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FRACTIONAL ORDER TUBE MODEL REFERENCE ADAPTIVE CONTROL FOR A CLASS OF FRACTIONAL ORDER LINEAR SYSTEMS

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We introduce a novel fractional order adaptive control design based on the tube model reference adaptive control (TMRAC) scheme for a class of fractional order linear systems. By considering an adaptive state feedback control configuration, the main idea is to replace the classical reference model with a single predetermined trajectory by a fractional order performance tube guidance model allowing a set of admissible trajectories. Besides, an optimization problem is formulated to compute an on-line correction control signal within specified bounds in order to update the system performance while minimizing a control cost criterion. The asymptotic stability of the closed loop fractional order control system is demonstrated using an extension of the Lyapunov direct method. The dynamical performance of the fractional order tube model reference adaptive control (FOTMRAC) is compared with the standard fractional order model reference adaptive control (FOMRAC) strategy, and the simulation results show the effectiveness of the proposed control method.

Keywords: fractional order linear system, model reference adaptive control, fractional adaptive control, optimization, performance tube, fractional order TMRAC.

1. Introduction

Fractional calculus is a topic involving a large scientific research community in various fields of science and engineering applications. Indeed, fractional order differential equations have proven to be more appropriate for the modeling of many physical systems, with a growing number of examples in the research literature. Their applications concern as well renewable energy systems (Neçaibia *et al.*, 2015), long transmission lines (Clarke *et al.*, 2004), robotics (Angel and

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Viola, 2018), economics (Dadras and Momeni, 2010), electromechanical systems (Cheng *et al*., 2002), biological systems (Ahmad and Abdel-Jabbar, 2006), cardiac behavior (Goldberger *et al*., 1985), modeling and identification (Benchellel *et al*., 2007), and so on.

In the meantime, differential operators of fractional order are becoming more and more prominent in control theory where the system to be controlled and/or the regulator are governed by fractional differential equations (Ladaci and Bensafia, 2016; He *et al*., 2019). Fractional order operators allow more freedom in designing controllers and setting parameters. A major advantage provided by the resulting controls is an improved robustness compared with classical regulators, as demonstrated in the CRONE (Commande Robuste d'Ordre Non Entier) approach introduced by Oustaloup (1991), the fractional order $PI^\lambda D^\mu$ control with different adjustment strategies (Bettayeb *et al*., 2017; Neçaibia and Ladaci, 2014; Mondal *et al*., 2020), the fractional order robust control based on the ideal Bode transfer function (Djouambi *et al*., 2008), the fractional order high-gain adaptive control (Charef *et al*., 2013), fractional order dynamic soft variable structure control (Kamal and Bandyopadhyay, 2015) and the model-free fractional order adaptive control (Yakoub *et al*., 2015).

Special attention has been paid to the fractional order adaptive control structures based on model reference adaptive control (MRAC) in the recent literature (Ladaci and Charef, 2012; Bourouba *et al*., 2019). Indeed, since the first works of Vinagre *et al*. (2002) as well as Ladaci and Charef (2006), many innovative research studies were developed around this promising adaptive control configuration and its applications.

Various FOMRAC control strategies were developed in the literature; one can cite the MIT adaptive control law based on the fractional order integral (Ladaci and Charef, 2003), the robust feed-forward based FOMRAC (Ladaci *et al*., 2009) and the composite FOMRAC control (Wei *et al*., 2016). Also, numerous applications of FOMRAC have been realized, such as electronic systems (Aguila-Camacho and Duarte-Mermoud, 2013), tank level supervision (Balaska *et al*., 2018), cruise control of an electrical vehicle (Balaska *et al*., 2019), a multi-source renewable energy system (Djebbri *et al*., 2020), etc. Some theoretical results on the stability of the FOMRAC control have consolidated this growing technique (Aguila-Camacho *et al*., 2014; Duarte-Mermoud *et al*., 2015).

Recently, a new concept of MRAC with performance tube was developed by Mirkin *et al*. (2011). It is called the tube based model reference adaptive control (TMRAC), where the control objective is not only to guaranty the closed loop stability, robustness and asymptotical tracking, but also to optimize the control cost with respect
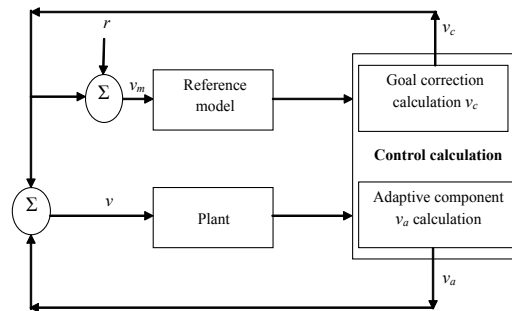

Fig. 1. Performance tube based MRAC configuration.

to some criterion. An application of this novel adaptive control scheme was performed for the water level of a cylindrical tank system (Chittillapilly and Hepsiba, 2015).

Fractional order MRAC with the performance tube constraint. The concept of tube reference model based adaptive control with real time input cost adaptation (TMRAC) was proposed and developed by Mirkin *et al.*, (2011; 2012) along with Mirkin and Gutman (2013). In contrast to the standard adaptive control design approach in which the desired performance of the closed loop control system is set using a guidance model with a predefined unique trajectory, in TMRAC control the reference is not imposed as a unique trajectory but as set of acceptable trajectories called tube reference model.

In the control principle detailed by Mirkin and Gutman (2013), a goal correction control is used to define the input signal for the guidance model which can evolve within a chosen range as represented in Fig. 1. By tuning this input signal with respect to the authorized interval, an acceptable set of orientation trajectories are induced. The feedback control signal is thus improved with the goal correction control. The resulting TMRAC strategy has two control aims: the first is the asymptotical stability and tracking, while the second is to satisfy an additional performance objective which can include decreasing the necessary control energy by varying the goal correction control within the permitted range.

In this paper, we propose a novel fractional order tube model reference adaptive control (FOTMRAC) approach to control a class of fractional order linear systems. Using a state feedback control configuration, we set two objectives: the first is to make the system states asymptotically track the states of a stable fractional order "guidance" model and the second goal is to reduce the control energy cost (Sajewski, 2017).

Compared with the literature (Mirkin and Gutman, 2013; Chittillapilly and Hepsiba, 2015), the main contribution is the generalization of the performance tube

technique from integer order systems to a larger class of plants involving fractional order derivatives of the states with an improvement of the performance indexes. Many preceding works have proven that fractional order MRAC is advantageously robust against additive noise and disturbances (Ladaci *et al.*, 2009). The challenging problem in this paper is to investigate the fractional order control behavior against model parameter uncertainties, within a fixed set of constraints.

The FOTMRAC design procedure can be summarized in two major stages. In the first one we compute a control law that guarantees the closed loop stability of the fractional order guidance model for any reference input, and the asymptotic tracking of the allowed orientation trajectory. In the second stage, we formulate an optimization problem in order to calculate an on line correction signal in order to minimize the control energy cost. Knowing this step is only feasible after succeeding in the first design stage.

