac{\left\|\mathbf{B}(\mathbf{u}_i^p) - \mathbf{B}(\mathbf{u}_j^p)\right\| + \left|\delta^{\mathbf{B}}(\mathbf{u}_i) - \delta^{\mathbf{B}}(\mathbf{u}_j)\right|}{\left\|\mathbf{u}_i^p - \mathbf{u}_j^p\right\|}, \quad (25)$$

$$\delta_{Cr} = \max_{i,j,i\neq j} \frac{\left\|\mathbf{C}(\mathbf{x}_i^p) - \mathbf{C}(\mathbf{x}_j^p)\right\| + \left|\delta^{\mathbf{C}}(\mathbf{x}_i) - \delta^{\mathbf{C}}(\mathbf{x}_j)\right|}{\left\|\mathbf{x}_i^p - \mathbf{x}_j^p\right\|}, \quad (26)$$

5. Control System

Control systems most often have either a linear feedback controller, represented by a state feedback or a nonlinear controller, for example fuzzy-logic.

A. Mathematical Description of Control

The most general description which can cover many different controllers may be written in a nonlinear state feedback operator or matrix form $\mathbf{F}(\mathbf{x}_k)$, as shown in Fig. 1.

It should be underlined that it is not assumed that all states have to be known in each time sample. If some states are difficult to determine, they can be either estimated by an appropriate state observer or excluded from the feedback, e.g. by simply multiplying them by zero in the feedback $\mathbf{F}(\mathbf{x})$.

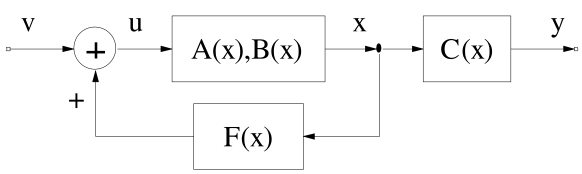

Fig. 1. Closed-loop control system with a state feedback.

On the other hand, the state feedback can be easily converted to an output feedback by the inclusion of the output state dependent matrix $\mathbf{C}(\mathbf{x})$ into the state dependent feedback, i.e. $\mathbf{F}(\mathbf{x}) = \mathbf{F}_y(\mathbf{C}(\mathbf{x}))$, as shown in Fig 2. The proposed methodology is applicable to the state feedback operator, but nevertheless, it can easily describe the output feedback. To model the output feedback, the output state dependent matrix must be included in the state feedback and the diagram from Fig. 2 is equivalent to a classical output feedback. Of course, for a practical implementation of the output feedback there must be known only the output dependent term $\mathbf{F_y}(\mathbf{y})$. The feedback opera-

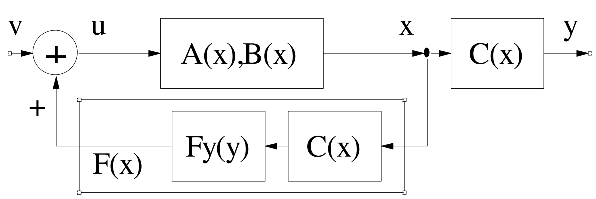

Fig. 2. Closed-loop control system with an output feedback .

tor $\mathbf{F}$ describes either a linear feedback with an invariant vector $\mathbf{F}$ or a nonlinear controller with a state dependent feedback $\mathbf{F}(\mathbf{y})$.

B. Closed-Loop Model

The input signal can be written as follows:

$$\mathbf{u}_k = \mathbf{v}_k + \mathbf{F}(\mathbf{x}_k)\mathbf{x}_k, \tag{27}$$

where $\mathbf{F} \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^m)$ and $k = 0, 1, \ldots, N-1$. After substituting (27) into (2), the system equations take the following state dependent form:

$$\begin{aligned} \mathbf{x}_{k+1} &= (\mathbf{A}(\mathbf{x}_k) + \mathbf{B}(\mathbf{u}_k)\mathbf{F}(\mathbf{x}_k))\,\mathbf{x}_k + \mathbf{B}(\mathbf{u}_k)\mathbf{v}_k, \\ \mathbf{y}_k &= \mathbf{C}(\mathbf{x}_k)\mathbf{x}_k, \qquad k = 0, 1, \ldots, N-1, \end{aligned} \tag{28}$$

where $\mathbf{u}_k$ is given by (27).

The system is asymptotically controllable to 0 if the pair (A, B) is stabilizable and the system is invertible (Albertini *et al.,* 1994).

