

DECENTRALIZED DESIGN OF INTERCONNECTED \mathcal{H}_∞ FEEDBACK CONTROL SYSTEMS WITH QUANTIZED SIGNALS

GUISHENG ZHAI *, NING CHEN **, WEIHUA GUI **

* Department of Mathematical Sciences
Shibaura Institute of Technology, Saitama 337-8570, Japan
e-mail: gs.zhai@gmail.com

** School of Information and Engineering
Central South University, Changsha, Hunan 410083, China
e-mail: ningchen@csu.edu.cn

In this paper, we consider the design of interconnected \mathcal{H}_∞ feedback control systems with quantized signals. We assume that a decentralized dynamic output feedback has been designed for an interconnected continuous-time LTI system so that the closed-loop system is stable and a desired \mathcal{H}_∞ disturbance attenuation level is achieved, and that the subsystem measurement outputs are quantized before they are passed to the local controllers. We propose a local-output-dependent strategy for updating the parameters of the quantizers, so that the overall closed-loop system is asymptotically stable and achieves the same \mathcal{H}_∞ disturbance attenuation level. Both the pre-designed controllers and the parameters of the quantizers are constructed in a decentralized manner, depending on local measurement outputs.

Keywords: interconnected systems, decentralized \mathcal{H}_∞ control, dynamic output feedback, quantizer, quantization, matrix inequality, LMI.

1. Introduction

Since quantized signals always exist in any computer based control systems (Bushnell, 2001; Ishii and Francis, 2002; Tatikonda and Mitter, 2004), many researchers have begun to study the analysis and design problems for control systems involving various quantization methods in the last two decades. Delchamps (1990) addressed the problem of stabilizing an unstable linear system by means of quantized state feedback, i.e., state feedback where the measurements of the system state are quantized. The quantizer in the work of Delchamps (1990) takes values in a countable set. Brockett and Liberzon (2000) defined a quantizer taking values in a finite set and considered quantized feedback stabilization for linear systems. While the approach of Brockett and Liberzon (2000) relies on the possibility of making discrete on-line adjustments of quantizer parameters, Liberzon (2003) extended it to more general nonlinear systems with general types of quantizers involving the states, the measurement outputs, and the control inputs of the system.

Later, Zhai *et al.* (2004) considered the stabilization problem for a discrete-time LTI system via state feedback

involving both quantized states and control inputs. In that context, a hybrid quantized state feedback strategy was proposed, where the values of the quantizer parameters are updated at discrete instants of time. Further, the authors extended the results to \mathcal{H}_∞ feedback control systems (Zhai *et al.*, 2005), dealing with both state feedback and dynamic output feedback. The key point is to propose a state-dependent (or an output-dependent) strategy for updating the quantizer's parameters, so that the system is asymptotically stable and achieves the same \mathcal{H}_∞ disturbance attenuation level.

It was also noted by Zhai *et al.* (2005) that the control strategies of updating the quantizer's parameter are dependent on time in the existing works (Brockett and Liberzon, 2000; Liberzon, 2003; Zhai *et al.*, 2004), and such control strategies cannot be applied for the case of \mathcal{H}_∞ control since the value of the disturbance inputs is not available and thus one cannot drive the state into an invariant region, as done by Liberzon (2003) and Zhai *et al.* (2004). In contrast, the control strategy of Zhai *et al.* (2005) is state- or output-dependent, and as such is usually regarded to have more robustness.

Zhai et al. (2010) extended their previous discussion (Zhai et al., 2005) to decentralized \mathcal{H}_∞ static output feedback of interconnected systems. For illustration, Fig. 1 gives an example of interconnected output feedback systems composed of two subsystems with quantized measurement outputs. It is well known that interconnected systems appear in many real applications such as large-scale power transfer systems, traffic networks, etc., and the typical control strategy is decentralized control. The existence of interconnections among subsystems leads to difficulties in decentralized control. Moreover, due to physical communication constraints, the signals among subsystems are generally quantized, which makes the entire control system design much more challenging.

For this purpose, the situation assumed by Zhai et al. (2010) is that, for each subsystem, a local static output feedback has been designed such that the overall system is stable and a certain \mathcal{H}_∞ disturbance attenuation level (in the sense of l_2 gain from disturbance input to controlled output in the overall closed-loop system) is achieved. However, although the data of the local controller go to the subsystem without loss, the subsystems' local outputs are quantized before they are passed to the controller. Due to the quantization effects, the desired system stability and \mathcal{H}_∞ disturbance attenuation level cannot be guaranteed.