The paper is organized as follows. In Section 2 we describe some necessary preliminaries and present some theoretical concepts on fractional order systems. The proposed FOTMRAC control design is exposed in Section 3, whereas Section 4 outlines the main results of this work and performs the stability analysis of the proposed adaptive control approach. Section 5 is dedicated to numerical simulation examples in order to illustrate the FOTMRAC efficiency. Finally, concluding remarks are given in Section 6.

2. Preliminaries

Fractional order differential equations are widely addressed in the mathematical and control literature (Podlubny, 1999). Their main drawback is generally the lack of exact solutions leading to the use of numerical and approximation techniques for their analysis and implementation. For the purpose of the proposed fractional order adaptive control design, we will present the Grünwald—Letnikov and Caputo derivative definitions with a numerical approximation algorithm for the Grünwald–Letnikov integral and derivative to be used later in our simulations. Also, we will recall some important results that are useful for the stability analysis of the proposed fractional order control system (Aguila-Camacho *et al.*, 2014; Alikhanov, 2013), and the fractional order extension of the Lyapunov theorem (Li and Podlubny, 2010; Kaczorek, 2019).

2.1. Caputo and Grünwald–Letnikov definitions.
The Caputo definition of the derivative of fractional order α has the following form (Kaczorek and Borawski, 2016):

$$
{}^{C}_{0}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau\right] \quad (1)
$$

where α is a real number such that $n - 1 \leq \alpha \leq n$ and $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,\mathrm{d}y$ is the Gamma function.

The Grünwald–Letnikov derivative is defined as

$$
{}^{GL}_{0}D^{\alpha}_{t}f(t) = \lim_{h\to 0} h^{-\alpha}\sum_{r=0}^{t/h}(-1)^r\binom{\alpha}{r}f(t-rh) \quad (2)
$$

where the binomial coefficients ($r > 0$) are given by

$$
\binom{\alpha}{0} = 1, \quad \binom{\alpha}{r} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!} \quad (3)
$$

with $r \in \mathbb{N}$ and h denotes the sampling period.

2.2. Grünwald–Letnikov numerical approximation.
For a causal function $f(t)$, with the time operator $t = kh$, the fractional order derivative is (Balaska *et al.*, 2019)

$$
D^{\alpha}_{t}f(t) = h^{-\alpha}\sum_{r=0}^{t/h}(-1)^r\binom{\alpha}{r}f(kh-rh) \quad (4)
$$

The fractional order integral is given by (Podlubny, 1999)

$$
I^{\alpha}_{t}f(t) = D^{-\alpha}_{t}f(kh) \quad (5)
$$
$$
\approx h^{\alpha}\sum_{r=0}^{t/h}(-1)^r\binom{-\alpha}{r}f(kh-rh).
$$

Besides, we have the following important property of the Caputo derivative of fractional order (Aguila-Camacho *et al.*, 2014).

Lemma 1. *Let $\xi(t) \in \mathbb{R}^n$ be a continuous and derivable function. Then for any time instant $t \geq t_0$*

$$
\frac{1}{2}{}^{C}_{t_0}D^{\alpha}_{t}\left(\xi^T(t)\xi(t)\right) \leq \xi^T(t){}^{C}_{t_0}D^{\alpha}_{t}\xi(t) \quad (6)
$$

2.3. Stability of fractional order systems.
Consider the fractional order nonlinear time varying systems of the form

$$
{}^{C}_{t_0}D^{\alpha}_{t}\xi(t) = f(\xi,t), \quad (7)
$$

where t represents time and $\alpha \in (0,1)$. Here the definition (1) is used. Then we have the following important result for the stability analysis of the system (7) using the direct Lyapunov theorem (Alikhanov, 2010).

Definition 1. A continuous function $\gamma : [0,t) \to [0,\infty)$ is said to be a *class-K function* if it is strictly increasing and $\gamma(0) = 0$ (Duarte-Mermoud *et al.*, 2015).

Theorem 1. *Consider a non autonomous fractional order system (7), and let $\xi = 0$ be its equilibrium point. Assume the existence of a Lyapunov function $V(t,\xi(t))$ and class-K functions γ_i $(i = 1,2,3)$ that satisfy*

$$
\gamma_1\left(\|\xi\|\right) \leq V(t,\xi(t)) \leq \gamma_2\left(\|\xi\|\right), \quad (8)
$$
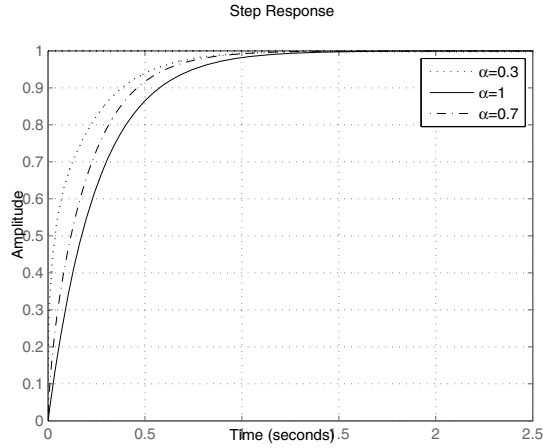
Fig. 2. Comparison of step responses for a first order-like system.

Fig. 3. Comparison of step responses for a second order-like system.

$$\,_{t_0}^{C}D_t^{\beta}V(t,\xi(t)) \leq -\gamma_3\left(\|\xi\|\right) \qquad (9)$$

with $\beta \in (0,1)$. *Then the system (7) is asymptotically stable.*

Proof. See the proof by Muñoz-Vázquez *et al.* (2019). ∎

2.4. Performance of fractional order systems. Advantageous properties of fractional order models compared with standard integer order ones have been pointed out by many previous studies as the best reference models for control applications (Ladaci and Charef, 2006; Mondal and Biswas, 2011). Indeed, fractional order systems offer better quality, in terms of time response and transition dynamic stability, in addition to their intrinsic robustness action against disturbances and noise (Ladaci *et al.*, 2006).

In order to illustrate this fact, consider two examples of systems with respectively first order and second order-like models given by

$$H_1(s) = \frac{1}{\left(1+\dfrac{s}{4}\right)^{\alpha}}, \qquad (10)$$

$$H_2(s) = \frac{1}{\left(\dfrac{s^2}{\omega_n^2} + 2\xi\dfrac{s}{\omega_n} + 1\right)^{\alpha}} \qquad (11)$$

with $\omega_n = 10\,\text{rad/s}$, $\xi = 0.95$.