C. Control Law

Let us assume that the cost functional is the worst case norm of the output trajectory deviation from the given reference trajectory. Generally, it can be written as

$$\tilde{J} = \max \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{r}(\cdot) \right\|. \tag{29}$$

By applying the following triangle inequality:

$$\begin{aligned} &\max \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{r}(\cdot) \right\| \\ &\qquad = \max \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot) + \mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot) \right\| \\ &\qquad \le \max \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot) \right\| + \left\| \mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot) \right\|, \end{aligned} \tag{30}$$

the above functional can be rewritten in the form

$$J = \max \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot) \right\| + \left\| \mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot) \right\| \ge \tilde{J}. \tag{31}$$

The nominal output deviation norm $\left\| \mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot) \right\|$ can be easily obtained by numerical simulations, while the output uncertainty norm $\max \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot) \right\|$ can only be estimated.

The optimization problem can be formulated as follows: For a given system, a fixed reference signal $\mathbf{y}^r$, a set of possible inputs $\mathbf{v} \in \mathbf{V}$ and a given form of the feedback function $\mathbf{F}(\mathbf{x}_k, a_1, \ldots, a_M)$, find values $a_1, \ldots, a_M$ which minimize the cost functional J. Due to the conservatism of the estimates, for practical evaluation a weighting factor α is introduced in the following three cost functions:

$$\begin{aligned} J_2 &= \max \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot) \right\|_2 \\ &\quad + \alpha \left\| \mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot) \right\|_2, && (32) \\ J_\infty &= \max \left\| \mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot) \right\|_\infty \\ &\quad + \alpha \left\| \mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot) \right\|_\infty, && (33) \\ J_1 &= \max \left\| \mathbf{y}^{\Delta}(N) - \mathbf{y}^{p}(N) \right\|_1 \\ &\quad + \alpha \left\| \mathbf{y}^{p}(N) - \mathbf{y}^{r}(N) \right\|_1. && (34) \end{aligned}$$

6. Describing a Nonlinear Feedback System Using Linear Operators

Every linear time-varying system can be described by linear invariant, recurrent operator equations (Orłowski 2001; 2004; 2006). The nonlinear system (2) can be described in a similar manner only in the case of a fixed trajectory, input and initial state vectors:

$$\begin{aligned} \mathbf{x}_k^p &= (\mathbf{N}^{\mathbf{F}}\mathbf{x}_0)(k) + (\mathbf{L}^{\mathbf{F}}(\mathbf{B}\mathbf{v}))(k), \\ \mathbf{y}_k^p &= \mathbf{C}_k^{\mathbf{F}}\mathbf{x}_k^p. \end{aligned} \tag{35}$$

For simplicity, we introduce operators $\mathbf{L^F} \in \mathcal{L}((\mathbb{R}^n)^N, (\mathbb{R}^n)^N)$ and $\mathbf{N^F} \in \mathcal{L}(\mathbb{R}^n, (\mathbb{R}^n)^N)$, defined by

$$(\mathbf{L^F h})(k) = \sum_{i=0}^{k-2} \left[\prod_{j=i+1}^{k-1} \mathbf{A^F}(j) \right] \mathbf{h}(i) + \mathbf{h}(k-1), \tag{36}$$

$$(\mathbf{N^F x}_0)(k) = \prod_{j=0}^{k-1} \mathbf{A^F}(j) \mathbf{x}_0, \tag{37}$$

where

$$\mathbf{A^F}(j) = \left(\mathbf{A}(\mathbf{x}_j) + \mathbf{B}(\mathbf{u}_j)\mathbf{F}(\mathbf{x}_j)\right), \qquad \mathbf{h}(i) \in \mathcal{L}(\mathbf{R}^n), \quad k = 2, 3, \dots, N.$$

Alternatively, operators can be rewritten in an equivalent matrix notation. In such a case, (35) takes the form

$$\hat{\mathbf{y}} = \hat{\mathbf{C}}\hat{\mathbf{L}}^{\mathbf{F}}\hat{\mathbf{B}}\mathbf{v} + \hat{\mathbf{C}}\hat{\mathbf{N}}^{\mathbf{F}}\mathbf{x}_0. \tag{38}$$

The operator $\hat{\mathbf{L}}^{\mathbf{F}}$ is given by the following $nN \times nN$ matrix, which simplifies calculating the operator norm:

$$\hat{\mathbf{L}}^{\mathbf{F}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{A^F}(1) & \mathbf{I} & \mathbf{0} & \vdots & \vdots \\ \vdots & \ddots & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{A^F}(N-2)\cdot\ldots\cdot\mathbf{A^F}(1) & \cdots & \mathbf{A^F}(N-2) & \mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{39}$$

The operators $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ are respectively $nN \times mN$ and $pN \times nN$ matrices and have the following diagonal forms:

$$\hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B}(0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}(N-1) \end{bmatrix},$$
$$\hat{\mathbf{C}} = \begin{bmatrix} \mathbf{C}(0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}(N-1) \end{bmatrix}. \tag{40}$$

For an LTV system, the operator $\hat{\mathbf{C}}\hat{\mathbf{L}}^{\mathbf{F}}\hat{\mathbf{B}}$ is a compact and Hilbert-Schmidt operator from l_2 into l_2 and it maps bounded signals $\mathbf{v}(k) \in V = l_2[0, N]$ into signals $y \in Y$. For SISO systems, the operator $\hat{\mathbf{C}}\hat{\mathbf{L}}^{\mathbf{F}}\hat{\mathbf{B}}$ is an $N \times N$ matrix.