The quantizers are supposed to take a generalized form where there is a zoom parameter that can be adjusted. Then, Zhai et al. (2010) proposed to update the quantizer parameters in a decentralized on-line manner, i.e., to change the parameter's value depending on each subsystem's measurement output, so that the overall closed-loop system is asymptotically stable and the same \mathcal{H}_∞ disturbance attenuation level is achieved. Recently, the approach of Zhai et al. (2010) has been further extended to dynamic output feedback by Chen et al. (2011a), who also dealt with various uncertainties and other quantized signals. However, the local dynamic output feedbacks of Chen et al. (2011a) did not have a general form, and the strategy of updating the quantizers' parameters is expressed as an equation, which is not desirable in real applications.

This paper aims to complement and improve the discussion by Zhai et al. (2010) and Chen et al. (2011a). The interconnected system under consideration is the same as in the works of Zhai et al. (2010) and Chen et al. (2011a), and its outline is depicted in Fig. 1, but the number of subsystems does not have to be two. Note that the local controller K_i in Fig. 1 is a dynamic output feedback now. We assume that a decentralized dynamic output feedback, composed of K_i 's, has been designed for the interconnected system so that the closed-loop system is stable and a desired \mathcal{H}_∞ disturbance attenuation level is achieved, and that the measurement outputs of the subsystems are quantized before they are passed to the local controllers. We then

propose a local-output-dependent strategy for updating the parameters of the quantizers, so that the overall closed-loop system is asymptotically stable and achieves the same \mathcal{H}_∞ disturbance attenuation level.

In contrast to the approach by Chen et al. (2011a), the quantizer updating strategy is expressed by an inequality, which has more robustness to small external disturbances and rounding errors. As desired, both the pre-designed controllers and the quantizer parameters are constructed in a decentralized manner, depending on the local measurement output.

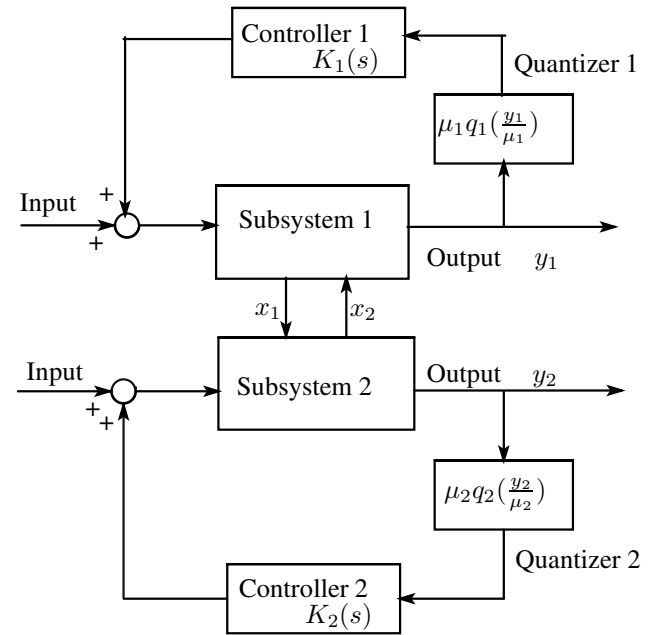


Fig. 1. Interconnected feedback systems with quantized measurement outputs.

The rest of this paper is organized as follows. Section 2 presents the definition and the property of a generalized quantizer. Section 3 describes the interconnected system, gives some comments on how to pre-design the controller in the case of decentralized dynamic output feedback, and formulates the control problem. Section 4 proposes a local-output-dependent strategy for updating the quantizer parameters, so that the overall closed-loop system is asymptotically stable and achieves the same \mathcal{H}_∞ disturbance attenuation level. Section 5 provides a simulation example, and Section 6 concludes the paper.

2. Preliminaries

2.1. Quantizer description. We first give the definition of a quantizer with general form as introduced by Liberzon (2003). Let $z \in \mathbb{R}^l$ be the variable being quantized. A *quantizer* is defined as a piecewise constant function $q : \mathbb{R}^l \rightarrow \mathcal{D}$, where \mathcal{D} is a finite subset of \mathbb{R}^l .