The step responses of systems $H_1(s)$ and $H_2(s)$ for the integer case ($\alpha = 1$) and the fractional order values of (α) are given in Figs. 2 and 3, respectively. They show the gain in the rise time in both the cases.
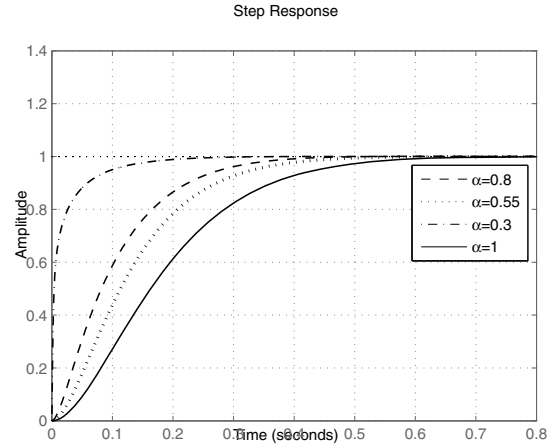
3. Problem statement

We consider the plant represented by the fractional order linear state model

$$D^{\alpha}\xi(t) = A\xi(t) + Bv(t), \qquad (12)$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times l}$ are unknown constant parameter matrices, α is a real number such that $0 < \alpha < 1$, $\xi(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^l$ are the system state and the input signal, respectively. We assume that the plant state variables are measured.

To simplify the notation, we write D^{α} for the fractional derivative of order α instead of $\,_{t_0}^{C}D_t^{\alpha}$.

The first objective of the proposed control approach is to determine the control signal $v(t)$ such that $\xi(t)$ tracks asymptotically the desired fractional order reference model specified by

$$D^{\alpha}\xi_m(t) = A_m\xi_m(t) + B_mv_m(t) \qquad (13)$$

with

$$v_m(t) = r(t) + v_c(t) \qquad (14)$$

where $A_m \in \mathbb{R}^{n \times n}$ is a stable matrix, $B_m \in \mathbb{R}^{n \times l}$, $\xi_m \in \mathbb{R}^n$ and $v_m \in \mathbb{R}^l$ are the reference model state and the reference model input signal, respectively.

Here $r(t) \in \mathbb{R}$ is the reference input signal, and $v_c(t) \in \mathbb{R}$ is the goal correction control signal varying in the following interval:

$$v_c(t) \in \left[v_c^-(t), v_c^+(t)\right]. \qquad (15)$$

Many practical processes can be modeled in the form (12). In fact, fractional order models have proven to be more accurate in representing many

physical phenomena than the classical "integer-order" methods. In (Stiassnie, 1979) and (Bagley, 1983), viscoelastic models are formulated by fractional order differential equations. Another typical fractional order system is the voltage-current relation of a semi-infinite lossy transmission line (Wang, 1987) and diffusion of the heat through a semi-infinite solid, where heat flow is equal to the half-derivative of the temperature (Podlubny, 1999). An automotive application of the fractional-order modeling of high-pressure fluid-dynamic flows is presented by Lino *et al.* (2015). Movahhed *et al.* (2016) proposed a fractional state space model for a fractional DC/DC Buck converter. Also a fractional order model is used to represent electrically coupled neuron systems by Moaddy *et al.* (2012) and electrical circuits by Kaczorek (2011). System identification for thermal dynamics of buildings uses fractional order models in the work of Chen *et al.* (2016). Even in biomedicine and biology engineering, a fractional order model is proposed to model immune cells influenced by cancer cells (Ucar *et al.*, 2019) and dynamics of HIV infection (Rihan, 2013) (see a more complete review by Ionescu *et al.* (2017)).

Besides, by (14) we define the fractional order reference model input signal, which can be changed within specified limits, and like that, we do not have unique reference trajectory but a set of admissible desired trajectories which constitute the performance tube reference model. In many applications, such as industrial process or flight control, it is not necessary to follow exactly a single reference trajectory, but some specified deviation from it is permitted (Duarte-Mermoud *et al.*, 2015).

Then, a second objective is to find the correction signal satisfying the condition (15) that minimizes the cost function,

$$J(v_c) = \int_0^t v^2(\tau)\, \mathrm{d}\tau \tag{16}$$

In order to be able to guarantee the first goal of the proposed fractional order adaptive control strategy, we assume that there exist a constant vector $\theta_\xi^\star \in \mathbb{R}^n$ and a nonzero constant scalar $\theta_m^\star$ such that the following conditions are satisfied:

$$A + B\theta_\xi^{\star T} = A_m, \tag{17}$$

$$B\theta_m^\star = B_m. \tag{18}$$

We assume that the structure of A and B is known and the pair (A, B) is controllable. We also assume that θ_m^* is positive. This last hypothesis is not restrictive because, first, it is usually used in similar designs (Ioannou and Sun, 1996) and we have the ability to choose A_m and B_m in a way that guaranties the existence of $\theta_\xi^\star \in \mathbb{R}^n$ and $\theta_m^\star$. We note that if we consider arbitrary matrices A, B, A_m, B_m, there may be no $\theta_\xi^\star \in \mathbb{R}^n$ and scalar $\theta_m^\star$ satisfying (17) and (18), and, therefore, the

control law may not have sufficient structural flexibility to realize the control objective.

In case the structure of A and B is known, we can design the matrices A_m and B_m so that the matching conditions (17) and (18) have a solution for some $\theta_\xi^\star$ and $\theta_m^\star$. In this study, we assume their existence which means that the controller has sufficient structural flexibility to realize the control objective.

The proposed control $v(t)$ is the sum of two signals: an adaptive signal $v_a(t)$, and a goal correction control signal $v_c(t)$,

$$v(t) = v_a(t) + v_c(t) \tag{19}$$

Defining the tracking error $e(t) = \xi(t) - \xi_m(t)$ for any $v(t)$ such that

$$\begin{aligned}
D^\alpha e(t) &= D^\alpha \xi(t) - D^\alpha \xi_m(t) \\
&= A\xi(t) + Bv(t) - A_m\xi_m(t) + B_m v_m(t) \\
&= \left(A_m - B\theta_\xi^{\star T}\right)\xi(t) + B\left(v_a(t) + v_c(t)\right) \\
&\quad - A_m\xi_m(t) - B_m\left(r(t) + v_c(t)\right) \\
&= A_m e(t) + B\left[v_a(t) - \theta_\xi^{\star T}\Psi(t) - \theta_c^\star v_c(t)\right]
\end{aligned} \tag{20}$$

with

$$\begin{cases}
\Psi(t) = [\xi(t)\ r(t)]^T, \\
\theta^\star = [\theta_\xi^{\star T}\ \theta_m^\star]^T, \\
\theta_c^\star = \theta_m^\star - 1,
\end{cases} \tag{21}$$

$$v_a(t) = \theta^T \Psi(t) + \theta_c v_c(t), \tag{22}$$

$$\tilde{\theta}(t) = \theta(t) - \theta^\star, \quad \tilde{\theta}_c(t) = \theta_c(t) - \theta_c^\star \tag{23}$$

which results in the fractional order error equation

$$D^\alpha e(t) = A_m e(t) + B\tilde{\theta}^T(t)\Psi(t) + B\tilde{\theta}_c(t)v_c(t). \tag{24}$$