6.1. Operator Description of the Perturbed System.

Theorem 1. *The perturbed nonlinear system (2) with the feedback control (27), a fixed input and an initial state, is always equal to the following equations:*

$$\begin{aligned} \mathbf{x}_k^{\Delta} = {} & \mathbf{L^F}\left(\Delta_{\mathbf{A}}(\mathbf{x}^p)\mathbf{x}^{\Delta}\right)(k) \\ & + \mathbf{L^F}\left(\Delta_{\mathbf{A}r}(\mathbf{x}^p)\left(\mathbf{x}^{\Delta} - \mathbf{x}^p\right)\right)(k) \\ & + \mathbf{L^F}\left(\Delta_{\mathbf{B}}(\mathbf{u}^p)(\mathbf{v} + \mathbf{F}(\mathbf{x}^{\Delta})\mathbf{x}^{\Delta}\right)(k) \\ & + \mathbf{L^F}\left(\Delta_{\mathbf{B}r}(\mathbf{u}^p)\left(\mathbf{F}(\mathbf{x}^{\Delta})\mathbf{x}^{\Delta} - \mathbf{F}(\mathbf{x}^p)\mathbf{x}^p\right)\right)(k) \\ & + \mathbf{x}_k^p, \end{aligned} \tag{41}$$

$$\begin{aligned} \mathbf{y}_k^{\Delta} = {} & \mathbf{C}_k^{\mathbf{F}}\mathbf{x}_k^{\Delta} + \Delta_{\mathrm{C}}(\mathbf{x}_k^p)\mathbf{x}_k^{\Delta} \\ & + \Delta'_{\mathrm{Cr}}(\mathbf{x}_k^p)\left(\mathbf{x}_k^{\Delta} - \mathbf{x}_k^p\right). \end{aligned} \tag{42}$$

Proof. The above equations can be proved using mathematical induction. For $k = 2$, the state equations (41) and (42) with the feedback control (27) are

$$\begin{aligned} \mathbf{x}_2^{\Delta} & \\ = {} & \mathbf{A}_1^{\mathbf{F}}\Big[\left(\mathbf{A}_0^{\mathbf{F}} + \Delta_{\mathbf{A}0} + \Delta_{\mathbf{B}0}\mathbf{F}_0\right)\mathbf{x}_0 + \left(\mathbf{B}_0^{\mathbf{F}} + \Delta_{\mathbf{B}0}\right)\mathbf{v}_0\Big] \\ & + \Big(\mathbf{B}_1^{\mathbf{F}} + \Delta_{\mathbf{B}1} + \Delta_{\mathbf{B}r1}\left(\mathbf{F}_1^{\Delta}\mathbf{x}_1^{\Delta} - \mathbf{F}_1^p\mathbf{x}_1^p\right)\Big)\mathbf{v}_1 \\ & + \Big(\Delta_{\mathbf{A}1} + \Delta_{\mathbf{A}r1}\left(\mathbf{x}_1^{\Delta} - \mathbf{x}_1^p\right) + \Delta_{\mathbf{B}1}\mathbf{F}_1 \\ & + \Delta_{\mathbf{B}r1}\left(\mathbf{F}_1^{\Delta}\mathbf{x}_1^{\Delta} - \mathbf{F}_1^p\mathbf{x}_1^p\right)\mathbf{F}_1^{\Delta}\Big)\mathbf{x}_1^{\Delta} \end{aligned}$$