This leads to a partition of \mathbb{R}^l into a finite number of quantization regions of the form $\{z \in \mathbb{R}^l : q(z) = i\}$, $i \in \mathcal{D}$. These quantization regions are not assumed to have any particular shapes. We assume that there exist positive real numbers M and Δ such that the following conditions (properties) hold:

- P1. If $|z| \leq M$ then $|q(z) - z| \leq \Delta$,
 P2. If $|z| > M$ then $|q(z)| > M - \Delta$. (1)

Condition P1 gives a bound on the quantization error when the quantizer does not saturate. Condition P2 provides a way to detect the possibility of saturation. We will refer to M and Δ as the *range of q* and the *quantization error*, respectively. We also assume that $q(x) = 0$ for x in some neighborhood of the origin. The example of satisfying the above requirements is given by the quantizer with rectangular quantization regions by Brockett and Liberzon (2000) as well as Liberzon (2000).

In the control strategy to be developed below, we will use quantized measurements of the form

$$q_\mu(z) \triangleq \mu q\left(\frac{z}{\mu}\right), \quad (2)$$

where $\mu > 0$ is the parameter. The extreme case of $\mu = 0$ is regarded as setting the output of the quantizer as zero. The range of this quantizer is $M\mu$ and the quantization error is $\Delta\mu$. We can view μ as a “zoom” variable: increasing μ corresponds to zooming out and essentially obtaining a new quantizer with a larger range and a larger quantization error, while decreasing μ corresponds to zooming in and obtaining a quantizer with a smaller range but also a smaller quantization error. We will update μ later depending on the system local measurement outputs. In this sense, it can be considered as another state of the resultant closed-loop system.

2.2. Notation. Throughout this paper, the superscript “ \top ” represents the transpose of a matrix, while the superscript “ -1 ” represents the inverse of a matrix. $W \succ 0$ (resp. $W \prec 0$) means W is symmetric and positive (resp. negative) definite, and $W_1 \succ W_2$ means $W_1 - W_2 \succ 0$. A matrix A is Hurwitz if all its eigenvalues have negative real parts.

Denote by $|\cdot|$ the standard Euclidean norm in the n -dimensional vector space \mathbb{R}^n , and denote by $\|\cdot\|$ the corresponding induced matrix norm in $\mathbb{R}^{n \times n}$. $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote the smallest and the largest eigenvalue of a symmetric matrix, respectively. Then, for any positive definite matrix W , the inequality $\lambda_m(W)|x|^2 \leq x^\top W x \leq \lambda_M(W)|x|^2$ holds for any vector x .

2.3. Bounded real lemma. In the end of this section, we state a preliminary lemma for the benefit of our

discussion later, which is the well-known bounded real lemma (Iwasaki *et al.*, 1998) concerning the \mathcal{H}_∞ analysis of continuous-time linear time-invariant systems.

Lemma 1. *The following three statements are equivalent:*

1. A is Hurwitz and $\|D + C(sI - A)^{-1}B\|_\infty < \gamma$.
2. There exists a positive definite matrix P satisfying

$$\begin{bmatrix} A^\top P + PA & PB & C^\top \\ B^\top P & -\gamma^2 I & D^\top \\ C & D & -I \end{bmatrix} \prec 0. \quad (3)$$

3. There exists a positive definite matrix P satisfying

$$\begin{bmatrix} A^\top P + PA + C^\top C & PB + C^\top D \\ B^\top P + D^\top C & -\gamma^2 I + D^\top D \end{bmatrix} \prec 0. \quad (4)$$

3. System description and problem formulation

3.1. Interconnected systems with decentralized control. The interconnected system we deal with is an input-output decentralized systems described by

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_{1i} w_i + B_{2i} u_i + \sum_{j=1, j \neq i}^N A_{ij} x_j, \\ z_i &= C_{1i} x_i + D_i w_i, \\ y_i &= C_{2i} x_i, \quad i = 1, 2, \dots, N, \end{aligned} \quad (5)$$

where $x_i \in \mathbb{R}^{n_i}$ is the state, $u_i \in \mathbb{R}^{m_i}$ is the control input, $w_i \in \mathbb{R}^{r_i}$ is the disturbance input, $z_i \in \mathbb{R}^{p_i}$ is the controlled output, $y_i \in \mathbb{R}^{q_i}$ is the measurement output of the i -th subsystem. N is the number of subsystems. The matrices $A_i, A_{ij}, B_{1i}, B_{2i}, C_{1i}, C_{2i}$ and D_i ($i, j = 1, 2, \dots, N$) are constant and of appropriate dimensions. It is obvious from (5) that the term $\sum_{j=1, j \neq i}^N A_{ij} x_j$ denotes the interconnection among the subsystems.