Following the results of Mirkin and Gutman (2013), we are able to propose the adaptive control of equations (19) and (22), with the following gain adaptation laws:

$$\begin{cases}
D^\alpha \theta(t) = -\Lambda_1 \Phi(t) - \Lambda_p D^\alpha \Phi(t), \\
\Phi(t) = \Omega(t)\Psi(t), \\
\Omega(t) = e^T(t)PB_m, \\
D^\alpha \theta_c(t) = -\Lambda_c \Omega(t)v_c(t) + \theta_{c0}(t),
\end{cases} \tag{25}$$

where

$$\theta_{c0}(t) = \begin{cases}
0 & \text{if } |\theta_c + 1| \geq \epsilon, \\
\Lambda_c \Omega(t)v_c(t) & \text{otherwise}
\end{cases} \tag{26}$$

is the projection function, $\Lambda_1 = \Lambda_1^T > 0$, $\Lambda_p = \Lambda_p^T > 0$ and $\Lambda_c > 0$ are the design parameters, ϵ is a known lower bound to $|\theta_c^\star + 1| = |\theta_m^\star| = \theta_m^\star$, P is the solution of the Lyapunov equation

$$A_m^T P + PA_m + Q = 0, \quad Q = Q^T > 0. \tag{27}$$

In the next section, we shall prove that the proposed control law guarantees the asymptotic stability in the sense of Lyapunov of the closed loop system for any choice of $v_c \in [v_c^-, v_c^+]$, and thus, the first control goal is achieved independently from the second one.

The objective for the proposed FOTMRAC control scheme it to design an updating mechanism for the goal correction control $v_c(t) \in [v_c^-(t), v_c^+(t)]$ aiming to minimize the control energy cost evaluated by the optimization criterion (16).

From (16) and (19) we get $J(v_c) = (v_a(t) + v_c(t))^2$. Thus, for any bounded $\theta^T(t)$, $\Psi(t)$ and $\theta_c(t)$, the optimization criterion (16) leads to the following solution:

$$
\begin{aligned}
& v_c^{\text{opt}}(t) \\
& = \begin{cases} -v_a(t) & \text{if } v_a \in [v_c^-, v_c^+], \\ \arg\min(J(v_c^-), J(v_c^+)) & \text{otherwise} \end{cases}
\end{aligned}
\tag{28}
$$

where $J(v_c^-$ and $J(v_c^+)$ are the values of $J(v_c) = (v_a(t)+ v_c(t))^2$ when $v_c = v_c^-$ and $v_c = v_c^+$, respectively.

From the given controller of Eqns. (19), (25), and (28), it results that the two control goals are realized and obviously, the first stage problem solution is indispensable to deal with the second design step.

Remark 1. Note that the solution of the optimization problem (16) in case when $v_a \in [v_c^-, v_c^+]$ has the form $-(1 + \theta_c(t))^{-1}\theta^T(t)\Psi(t)$, so the adaptive gain $\theta_c(t)$ has a point of singularity. In order to prevent $\theta_c(t)$ from taking the value -1, we suggest the parameter projection algorithm for updating this gain (Ioannou and Sun, 1996).

4. Main results

In order to be able to perform the stability analysis of the proposed performance tube based adaptive control scheme for the discussed class of fractional order systems represented by equation (12) for any bounded choice of $v_c(t) \in [v_c^-(t), v_c^+(t)]$, we need to recall the following lemma (Duarte-Mermoud *et al.*, 2015).

Lemma 2. *Consider a continuous and differentiable function $\xi(t) \in \mathbb{R}^n$, with $Q = Q^T \geq 0 \in \mathbb{R}^{n \times n}$. For any time instant $t \geq t_0$, we have*

$$
\frac{1}{2} {}_{t_0}^{C}D_t^\alpha \xi^T(t) Q \xi(t) \leq \xi(t) Q {}_{t_0}^{C}D_t^\alpha \xi(t), \quad \forall \alpha \in (0, 1).
\tag{29}
$$

Proof. We assume that there exist a matrix $P \in \mathbb{R}^{n \times n}$ and a vector $y(t) \in \mathbb{R}^n$ such that

$$
Q = P^T P, \quad P\xi(t) = y(t).
\tag{30}
$$

Thus

$$
\frac{1}{2} {}_{t_0}^{C}D_t^\alpha \xi^T(t) P^T P \xi(t) = \frac{1}{2} {}_{t_0}^{C}D_t^\alpha y^T(t) y(t).
\tag{31}
$$

From (31) and Lemma 1, we get $\forall \alpha \in (0, 1)$,

$$
\frac{1}{2} {}_{t_0}^{C}D_t^\alpha \xi^T(t) Q \xi(t) \leq y^T(t) {}_{t_0}^{C}D_t^\alpha y(t),
\tag{32}
$$

$$
\frac{1}{2} {}_{t_0}^{C}D_t^\alpha \xi^T(t) Q \xi(t) \leq \xi^T(t) P^T P {}_{t_0}^{C}D_t^\alpha \xi(t).
\tag{33}
$$

Consequently,

$$
\frac{1}{2} {}_{t_0}^{C}D_t^\alpha \xi^T(t) Q \xi(t) \leq \xi^T(t) Q {}_{t_0}^{C}D_t^\alpha \xi(t).
\tag{34}
$$

$\blacksquare$

Now we present the main result of this paper concerning the Lyapunov stability of the proposed FOTMRAC control strategy.

Theorem 2. *Consider the class of fractional order systems given by (12), and the designed reference model given by (13), with the reference input (14) and the goal correction control satisfying (15). The FOTMRAC control law given by (19) and (22) with the gain adaptation laws (25) and (28) guarantees that all the closed-loop signals are bounded, the tracking error tends asymptotically to zero, and also minimizes the control cost defined by (16).*

Proof. Define the Lyapunov candidate function

$$
V = e^T(t) P e(t) + \frac{1}{\theta_m^\star} \left[\tilde{\Phi}^T(t) \Lambda_1^{-1} \tilde{\Phi}(t) + \Lambda_1^{-1} \tilde{\theta}_c^2(t) \right],
\tag{35}
$$

where

$$
\tilde{\Phi}(t) = \tilde{\theta} + \Lambda_p \Phi(t).
\tag{36}
$$

Then

$$
\begin{aligned}
D^\alpha V = {} & D^\alpha \left[e^T(t) P e(t) \right] \\
& + \frac{1}{\theta_m^\star} \Big(D^\alpha \left[\tilde{\Psi}^T(t) \Lambda_1^{-1} \tilde{\Psi}(t) \right] \\
& + \Lambda_1^{-1} D^\alpha \left[\tilde{\theta}_c^2(t) \right] \Big).
\end{aligned}
\tag{37}
$$