Substituting (41) and (27) in (28) for $k+1$ yields

$$\begin{aligned} \mathbf{x}_{k+1}^{\Delta} = {} & \mathbf{B}_k^{\mathbf{F}\Delta}\mathbf{v}_k + \left(\mathbf{A}_k^{\mathbf{F}} + \Delta_{\mathbf{A}k} + \Delta_{\mathbf{B}k}\mathbf{F}_k\right) \\ & \cdot \begin{bmatrix} \left(\mathbf{N^F x}_0\right)(k) + \left(\mathbf{L^F B^F v}\right)(k) \\ + \mathbf{L^F}\left(\Delta_{\mathbf{A}}\mathbf{x}^{\Delta}\right)(k) \\ + \mathbf{L^F}\left(\Delta_{\mathbf{A}r}\left(\mathbf{x}^{\Delta} - \mathbf{x}^p\right)\right)(k) \\ + \mathbf{L^F}\Big(\Delta_{\mathrm{B}}(\mathbf{v} + \mathbf{F}\mathbf{x}^{\Delta})\Big)(k) \\ + \mathbf{L^F}\Big(\Delta_{\mathrm{Br}}(\mathbf{F}\mathbf{x}^{\Delta} - \mathbf{F}\mathbf{x}^p)\Big)(k) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \mathbf{x}_{k+1}^{\Delta} = {} & \left(\mathbf{N^F x}_0\right)(k+1) + \left(\mathbf{L^F B^F v}\right)(k+1) \\ & + \mathbf{L^F}\left(\Delta_{\mathbf{A}}\mathbf{x}^{\Delta}\right)(k+1) \\ & + \mathbf{L^F}\Big(\Delta_{\mathbf{A}r}\left(\mathbf{x}^{\Delta} - \mathbf{x}^p\right)\Big)(k+1) \\ & + \mathbf{L^F}\Big(\Delta_{\mathrm{B}}(\mathbf{v} + \mathbf{F}\mathbf{x}^{\Delta})\Big)(k+1) \\ & + \mathbf{L^F}\Big(\Delta_{\mathrm{Br}}\left(\mathbf{F}\mathbf{x}^{\Delta} - \mathbf{F}\mathbf{x}^p\right)\Big)(k+1). \end{aligned}$$

Detailed conversions are simple but laborious. A sketch of the proof is presented above. ■

6.2. Output Perturbation Estimates. The result of Theorem 1 is very useful to find the estimate of

$\|\mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot)\|$. The physical interpretation of the estimate depends on whether the norm is 2-norm, ∞-norm or 1-norm.

Theorem 2. *For any additive perturbations Δ_A, Δ_B, Δ_C and differential perturbations $\Delta_{Ar}, \Delta_{Br}, \Delta_{Cr}$ with the conditions (11)–(16) and*

$$\left(\delta_{\mathrm{Axz}} + \delta_{\mathrm{Arz}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\|\right) < \left\|\mathbf{L}^{\mathbf{F}}\right\|^{-1}, \tag{43}$$

$$\left(\delta_{\mathrm{Axz}} + \delta_{\mathrm{Arz}} \left\|\mathbf{x}^{p}\right\| + \delta_{\mathrm{ab}}\right) < \left\|\mathbf{L}^{\mathbf{F}}\right\|^{-1}, \tag{44}$$

the difference norms $\|\mathbf{x}^{\Delta}(\cdot) - \mathbf{x}^{p}(\cdot)\|_{(\mathbb{R}^n)^N}$ and $\|\mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot)\|_{(\mathbb{R}^p)^N}$ are

$$\left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| \le \frac{\left\|\mathbf{L}^{\mathbf{F}}\right\| \left(\delta_{\mathrm{aoz}} + \delta_{\mathrm{Axz}} \left\|\mathbf{x}^{p}\right\|\right)}{1 - \left\|\mathbf{L}^{\mathbf{F}}\right\| \left(\delta_{\mathrm{Axz}} + \delta_{\mathrm{Arz}} \left\|\mathbf{x}^{p}\right\| + \delta_{\mathrm{ab}}\right)}, \tag{45}$$

$$\begin{aligned}\left\|\mathbf{y}^{\Delta} - \mathbf{y}^{p}\right\| &\le \delta_{\mathrm{C}} \left\|\mathbf{x}_{p}\right\| \\ &+ \frac{\left[\left\|\mathbf{C}\mathbf{L}^{\mathbf{F}}\right\| + \left\|\mathbf{L}^{\mathbf{F}}\right\| (\delta_{\mathrm{C}} + \delta_{\mathrm{Cr}})\right]\left[\delta_{\mathrm{aoz}} + \delta_{\mathrm{Axz}} \left\|\mathbf{x}_{p}\right\|\right]}{1 - \left\|\mathbf{L}^{\mathbf{F}}\right\| (\delta_{\mathrm{Axz}} + \delta_{\mathrm{Arz}} \left\|\mathbf{x}_{p}\right\| + \delta_{\mathrm{ab}})},\end{aligned} \tag{46}$$

where

$$\delta_{\mathrm{Axz}} = \delta_{\mathrm{A}} + \delta_{\mathrm{B}} \left\|\mathbf{F}\right\|, \tag{47}$$

$$\delta_{\mathrm{Arz}} = \delta_{\mathrm{Ar}} + \delta_{\mathrm{Br}} \left\|\mathbf{F}\right\|^{2}, \tag{48}$$

$$\delta_{\mathrm{aoz}} = \delta_{\mathrm{B}} \left\|\mathbf{v}\right\|, \tag{49}$$

$$\delta_{\mathrm{ab}} = \delta_{\mathrm{Br}} \left\|\mathbf{v}\right\| \left\|\mathbf{F}\right\|. \tag{50}$$