Suppose that, for the system (5), we have designed a decentralized dynamic output feedback controller which is composed of N local output feedbacks

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{A}_i \hat{x}_i + \hat{B}_i y_i \\ u_i &= \hat{C}_i \hat{x}_i + \hat{D}_i y_i, \quad i = 1, 2, \dots, N, \end{aligned} \quad (6)$$

where $\hat{x}_i \in \mathbb{R}^{\hat{n}_i}$ is the state of the local controller and $\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i, i = 1, 2, \dots, N$, are coefficient matrices.

The overall closed-loop system obtained by applying the decentralized controller (6) to the interconnected

system (5) is described as

$$\begin{aligned} \dot{\tilde{x}}_i &= \begin{bmatrix} A_i + B_{2i}\hat{D}_iC_{2i} & B_{2i}\hat{C}_i \\ \hat{B}_iC_{2i} & \hat{A}_i \end{bmatrix} \tilde{x}_i \\ &+ \begin{bmatrix} B_{1i} \\ 0 \end{bmatrix} w_i + \sum_{j=1, j \neq i}^N \begin{bmatrix} A_{ij} & 0 \\ 0 & 0 \end{bmatrix} \tilde{x}_j \quad (7) \\ z_i &= [C_{1i} \ 0] \tilde{x}_i + D_i w_i, \quad i = 1, 2, \dots, N, \end{aligned}$$

where $\tilde{x}_i = [x_i^\top \ \hat{x}_i^\top]^\top$ is the state of the closed-loop subsystem. We introduce the compact notation

$$\begin{aligned} \tilde{A}_i &= \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_{ij} = \begin{bmatrix} A_{ij} & 0 \\ 0 & 0 \end{bmatrix}, \\ \tilde{B}_{1i} &= \begin{bmatrix} B_{1i} \\ 0 \end{bmatrix}, \quad \tilde{B}_{2i} = \begin{bmatrix} B_{2i} & 0 \\ 0 & I \end{bmatrix}, \quad (8) \\ \tilde{C}_{1i} &= [C_{1i} \ 0], \quad \tilde{C}_{2i} = \begin{bmatrix} C_{2i} & 0 \\ 0 & I \end{bmatrix}, \end{aligned}$$

and write the controller parameters \hat{A}_i , \hat{B}_i , \hat{C}_i , and \hat{D}_i into a single matrix as

$$K_i = \begin{bmatrix} \hat{D}_i & \hat{C}_i \\ \hat{B}_i & \hat{A}_i \end{bmatrix}. \quad (9)$$

Then, the closed-loop system (7) is rewritten as

$$\begin{aligned} \dot{\tilde{x}}_i &= (\tilde{A}_i + \tilde{B}_{2i}K_i\tilde{C}_{2i})\tilde{x}_i + \tilde{B}_{1i}w_i + \sum_{j=1, j \neq i}^N \tilde{A}_{ij}\tilde{x}_j, \quad (10) \\ z_i &= \tilde{C}_{1i}\tilde{x}_i + D_i w_i, \quad i = 1, 2, \dots, N \end{aligned}$$

and, equivalently, in a compact form as

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}_{1d}w, \\ z &= \tilde{C}_{1d}\tilde{x} + D_d w, \end{aligned} \quad (11)$$

where

$$\begin{aligned} \tilde{x} &= [\tilde{x}_1^\top \ \tilde{x}_2^\top \ \dots \ \tilde{x}_N^\top]^\top, \\ w &= [w_1^\top \ w_2^\top \ \dots \ w_N^\top]^\top, \\ z &= [z_1^\top \ z_2^\top \ \dots \ z_N^\top]^\top, \\ \tilde{A} &= \tilde{A}_d + \tilde{A}_c + \tilde{B}_{2d}K_d\tilde{C}_{2d}, \\ \tilde{A}_d &= \text{diag}\{\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_N\}, \\ \tilde{A}_c &= [\tilde{A}_{ij}]_{N \times N}, \quad \tilde{A}_{ii} \triangleq 0, \quad (12) \\ \tilde{B}_{1d} &= \text{diag}\{\tilde{B}_{11}, \tilde{B}_{12}, \dots, \tilde{B}_{1N}\}, \\ \tilde{B}_{2d} &= \text{diag}\{\tilde{B}_{21}, \tilde{B}_{22}, \dots, \tilde{B}_{2N}\}, \\ \tilde{C}_{1d} &= \text{diag}\{\tilde{C}_{11}, \tilde{C}_{12}, \dots, \tilde{C}_{1N}\}, \\ \tilde{C}_{2d} &= \text{diag}\{\tilde{C}_{21}, \tilde{C}_{22}, \dots, \tilde{C}_{2N}\}, \\ D_d &= \text{diag}\{D_1, D_2, \dots, D_N\}, \\ K_d &= \text{diag}\{K_1, K_2, \dots, K_N\}. \end{aligned}$$