Applying Lemma 2 for each fractional order derivative term, we get

$$
\begin{aligned}
& D^\alpha \left[e^T(t) P e(t) \right] \\
& \leq 2 e^T(t) P D^\alpha e(t) \\
& \leq 2 \Big[e^T(t) P A_m e(t) \\
& \quad + \frac{1}{\theta_m^\star} \left[\tilde{\theta}^T(t) \Psi(t) + \tilde{\theta}_c(t) \Omega(t) v_c(t) \right] \Big],
\end{aligned}
\tag{38}
$$

$$
D^\alpha \left[\tilde{\Psi}^T(t) \Lambda_1^{-1} \tilde{\Psi}(t) \right] \leq 2 \tilde{\Psi}^T(t) \Lambda_1^{-1} D^\alpha \tilde{\Psi}(t).
\tag{39}
$$

But

$$D^\alpha \tilde{\Psi}(t) = D^\alpha \theta(t) + \Lambda_p D^\alpha \Psi(t) \quad (40)$$
$$= -\Gamma_1 \Psi(t),$$

which yields

$$D^\alpha \left[\tilde{\Psi}^T(t) \Lambda_1^{-1} \Psi(t) \right]$$

$$\leq 2 \left[\tilde{\theta}^T(t) + \Psi^T(t) \Lambda_p \right] \Lambda_1^{-1} \left[-\Lambda_1 \Psi(t) \right] \quad (41)$$

$$\leq -2 \left[\tilde{\theta}^T(t) + \Psi^T(t) \Lambda_p \right] \Psi(t),$$

$$D^\alpha \left(\tilde{\theta}_c^2(t) \right) \leq 2 \tilde{\theta}_c(t) D^\alpha \tilde{\theta}_c(t) \quad (42)$$

$$\leq 2 \tilde{\theta}_c(t) \left[-\Lambda_c \Omega(t) v_c(t) + \theta_{c0}(t) \right].$$

From (38), (41), and (42), we get

$$D^\alpha V \leq 2 e^T(t) P A_m e(t) \quad (43)$$

$$+ \frac{2}{\theta_m^\star} \left[\tilde{\theta}^T(t) \Psi(t) + \tilde{\theta}_c(t) \Omega(t) v_c(t) \right]$$

$$+ \frac{2}{\theta_m^\star} \left[-\tilde{\theta}^T(t) \Psi(t) - \Psi^T(t) \Lambda_p \Psi(t) \right]$$

$$+ \frac{2\Lambda_c^{-1}}{\theta_m^\star} \left[-\Lambda_c \tilde{\theta}_c(t) \Omega(t) v_c(t) + \tilde{\theta}_c(t) \theta_{c0}(t) \right],$$

$$D^\alpha V \leq 2 e^T(t) P A_m e(t) \quad (44)$$

$$- \frac{2}{\theta_m^\star} \Psi^T(t) \Lambda_p \Psi(t) + \frac{2}{\theta_m^\star \Lambda_c} \tilde{\theta}_c(t) \theta_{c0}(t).$$

Since

$$2 e^T(t) P A_m e(t) = e^T(t) P A_m e(t) \quad (45)$$

$$+ e^T(t) A_m^T P e(t)$$

$$= e^T(t) \left[A_m^T P + P A_m \right] e(t)$$

$$= -e^T(t) Q e(t),$$

we have

$$D^\alpha V \leq -e^T(t) Q e(t) \quad (46)$$

$$- \frac{2}{\theta_m^\star} \Psi^T(t) \Lambda_p \Psi(t) + \frac{2}{\theta_m^\star \Lambda_c} \tilde{\theta}_c(t) \theta_{c0}(t)$$

and the last term satisfies $\tilde{\theta}_c(t) \theta_{c0}(t) \leq 0$ (Ioannou and Sun, 1996).

From (46), we have $D^\alpha V \leq 0$, so that the Lyapunov stability of the proposed control scheme is proved for any $v_c(t) \in [v_c^-(t), v_c^+(t)]$. Consequently, the first goal stage is achieved. It is clear that choosing $v_c^{\text{opt}}(t)$ satisfying (28) allows minimizing the control cost defined by (16). Thus, the two control goals are achieved; we guarantee the stability of the closed loop, the tracking convergence and also the minimization of the control cost.

The two control goals are achieved separately, but the solution of the first stage is necessary for the solution of the second one. ∎
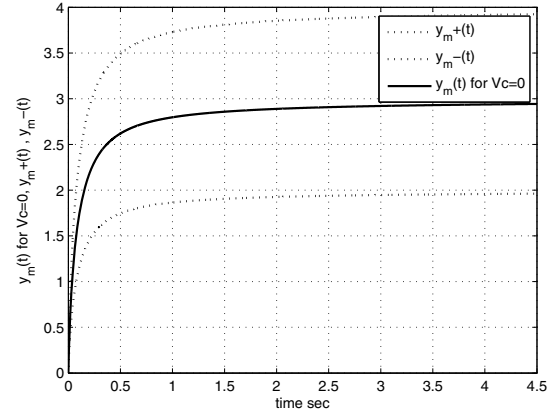

Fig. 4. Reference model response for $v_c(t) = 0$ (solid line) and the performance tube (dotted line) for Example 1.

5. Simulation examples

In this section, we propose two numerical simulation examples to illustrate the efficiency of the proposed FOTMRAC adaptive control strategy. As discussed before, the proposed fractional order models can represent a wide range of practical processes.

5.1. Example 1. Let the unstable fractional order scalar plant be modeled as

$$D^\alpha y(t) = a y(t) + b v(t), \quad y(0) = y_0. \quad (47)$$

Assume that the desired reference model is

$$D^\alpha y_m(t) = a_m y_m(t) + b_m v_m(t), \quad (48)$$
$$v_m(t) = r(t) + v_c(t),$$

where $a = 1$, $b = 0.5$, $a_m = -5$, $b_m = 5$, $v_c^- = -1$, $v_c^+ = +1$, $\alpha = 0.75$ and $r(t)$ is a step signal with amplitude $+3$.

By the choice of a_m and b_m we describe the performance of the reference model. In this numerical example, we choose a_m as a stable pole, and b_m in order to have a unitary gain.

Since the reference signal $r(t)$ is a step of $+3$ in amplitude, we choose the range of $v_c(t)$ in $[-1, 1]$, and thus, we define the admissible range for the reference trajectories. The variables $y_m^+(t)$ and $y_m^-(t)$ in Fig. 4, represent the reference model response for the inputs $v(t) = r(t) + v_c^+$ and $v(t) = r(t) + v_c^-$, respectively.

These variables define the tube of the admissible reference trajectories. The reference model response for $v_c(t) = 0$ is represented in Fig. 4 (the solid line).