Proof. The linear space with the defined norm satisfies all axioms of a metric space, and thus the triangle inequality follows,

$$\begin{aligned}\left\|\mathbf{x}_{k}^{\Delta} - \mathbf{x}_{k}^{p}\right\| &\le \left\|\mathbf{L}^{\mathbf{F}} (\Delta_{\mathrm{B}}(\mathbf{u}^{p}) \left(\mathbf{v}^{\Delta} + \mathbf{F}(\mathbf{x}^{\Delta})\mathbf{x}^{\Delta}\right) (k)\right\| \\ &+ \left\|\mathbf{L}^{\mathbf{F}} \left(\Delta_{\mathrm{Ar}}(\mathbf{x}^{p}) \left(\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right) \mathbf{x}^{\Delta}\right) (k)\right\| \\ &+ \left\|\mathbf{L}^{\mathbf{F}} \left(\Delta_{\mathrm{A}}(\mathbf{x}^{p})\mathbf{x}^{\Delta}\right) (k)\right\| \\ &+ \Big\|\mathbf{L}^{\mathbf{F}} \Big(\Delta_{\mathrm{Br}}(\mathbf{u}^{p})\left(\mathbf{F}(\mathbf{x}^{\Delta})\mathbf{x}^{\Delta} - \mathbf{F}(\mathbf{x}^{p})\mathbf{x}^{p}\right) \\ &\quad \left(\mathbf{v}^{\Delta} + \mathbf{F}(\mathbf{x}^{\Delta})\mathbf{x}^{\Delta}\right)\Big)(k)\Big\|.\end{aligned}$$

Then

$$\begin{aligned}\left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| &\le \left\|\mathbf{L}^{\mathbf{F}}\right\| \delta_{\mathrm{A}} \left\|\mathbf{x}^{\Delta}\right\| \\ &+ \left\|\mathbf{L}^{\mathbf{F}}\right\| \delta_{\mathrm{Ar}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| \left\|\mathbf{x}^{\Delta}\right\| \\ &+ \left\|\mathbf{L}^{\mathbf{F}}\right\| \delta_{\mathrm{B}} \left(\left\|\mathbf{v}^{\Delta}\right\| + \left\|\mathbf{F}\right\| \left\|\mathbf{x}^{\Delta}\right\|\right) \\ &+ \left\|\mathbf{L}^{\mathbf{F}}\right\| \delta_{\mathrm{Br}} \left\|\mathbf{F}\right\| \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| \left\|\mathbf{v}^{\Delta} + \mathbf{F}(\mathbf{x}^{\Delta})\mathbf{x}^{\Delta}\right\|.\end{aligned} \tag{51}$$

Assuming that (47)–(50) hold, the above equation can be simplified as follows:

$$\begin{aligned}\left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| \le \left\|\mathbf{L}^{\mathbf{F}}\right\| \left\|\mathbf{x}^{\Delta}\right\| \left(\delta_{\mathrm{Axz}} + \delta_{\mathrm{Arz}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\|\right) \\ + \left\|\mathbf{L}^{\mathbf{F}}\right\| \left(\delta_{\mathrm{ab}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| + \delta_{\mathrm{aoz}}\right).\end{aligned} \tag{52}$$

The uncertain state norm can be estimated by rearranging (41) and applying the triangle inequality again:

$$\begin{aligned}&\left\|\mathbf{x}^{\Delta}\right\| \\ &\le \left\|\mathbf{x}^{p}\right\| + \left\|\mathbf{L}^{\mathbf{F}}\right\| \Big[\delta_{\mathrm{Axz}} \left\|\mathbf{x}^{\Delta}\right\| + \delta_{\mathrm{Arz}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| \left\|\mathbf{x}^{\Delta}\right\| \\ &\quad + \delta_{\mathrm{ab}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| + \delta_{\mathrm{aoz}}\Big],\end{aligned}$$

$$\begin{aligned}\left\|\mathbf{x}^{\Delta}\right\| \left[1 - \left(\delta_{\mathrm{Axz}} + \delta_{\mathrm{Arz}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\|\right)\right] \\ \le \left\|\mathbf{x}^{p}\right\| + \left\|\mathbf{L}^{\mathbf{F}}\right\| \left[\delta_{\mathrm{ab}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| + \delta_{\mathrm{aoz}}\right],\end{aligned}$$

$$\left\|\mathbf{x}^{\Delta}\right\| \le \frac{\left\|\mathbf{x}^{p}\right\| + \left\|\mathbf{L}^{\mathbf{F}}\right\| \left[\delta_{\mathrm{ab}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| + \delta_{\mathrm{aoz}}\right]}{1 - \left(\delta_{\mathrm{Axz}} + \delta_{\mathrm{Arz}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\|\right)}. \tag{53}$$