Notice that w and z are the disturbance input and the controlled output of the overall system, respectively, and the transfer function from w to z in the closed-loop system (11) is $T_{zw}(s) = D_d + \tilde{C}_{1d}(sI - \tilde{A})^{-1}\tilde{B}_{1d}$. Then, the hypothesis in this paper is that, without taking quantization into consideration, the decentralized controller (6) (or the feedback gain matrices K_i 's in (9) and thus K_d in (12)), has been designed so that \tilde{A} is Hurwitz stable, and the \mathcal{H}_∞ norm of the transfer function $T_{zw}(s)$ is less than a specified level γ . Therefore, according to Lemma 1, there exists a positive definite matrix \tilde{P} satisfying the Linear Matrix Inequality (LMI)

$$\begin{bmatrix} \tilde{A}^\top \tilde{P} + \tilde{P} \tilde{A} & \tilde{P} \tilde{B}_{1d} & \tilde{C}_{1d}^\top \\ \tilde{B}_{1d}^\top \tilde{P} & -\gamma^2 I & D_d^\top \\ \tilde{C}_{1d} & D_d & -I \end{bmatrix} \prec 0. \quad (13)$$

3.2. Controller design without involving quantization. Corresponding to the decentralized structure of the interconnected system, we assume that the positive definite matrix \tilde{P} takes a block-diagonal structure as

$$\tilde{P} = \text{diag}\{\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_N\}. \quad (14)$$

To say it in other words, since the feasibility of the LMI (13) is equivalent to solvability of the decentralized \mathcal{H}_∞ control problem for (5), we can set the structure of \tilde{P} as in (14) with $\tilde{P}_i \succ 0$ and then solve the matrix inequality (13) with respect to \tilde{P} and K_d , to obtain all the coefficient matrices of the controllers.

However, since (13) is a Bilinear Matrix Inequality (BMI) with respect to \tilde{P} and K_d , and there is structure limitation on the matrix variables, there is no globally effective method for solving it in general. For integrity, we here briefly review a practical method of solving the matrix inequality (13) with respect to \tilde{P}_i 's and K_i 's, which is based on the approach using the idea of the homotopy method (Ikeda et al., 1996; Zhai et al., 2001). Rewrite (13) as

$$F_0(K_d, \tilde{P}) + F_1(\tilde{P}) \prec 0, \quad (15)$$

where

$$\begin{aligned} F_0(K_d, \tilde{P}) &= \begin{bmatrix} \tilde{A}_d^\top \tilde{P} + \tilde{P} \tilde{A}_d & \tilde{P} \tilde{B}_{1d} & \tilde{C}_{1d}^\top \\ \tilde{B}_{1d}^\top \tilde{P} & -\gamma^2 I & D_d^\top \\ \tilde{C}_{1d} & D_d & -I \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{P} \tilde{B}_{2d} \\ 0 \\ 0 \end{bmatrix} K_d [\tilde{C}_{2d} \ 0 \ 0] \end{aligned}$$

$$+ \left\{ \begin{bmatrix} \tilde{P}\tilde{B}_{2d} \\ 0 \\ 0 \end{bmatrix} K_d \begin{bmatrix} \tilde{C}_{2d} & 0 & 0 \end{bmatrix} \right\}^\top, \quad (16)$$

$$F_1(\tilde{P}) = \begin{bmatrix} \tilde{A}_c^\top \tilde{P} + \tilde{P}\tilde{A}_c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (17)$$

and define the homotopy

$$H(K_d, \tilde{P}, \lambda) = F_0(K_d, \tilde{P}) + \lambda F_1(\tilde{P}), \quad (18)$$

where λ takes values in the interval $[0, 1]$. By this introduction, the problem of finding a solution to (13) can be embedded in the parametrized family of problems $H(K_d, \tilde{P}, \lambda) \prec 0$ with λ changing from 0 to 1.