The fractional order tube model reference adaptive controller of (19) and (22) with the gain adaptation
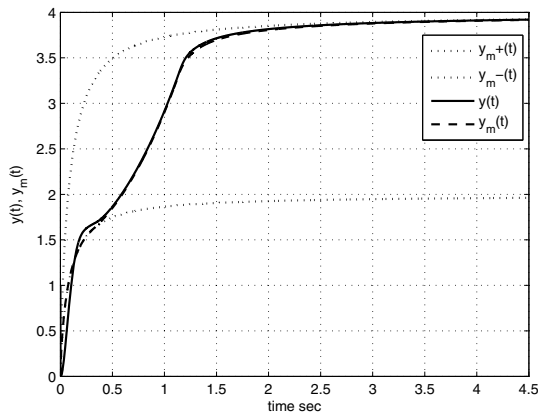
Fig. 5. Controlled system output (solid line) and the reference trajectory (dotted line) within the performance tube (dashed line).
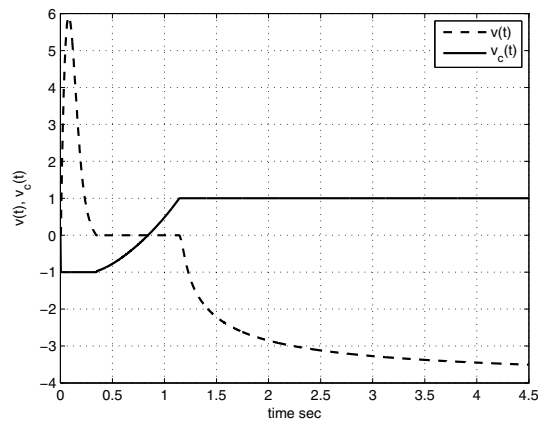

Fig. 6. FOTMRAC control signal (solid line) and the goal correction control signal (dashed line).
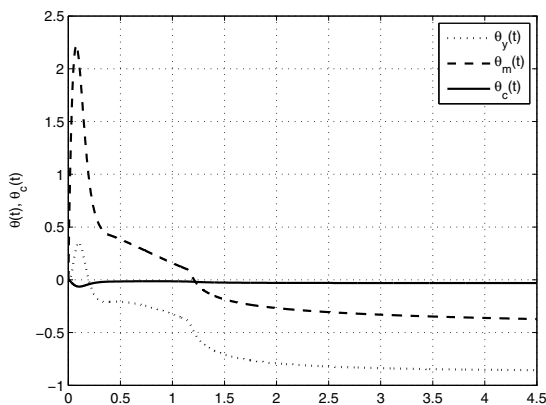

Fig. 7. Adaptive gain vectors.

laws (25) and the optimization problem solution (28) are applied to the plant (47) with the controller parameters

$$\Lambda_1 = 9, \quad \Lambda_p = 0.6, \quad P = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/5 \end{bmatrix}. \quad (49)$$

P is the solution of the Lyapunov equation (27). Besides, in order to compare the performance of the proposed FOTMRAC strategy with the FOMRAC scheme (i.e., $v_c = 0$ and $\theta_c = 0$ in (19) and (22)), we adjust the value of the reference model input in order to have the same steady state trajectory reaching.

For the needs of numerical simulation, we use the Gründwald–Letnikov approximation of the derivative and integrator operators (4) and (5), respectively.

Our experimental setting for the proposed control application includes two cases with two different values for the reference model inputs $r(t) = +3$ and $r(t) = -3$. First we take the reference model input $r(t) = +3$. The simulation results for the obtained closed loop control system output and the guidance trajectory within the performance tube are presented in Fig. 5.

The FOTMRAC control signal, the goal correction control signal and the adaptive gain vectors are respectively represented in Figs. 6 and 7.

The closed loop control system outputs for both the cases of fractional order adaptive control (FOMRAC) with optimization and without optimization are shown in Fig. 8, whereas the absolute control signals are illustrated in Fig. 9.

By taking the reference model input $r(t) = -3$, we obtain the results summarized in Figs. 10–14, illustrating, respectively, the controlled system output and the reference trajectory within the performance tube (Fig. 10), the FOTMRAC control signal and the goal correction control signal (Fig. 11), the adaptive gain vectors (Fig. 12), the system outputs for the cases of fractional order adaptive control with and without optimization (Fig. 13), and the absolute control signals for the same cases (Fig. 14).

From these simulation results, it is obvious that the correction control signal is updated at each instant of time, and varies within the preset limits in order to reduce the energy control criterion. It implies also that the reference trajectory is updated in real time and remains in the predefined model reference tube, whereas the system output tracks perfectly the updated trajectory.

Figures 9 and 14 show the absolute value of the control signals $|v(t)|$ for the cases of fractional order adaptive control with and without optimization, respectively. We notice that the control signal cost is clearly lower with optimization, and thus, we have the same steady response space with a minimum control effort.
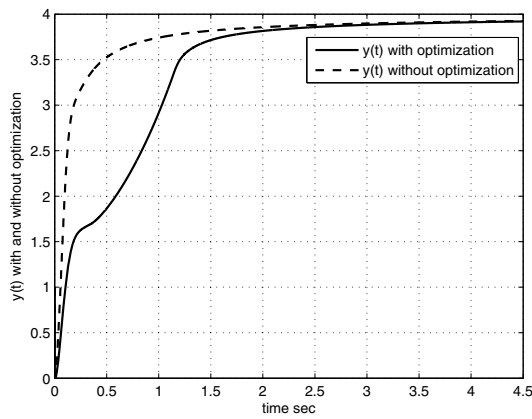
Fig. 8. System outputs for case of fractional order adaptive control with and without optimization.
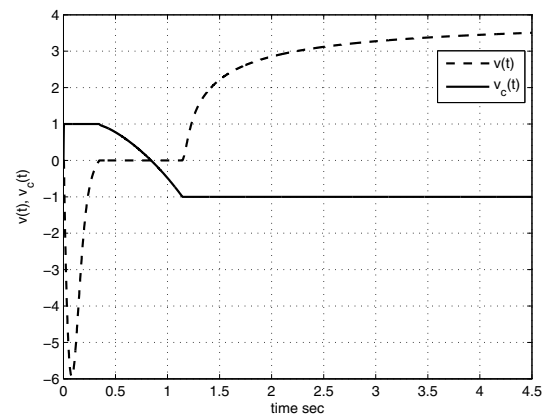

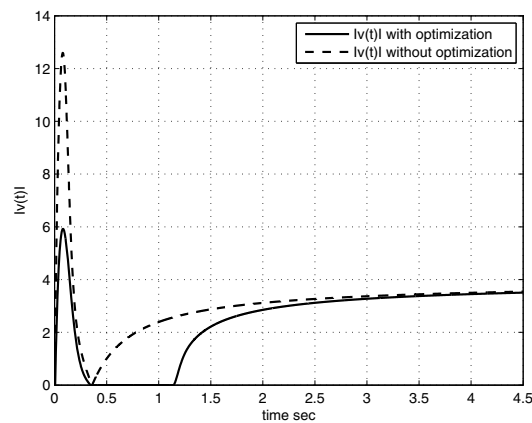Fig. 11. FOTMRAC control signal and the goal correction control signal.