Substituting (53) into (52) yields

$$\begin{aligned}\left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| &- \left\|\mathbf{L}^{\mathbf{F}}\right\| \delta_{\mathrm{Axz}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| \\ &- \left\|\mathbf{L}^{\mathbf{F}}\right\| \delta_{\mathrm{Axz}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| \\ &- \left\|\mathbf{L}^{\mathbf{F}}\right\| \delta_{\mathrm{Arz}} \left\|\mathbf{x}^{p}\right\| \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| \\ &- \left\|\mathbf{L}^{\mathbf{F}}\right\| \delta_{\mathrm{Arz}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\|^{2} \\ &\le \left\|\mathbf{L}^{\mathbf{F}}\right\| \left\|\mathbf{x}^{p}\right\| \delta_{\mathrm{Axz}} \\ &\quad + \left\|\mathbf{L}^{\mathbf{F}}\right\| \Big(\delta_{\mathrm{ab}} \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| + \delta_{\mathrm{aoz}}\Big).\end{aligned}$$

When the difference $\left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\|^{2}$ is small enough, it can be neglected and the inequality takes the form

$$\begin{aligned}\left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| \left[1 - \left\|\mathbf{L}^{\mathbf{F}}\right\| \left(\delta_{\mathrm{Axz}} + \delta_{\mathrm{Arz}} \left\|\mathbf{x}^{p}\right\| + \delta_{\mathrm{ab}}\right)\right] \\ \le \left\|\mathbf{L}^{\mathbf{F}}\right\| \left(\delta_{\mathrm{Axz}} \left\|\mathbf{x}^{p}\right\| + \delta_{\mathrm{aoz}}\right).\end{aligned} \tag{54}$$

It is equivalent to (45). The output difference trajectory can be we written as

$$\begin{aligned}\mathbf{y}^{\Delta} - \mathbf{y}^{p} &= \mathbf{C}(\mathbf{x}^{p})\mathbf{x}^{\Delta} + \Delta_{\mathbf{C}r}(\mathbf{x}^{p}) \left(\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right) \mathbf{x}^{\Delta} \\ &\quad + \Delta_{\mathbf{C}}(\mathbf{x}^{p})\mathbf{x}^{\Delta} - \mathbf{C}(\mathbf{x}^{p})\mathbf{x}^{p} \\ &= \left(\mathbf{C}(\mathbf{x}^{p}) + \Delta_{\mathbf{C}r}(\mathbf{x}^{p})\mathbf{x}^{\Delta}\right) \left(\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right) \\ &\quad + \Delta_{\mathbf{C}}(\mathbf{x}^{p})\mathbf{x}^{\Delta}, \\ \left\|\mathbf{y}^{\Delta} - \mathbf{y}^{p}\right\| &\le \left(\left\|\mathbf{C}\right\| + \delta_{\mathrm{C}r} \left\|\mathbf{x}^{\Delta}\right\|\right) \left\|\mathbf{x}^{\Delta} - \mathbf{x}^{p}\right\| \\ &\quad + \delta_{\mathrm{C}} \left\|\mathbf{x}^{\Delta}\right\|.\end{aligned} \tag{55}$$

Equation (46) can be proven by rearranging Eqns. (42) and (35), and then by substituting the uncertain state norm from (45), which completes the proof. ■

7. Numerical Example—A Water Tank System

Consider the system described by the following one-dimensional nonlinear model:

$$x_{k+1} = x_k - \frac{aT_p}{A}\sqrt{x_k} + \frac{bT_p}{A}u_k, \qquad y_k = x_k. \quad (56)$$

The equation describes the level of water x as a function

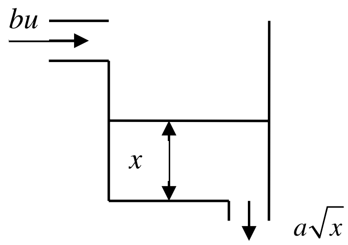

Fig. 3. Water tank system.