The algorithm starts computing the solution to $H(K_d, \tilde{P}, \lambda) \prec 0$ with $\lambda = 0$, i.e., $F_0(K_d, \tilde{P}) \prec 0$, which is exactly the \mathcal{H}_∞ control problem for each separated subsystem, and thus can be easily solved. Then, using the idea of the homotopy method, we increase λ gradually from 0 to 1 to solve $H(K_d, \tilde{P}, \lambda) \prec 0$ by fixing K_d or \tilde{P} at each step, until λ reaches 1, which means we have found a solution to (15).

For the detailed algorithm and other possible extensions (e.g., local controllers' order reduction), refer to the works of Ikeda *et al.* (1996) and Zhai *et al.* (2001).

Another practical method of solving (13) is proposed by setting a specified structure of the matrix variable \tilde{P} and making a similarity transformation for \tilde{B}_{2i} or \tilde{C}_{2i} , so that (13) is reduced to an LMI. Refer to the work of Murao *et al.* (2002) for a detailed discussion.

3.3. Problem formulation. Now, we are ready to formulate our control problem. The above mentioned decentralized controller design has been done in the case where there is no quantization. Here, as depicted in Fig. 1, we deal with the case where only quantized local output is available. For this reason, we modify the decentralized dynamic output feedback (6) by replacing y_i with its quantized value $\mu_i q_i(y_i/\mu_i)$ as

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{A}_i \hat{x}_i + \hat{B}_i y_i + \hat{B}_i F_i(\mu_i, y_i), \\ u_i &= \hat{C}_i \hat{x}_i + \hat{D}_i y_i + \hat{D}_i F_i(\mu_i, y_i), \end{aligned} \quad (19)$$

where

$$F_i(\mu_i, y_i) = \mu_i \left(q_i \left(\frac{y_i}{\mu_i} \right) - \frac{y_i}{\mu_i} \right).$$

For fixed positive scalars μ_i , the closed-loop system composed of the system (5) and the modified output

feedback (19) is given by

$$\begin{aligned} \dot{\hat{x}}_i &= (\tilde{A}_i + \tilde{B}_{2i} K_i \tilde{C}_{2i}) \hat{x}_i + \tilde{B}_{1i} w_i \\ &\quad + \sum_{j=1, j \neq i}^N \tilde{A}_{ij} \hat{x}_j + \tilde{B}_{2i} K_i \tilde{F}_i(\mu_i, y_i) \\ z_i &= \tilde{C}_{1i} \hat{x}_i + D_i w_i, \quad i = 1, 2, \dots, N \end{aligned} \quad (20)$$

and, equivalently, in a compact form as

$$\begin{aligned} \dot{\hat{x}} &= \tilde{A} \hat{x} + \tilde{B}_{1d} w + \tilde{B}_{2d} K_d \tilde{F}(\mu, y), \\ z &= \tilde{C}_{1d} \hat{x} + D_d w, \end{aligned} \quad (21)$$

where

$$\begin{aligned} \tilde{F}(\mu, y) &= \left[\tilde{F}_1^\top(\mu_1, y_1) \cdots \tilde{F}_N^\top(\mu_N, y_N) \right]^\top, \\ \tilde{F}_i(\mu_i, y_i) &= \begin{bmatrix} F_i(\mu_i, y_i) \\ 0 \end{bmatrix}. \end{aligned} \quad (22)$$

Now, the control problem is very natural. Due to the existence of the quantization error, the stability of the closed-loop system and the desired \mathcal{H}_∞ disturbance attenuation level γ are not guaranteed. Here, as defined in many references, the \mathcal{H}_∞ disturbance attenuation level γ means that the l_2 gain of the controlled output z to the disturbance input w is less than γ in the closed-loop system.

With the above preparation, we formulate our control problem as follows.

Decentralized \mathcal{H}_∞ control via quantized output feedback. Design a decentralized control strategy which adjusts the quantizer parameters μ_i , depending on the local measurement outputs y_i , so that the overall closed-loop system (21) is asymptotically stable and the \mathcal{H}_∞ disturbance attenuation level γ is achieved.

The above control specification of requiring the quantizer parameters μ_i adjusted by local measurement outputs y_i (i.e., a local-output-dependent strategy) is desired in the framework of any decentralized control systems. It is obvious from Fig. 1 and the system description that one cannot obtain all the outputs so as to adjust the local quantization parameters.