Fig. 9. Absolute control signals for case of fractional order adaptive control with and without optimization.
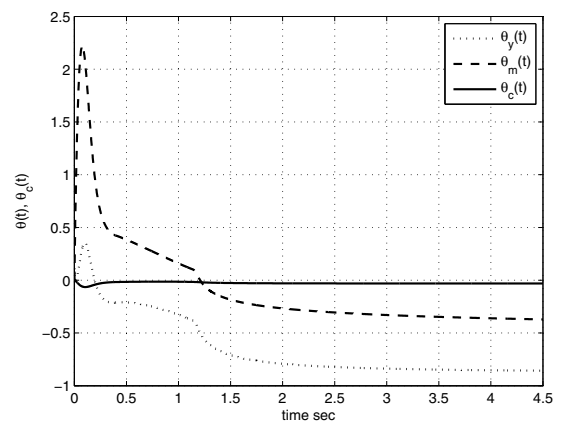

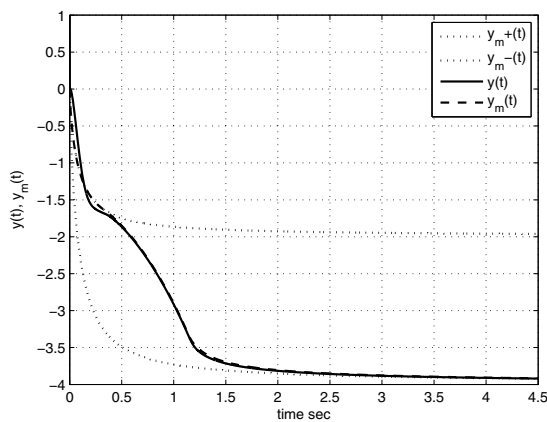Fig. 12. Adaptive gain vectors.

Fig. 10. Controlled system output and the reference trajectory within the performance tube.
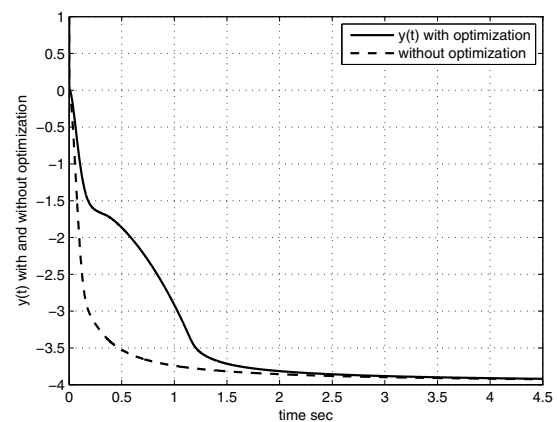

Fig. 13. System outputs for the case of fractional order adaptive control with and without optimization.
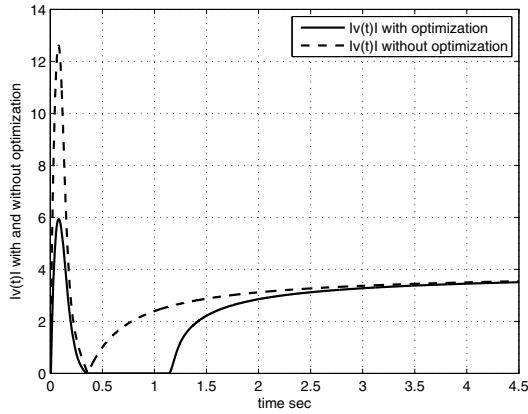
Fig. 14. Absolute control signals for the case of fractional order adaptive control with and without optimization.
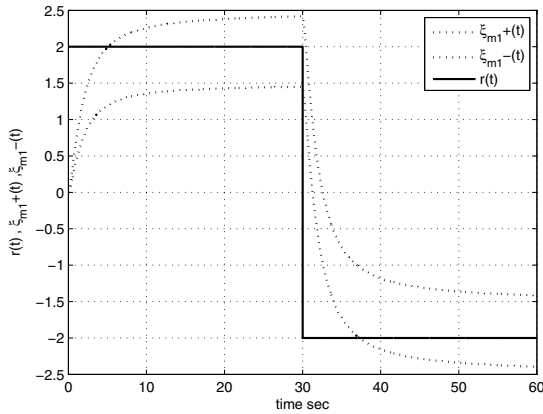

Fig. 15. Model reference tube and the reference input signal for Example 2.

Table 1. Comparison of control cost evaluations for Example 1.

Cost	FOTMRAC strategy (with optimization)	FOMRAC strategy (without optimization)
$J = \sum_k v^2(k)$	35171	54100

To evaluate the quadratic criterion J given in (16), we compare for the two cases (with and without optimization) the control cost function given in Table 1.

We observe that the control cost of the proposed method is much lower (about 35%) than in the case of the standard model reference adaptive control.

5.2. Example 2. Let us now consider the unstable SIMO system defined by

$$\left[\begin{array}{c} D^\alpha \xi_1(t) \\ D^\alpha \xi_2(t) \end{array} \right] = A \left[\begin{array}{c} \xi_1(t) \\ \xi_2(t) \end{array} \right] + B\, v(t), \quad (50)$$

where

$$A = \left[\begin{array}{cc} 0 & 1 \\ 1 & 2 \end{array} \right], \quad B = \left[\begin{array}{c} 0 \\ 0.5 \end{array} \right],$$

and the order $\alpha = 0.8$.

Our objective is to design a state feedback control law $v(t)$ according to our proposed FOTMRAC strategy, in order to control the first state variable $\xi_1(t)$. For that purpose we choose the following stable reference model:

$$\left[\begin{array}{c} D^\alpha \xi_m 1(t) \\ D^\alpha \xi_m 2(t) \end{array} \right] = A_m \left[\begin{array}{c} \xi_m 1(t) \\ \xi_m 2(t) \end{array} \right] + B_m\, v_m(t), \quad (51)$$

where

$$A_m = \left[\begin{array}{cc} 0 & 1 \\ -1 & -2 \end{array} \right], \quad B_m = \left[\begin{array}{c} 0 \\ 1 \end{array} \right],$$

and

$$v_m(t) = r(t) + v_c(t).$$

The controller parameter values are chosen as follows:

$$P = \left[\begin{array}{cc} 3/2 & 1/2 \\ 1/2 & 1/2 \end{array} \right], \quad \Lambda_1 = \left[\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{array} \right],$$

$$\Lambda_p = \left[\begin{array}{ccc} 0.9 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.9 \end{array} \right], \quad \Lambda_c = 2. \quad (52)$$

The reference input signal $r(t)$ is represented in Fig. 15. The variables $\xi_{m1}^+(t)$ and $\xi_{m1}^-(t)$ in Fig. 15 represent the first state variable of the reference model for the input $v(t) = r(t) + v_c^+$ and $v(t) = r(t) + v_c^-$, respectively, where $v_c^- = -0.5$, $v_c^+ = +0.5$. These variables define the tube of the admissible reference trajectories.

Figure 16 shows the controlled output system, tracking perfectly the model reference trajectory within the reference tube. This reference trajectory is updated by the adaptive control law such that the control effort is minimized.

The control signal and the goal correction control signal are shown in Fig. 17, whereas the adaptive gain vectors are represented in Fig. 18.