of time due to the differences between the flow rates into and out of the tank, bT_pu_k and $aT_p\sqrt{x_k}$, respectively. A is a cross-sectional area of the tank, $b \in [b_-, b_+]$ is an uncertain constant related to the flow rate into the tank, $a \in [a_-, a_+]$ is an uncertain constant related to the flow rate out of the tank, $u_k \in \{0, 1\}$ is a logical variable which signifies the open (1) or closed (0) input valve, and T_p is a sampling pariod. A simple scheme of the system is shown in Fig. 3. The nonlinear state space model can be written using state and input dependent matrices and Eqn. (3):

$$\begin{aligned}\mathbf{A}(x_k) &= \begin{cases} 1 - \frac{aT_p}{A}\frac{\sqrt{x_k}}{x_k} & \text{for} \quad x_k \neq 0, \\ 1 & \text{for} \quad x_k = 0, \end{cases} \\ \mathbf{B}(u_k) &= \frac{bT_p}{A}, \qquad \mathbf{C}(x_k) = 1. \end{aligned} \quad (57)$$

The water level can be controlled using a simple bistable controller with hysteresis. The output of the controller opens or closes the input valve. The valve opens when the level is lower than $L_n - \Delta L$ and remains open until the water reaches the level $L_n + \Delta L$, otherwise the valve is closed. Here L_n is a nominal setpoint value, usually equal to the reference output y_k^r, and ΔL is the width of the one-side hysteresis. The controller has the following mathematical description:

$$\begin{aligned} u_i &= \mathbf{F}\mathbf{x}_i \\ &= \begin{cases} 1 & \text{if } x_i < L_n - \Delta L, \\ & \text{or } x_i < L_n + \Delta L \text{ and } u_{i-1} = 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (58)$$

A larger value of ΔL reduces the controller sensitivity and the switching frequency of the valve. The norm of the operator $\mathbf{F}$ can be estimated from (58) in the following way:

$$\|\mathbf{F}\| = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{F}\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{x\neq 0} \frac{1}{|x|} \approx \frac{1}{\min_k |x_k^p|}. \quad (59)$$

The following substitution can be used to decompose the system into the form (8)–(10):

$$\begin{aligned} \mathbf{A}\left(x_k^p\right) &= 1 - \frac{a_nT_p}{A\sqrt{x_k^p}}, \\ \Delta_{\mathbf{A}}\left(x_k^p\right) &= -\frac{(a-a_n)T_p}{A\sqrt{x_k^p}}, \\ \Delta_{\mathbf{A}r}\left(x_k^p\right) &= \frac{a_nT_p}{2Ax_k^p\sqrt{x_k^p}}, \\ a_n &= \frac{a_+ + a_-}{2} \end{aligned} \quad (60)$$

$$\begin{aligned} \mathbf{B}\left(u_k^p\right) &= \frac{b_nT_p}{A}, \\ \Delta_{\mathbf{B}}\left(u_k^p\right) &= \frac{(b-b_n)T_p}{A}, \\ \Delta_{\mathbf{B}r}\left(u_k^p\right) &= 0, \\ b_n &= \frac{b_+ + b_-}{2}, \end{aligned} \quad (61)$$

$$\begin{aligned} \mathbf{C}\left(x_k^p\right) &= 1, \\ \Delta_{\mathbf{C}}\left(x_k^p\right) &= 0, \\ \Delta_{\mathbf{C}r}\left(x_k^p\right) &= 0. \end{aligned} \quad (62)$$

The parameters assumed for computations are $A = 50$, $a_- = 4, a_+ = 6, b_- = 15, b_+ = 20, T_p = 0.1\ s$, $y_k^r = x_k^r = L_n = 4.5$, $\Delta L = 0.5$. The computed estimates and norms are collected in Table 1.

Transient responses simulated for nominal and uncertain systems for $N = 1000$, $x_0 = 0$, $L_n = 4.5$ and three different values of L are shown in Fig. 4. The computed values of the nominal output deviation norm $\|\mathbf{y}^p(\cdot) - \mathbf{y}^r(\cdot)\|$ and the output uncertainty norm $\left\|\mathbf{y}^\Delta(\cdot) - \mathbf{y}^p(\cdot)\right\|$ are annotated on the plot. The functional (32) with $\alpha = 1$ is minimized for the most frequent switching $\Delta L \to 0$.

The quality of the estimates is determined by the following estimation error coefficient:

$$\varepsilon = \frac{\left\|\mathbf{y}^\Delta(\cdot) - \mathbf{y}^p(\cdot)\right\|_{\text{estimated}}}{\left\|\mathbf{y}^\Delta(\cdot) - \mathbf{y}^p(\cdot)\right\|_{\text{simulated}}} - 1. \quad (63)$$

The plot of the estimation error as a function of the simulation horizon N and the initial condition x_0 is shown in Fig. 5. The minimal value of ε is 0.37 ($x_0 = 3.5,\ N = 9$) and the median value is approximately equal to 2. For longer time horizons, the condition (44) is not satisfied and (45) cannot be used for the estimation of the output deviation.

Table 1. Computed estimates and norms for the water-tank system.