4. Decentralized quantizer design

Since (13) is a strict matrix inequality in the sense of negative definite, we can always find a block diagonal positive definite matrix $R = \text{diag}\{R_1, R_2, \dots, R_N\}$ with $R_i \succ 0$, $i = 1, 2, \dots, N$, such that

$$\begin{bmatrix} \tilde{A}^\top \tilde{P} + \tilde{P}\tilde{A} + R & \tilde{P}\tilde{B}_{1d} & \tilde{C}_{1d}^\top \\ \tilde{B}_{1d}^\top \tilde{P} & -\gamma^2 I & D_d^\top \\ \tilde{C}_{1d} & D_d & -I \end{bmatrix} \prec 0, \quad (23)$$

which is equivalent to

$$\begin{bmatrix} \left(\begin{array}{c} \tilde{A}^\top \tilde{P} + \tilde{P} \tilde{A} + R \\ + \tilde{C}_{1d}^\top \tilde{C}_{1d} \end{array} \right) & \tilde{P} \tilde{B}_{1d} + \tilde{C}_{1d}^\top D_d \\ \tilde{B}_{1d}^\top \tilde{P} + D_d^\top \tilde{C}_{1d} & -\gamma^2 I + D_d^\top D_d \end{bmatrix} \prec 0. \quad (24)$$

We are in a position to state and prove the main result in this paper.

Theorem 1. Assume that for each local quantizer, M_i is chosen large enough compared to Δ_i so that

$$M_i > 2\Delta_i \frac{\|\tilde{P}_i \tilde{B}_{2i} K_i\| \cdot \|C_{2i}\|}{\lambda_m(R_i)}, \quad i = 1, 2, \dots, N. \quad (25)$$

Then, there exists a decentralized control strategy for updating μ_i , which is dependent on the local measurement output y_i , such that the closed-loop system (21) is asymptotically stable and the \mathcal{H}_∞ disturbance attenuation level γ is achieved.

Proof. Since

$$\frac{y_i}{\mu_i} = \frac{C_{2i} x_i}{\mu_i}, \quad i = 1, 2, \dots, N$$

is quantized before being passed to the controller, we obtain by using the properties of general quantizers in (1) that, whenever $|y_i| \leq M_i \mu_i$, the inequality

$$\left| \frac{y_i}{\mu_i} - q\left(\frac{y_i}{\mu_i}\right) \right| \leq \Delta_i \iff |F_i(\mu_i, y_i)| \leq \mu_i \Delta_i \quad (26)$$

is true. We consider the Lyapunov function candidate

$$V(\tilde{x}) = \tilde{x}^\top \tilde{P} \tilde{x} \quad (27)$$

for the closed-loop system (21). By using the matrix inequality (24), we obtain that, when $|y_i| \leq M_i \mu_i$, the derivative of $V(x)$ along the trajectories of (21) satisfies

$$\begin{aligned} \dot{V} &= \left(\tilde{A} \tilde{x} + \tilde{B}_{1d} w + \tilde{B}_{2d} K_d \tilde{F}(\mu, y) \right)^\top \tilde{P} \tilde{x} \\ &\quad + \tilde{x}^\top \tilde{P} \left(\tilde{A} \tilde{x} + \tilde{B}_{1d} w + \tilde{B}_{2d} K_d \tilde{F}(\mu, y) \right) \\ &= \begin{bmatrix} \tilde{x} \\ w \end{bmatrix}^\top \begin{bmatrix} \tilde{A}^\top \tilde{P} + \tilde{P} \tilde{A} & \tilde{P} \tilde{B}_{1d} \\ \tilde{B}_{1d}^\top \tilde{P} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ w \end{bmatrix} \\ &\quad + \tilde{F}^\top(\mu, y) K_d^\top \tilde{B}_{2d}^\top \tilde{P} \tilde{x} + \tilde{x}^\top \tilde{P} \tilde{B}_{2d} K_d \tilde{F}(\mu, y) \end{aligned}$$