In order to compare the performance of the proposed FOTMRAC strategy with the FOMRAC scheme (i.e., $v_c(t) = 0$ and $\theta_c = 0$ in (19) and (22)), we adjust the value of the reference model input in order to have the same steady state trajectory reaching. The controlled output system and the absolute value of the control signals for the
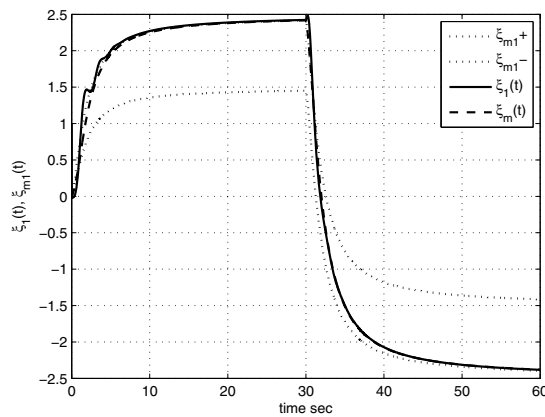
Fig. 16. Controlled output system and the reference trajectory within the performance tube.
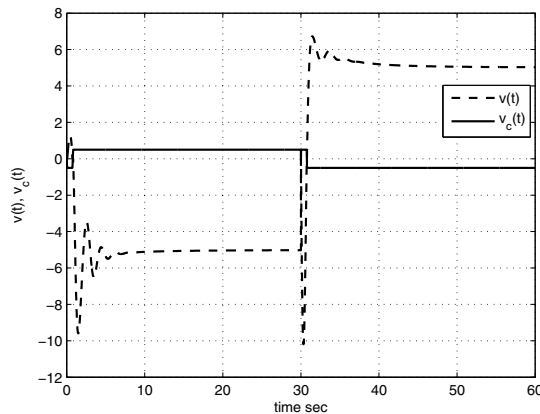

Fig. 17. FOTMRAC control signal and the goal correction control signal.

Table 2. Comparison of control cost evaluations for Example 2.

Cost	FOTMRAC strategy (with optimization)	FOMRAC strategy (without optimization)
$J = \sum_k v^2(k)$	162190	165500

scheme of adaptive control with and without optimization are respectively illustrated in Figs. 19 and 20.

For the same steady state response, we remark that the proposed FOTMRAC controller finds the reference trajectory within the performance tube so that the control effort is minimized.

From Fig. 20, it is obvious that the control cost with FOTMRAC strategy is lower than the control cost applying a simple FOMRAC control without optimization. We notice that the gain in control cost compared with the classical scheme is realized in the transitory phase, because in the steady state phase the curves of the proposed and the standard control converge to the same value. From the instant 0 [s] to 10 [s], the gain equals 5.75%. Notice that this improvement in control cost could be increased by augmenting the interval $[v_c^-, v_c^+]$. For example, if we take the interval $[-1.5, +1.5]$, the gain in command cost becomes 13.98%.

Also to evaluate the criterion J, we present a comparative evaluation of the control cost function in the two cases (with and without optimization) in Table 2.

In regard of the obtained simulation results, we can say that our proposed FOTMRAC strategy gives a better control performance since all the closed loop signals are bounded, the plant response tracks the reference trajectory within the admissible range (the performance tube) with a minimum control effort. Even if this performance quantification is not linearly related to the fractional order, we can always find a result improvement using a fractional order tube MRAC control.

6. Conclusion

In this article, a novel fractional order tube model reference adaptive control (FOTMRAC) has been developed for a class of fractional order linear systems. Based on the TMRAC control scheme proposed by Mirkin and Gutman (2013), the proposed adaptive control scheme generalizes the performance tube technique to the class of arbitrary order systems.

Using an adaptive state feedback configuration, this approach has two main objectives: the first one is to ensure the stability of the resulting fractional order closed loop system and the asymptotic tracking of the guidance trajectory within the performance tube. The second objective is the minimization of a control cost. The stability analysis has been performed in Theorem 2 using an extended version of the Lyapunov theorem to fractional order systems.

Two illustrative examples have been presented to demonstrate the effectiveness and the accuracy of the proposed method. Applying the FOTMRAC control scheme has given satisfactory performance for the two plants considered, namely:

- The asymptotic stability is guaranteed in both the cases despite the fact that the model parameters are unknown.

- The systems' outputs lay in the reference model tube domain.

- The control energy cost is minimized.

- The performance indexes are much better than those of the classical TMRAC control scheme.

Future research will concern the extension of this adaptive control approach to fractional order nonlinear and MIMO systems.

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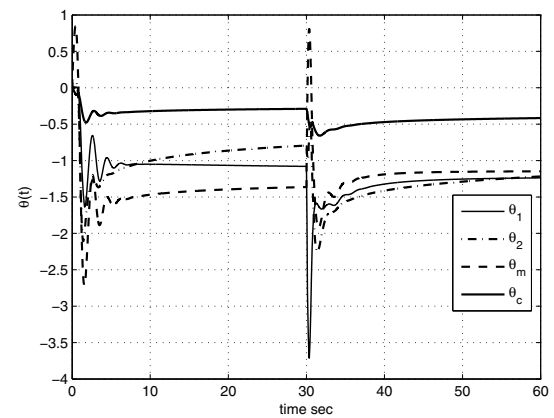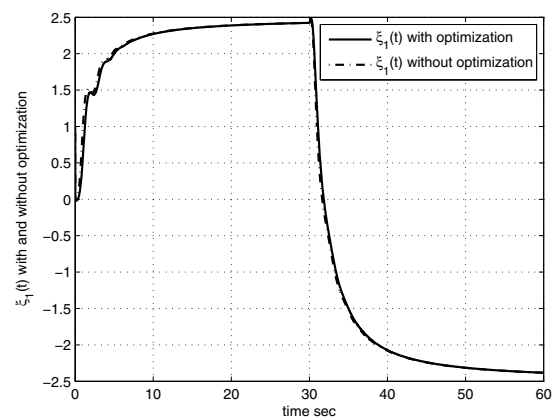
Fig. 18. Adaptive gain vectors.

Fig. 19. System output for the cases of fractional order control with and without optimization.
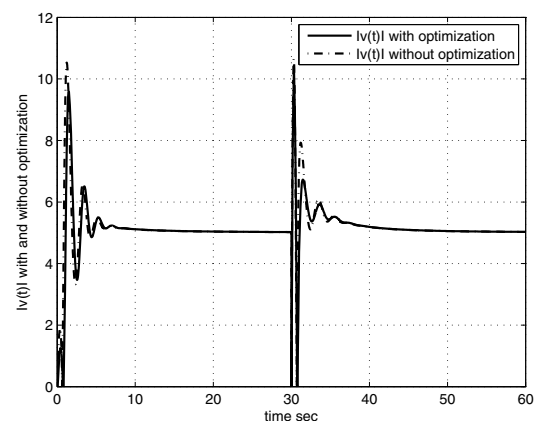

Fig. 20. Absolute control signals for the cases of fractional order control with and without optimization.

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