$\delta_A \approx 0.002\left(\min_k (x_k^p)\right)^{-0.5}$	$\delta_{Axz} \approx 0.002\left(\min_k (x_k^p)\right)^{-0.5} + 0.01\left(\min_k (x_k^p)\right)^{-1}$
$\delta_{Ar} \approx 0.005\left(\min_k (x_k^p)\right)^{-1.5}$	$\delta_{Arz} \approx 0.005\left(\min_k (x_k^p)\right)^{-1.5}$
$\Vert\mathbf{F}\Vert \approx \left(\min_k (x_k^p)\right)^{-1}$	$\Vert\mathbf{L}^{\mathbf{F}}\Vert \approx 0.636N$
$\Vert\mathbf{C}\Vert = 1$	$\Vert\mathbf{x}^p\Vert \approx \sqrt{N}\max_k \vert x_k^p\vert$
$\delta_B = 0.01, \delta_{Br} = 0$	$\Vert\mathbf{v}^p\Vert = 0$
$\delta_C = 0, \delta_{Cr} = 0$	$\delta_{aoz} = 0, \delta_{ab} = 0$

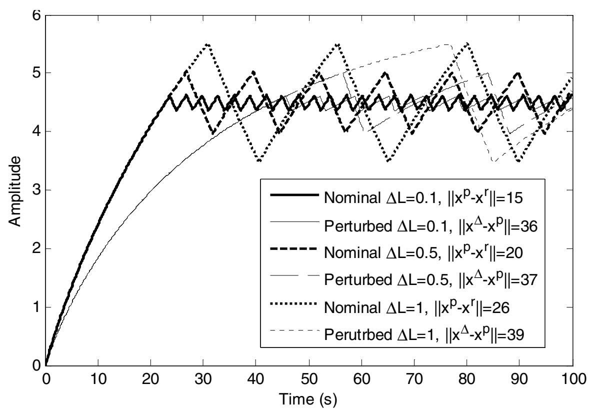

Fig. 4. Transient responses and computed norms for the water tank system.

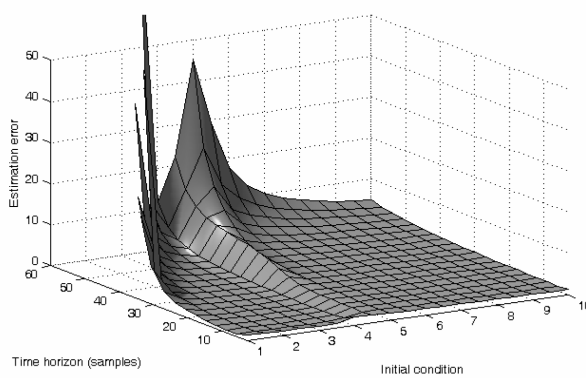

Fig. 5. Estimation error vs. simulation horizon and initial condition for the water tank system.

8. Conclusions

The main aim of this paper was to propose a new method for estimating the norm of the output deviation for uncertain, nonlinear, discrete-time systems. The method can be an interesting alternative to two existing numerical methods for estimating $\max\left\Vert\mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot)\right\Vert$. The first is to estimate the norm on the basis of simulations of the uncertain system for a specified input and the initial state on the assumption of extreme positive and negative values of perturbation matrices $\Delta_A, \Delta_B, \Delta_C,\ \Delta_{Ar}, \Delta_{Br}, \Delta_{Cr}$. The maximal deviation norm of all simulations is an estimate of the norm $\max\left\Vert\mathbf{y}^{\Delta} - \mathbf{y}^{p}\right\Vert$. The number of simulations n_s grows exponentially with the number of nonzero coefficients of the additive perturbation, e.g. $n_s = 2^{\mathrm{nzcoeff}}$. Extreme values of parameters do not guarantee a maximal deviation of the output. Nevertheless, the results are often close to the global maximum. The second method takes advantage of numerical optimization methods, most of them implemented in Matlab. Such an algorithm requires considerable computational power. Moreover, the convergence to the worst-case solution, i.e. the maximal norm, is not guaranteed. The number of variables is equal to the sum of all nonzero coefficients of additive perturbations. The proposed operator-based method guarantees that the estimated output difference norm is not lower than the worst possible real case norm and requires low computational power. The main disadvantage of the method is the conservatism in the estimates.

An iterative two-stage process can be used to find the optimal control solution. The first stage is to find the appropriate structure of the controller such that the maximal trajectory deviation from the nominal trajectory for the uncertain system $\left\Vert\mathbf{y}^{\Delta}(\cdot) - \mathbf{y}^{p}(\cdot)\right\Vert$ is minimal. The second is to find a control which minimizes the deviation of the nominal system from the reference trajectory $\Vert\mathbf{y}^{p}(\cdot) - \mathbf{y}^{r}(\cdot)\Vert$. The procedure must be repeated until the assumed accuracy is approached.

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