$$\begin{aligned} &\leq \begin{bmatrix} \tilde{x} \\ w \end{bmatrix}^\top \begin{bmatrix} -R - \tilde{C}_{1d}^\top \tilde{C}_{1d} & -\tilde{C}_{1d}^\top D_d \\ -D_d^\top \tilde{C}_{1d} & \gamma^2 I - D_d^\top D_d \end{bmatrix} \\ &\quad \times \begin{bmatrix} \tilde{x} \\ w \end{bmatrix} + 2\tilde{x}^\top \tilde{P} \tilde{B}_{2d} K_d \tilde{F}(\mu, y) \\ &= -z^\top z + \gamma^2 w^\top w \\ &\quad - \sum_{i=1}^N \left(\tilde{x}_i^\top R_i \tilde{x}_i - 2\tilde{x}_i^\top \tilde{P}_i \tilde{B}_{2i} K_i \tilde{F}_i(\mu_i, y_i) \right) \\ &\leq -z^\top z + \gamma^2 w^\top w - \sum_{i=1}^N \lambda_m(R_i) |\tilde{x}_i| \\ &\quad \times \left(|\tilde{x}_i| - 2\mu_i \Delta_i \frac{\|\tilde{P}_i \tilde{B}_{2i} K_i\|}{\lambda_m(R_i)} \right). \quad (28) \end{aligned}$$

Since

$$\begin{aligned} |\tilde{x}_i| &= \left| \begin{bmatrix} x_i \\ \hat{x}_i \end{bmatrix} \right| \geq |x_i|, \\ |y_i| &= |C_{2i} x_i| \leq \|C_{2i}\| \cdot |x_i|, \end{aligned}$$

we obtain

$$|\tilde{x}_i| \geq \frac{|y_i|}{\|C_{2i}\|}.$$

Using this fact in the final inequality of (28) leads to

$$\begin{aligned} \dot{V} &\leq -z^\top z + \gamma^2 w^\top w \\ &\quad - \sum_{i=1}^N \lambda_m(R_i) |\tilde{x}_i| \left(\frac{|y_i|}{\|C_{2i}\|} - 2\mu_i \Delta_i \frac{\|\tilde{P}_i \tilde{B}_{2i} K_i\|}{\lambda_m(R_i)} \right) \\ &= -z^\top z + \gamma^2 w^\top w - \sum_{i=1}^N \lambda_m(R_i) \frac{|\tilde{x}_i|}{\|C_{2i}\|} \\ &\quad \times \left(|y_i| - 2\mu_i \Delta_i \frac{\|\tilde{P}_i \tilde{B}_{2i} K_i\| \cdot \|C_{2i}\|}{\lambda_m(R_i)} \right). \quad (29) \end{aligned}$$

According to (25), we can always find a scalar $\epsilon \in (0, 1)$ such that

$$M_i > 2\Delta_i \frac{\|\tilde{P}_i \tilde{B}_{2i} K_i\| \cdot \|C_{2i}\|}{\lambda_m(R_i)} \frac{1}{1 - \epsilon}, \quad (30)$$

which is equivalent to

$$\frac{1}{1 - \epsilon} 2\mu_i \Delta_i \frac{\|\tilde{P}_i \tilde{B}_{2i} K_i\| \cdot \|C_{2i}\|}{\lambda_m(R_i)} < M_i \mu_i. \quad (31)$$

Therefore, for any nonzero y_i , we can choose a positive scalar μ_i such that

$$\frac{1}{1 - \epsilon} 2\mu_i \Delta_i \frac{\|\tilde{P}_i \tilde{B}_{2i} K_i\| \cdot \|C_{2i}\|}{\lambda_m(R_i)} \leq |y_i| \leq M_i \mu_i. \quad (32)$$

This is also true in the case of $y_i = 0$, where we set $\mu = 0$ as an extreme case and consider the output of the quantizer as zero.

In other words, since we can always choose μ_i 's so that (32) is satisfied, (29) holds since $|y_i| \leq M_i \mu_i$. It is further obtained from (32) and (29) that

$$\dot{V} \leq -z^\top z + \gamma^2 w^\top w - \epsilon \sum_{i=1}^N \lambda_m(R_i) \frac{|\tilde{x}_i| |y_i|}{\|C_{2i}\|}. \quad (33)$$

First, by setting $w = 0$ in (33), we see clearly that the system is asymptotically stable.

Next, we integrate both the sides of (33) from the initial time t_0 to any time instant $t > t_0$ to obtain

$$V(t) - V(t_0) \leq \int_{t_0}^t (-z^\top(\tau)z(\tau) + \gamma^2 w^\top(\tau)w(\tau)) d\tau. \quad (34)$$

Using $V(t) \geq 0$, we obtain

$$\int_{t_0}^t z^\top(\tau)z(\tau) d\tau \leq V(t_0) + \gamma^2 \int_{t_0}^t w^\top(\tau)w(\tau) d\tau, \quad (35)$$

which implies that the \mathcal{H}_∞ disturbance attenuation level γ is achieved. This completes the proof. ■

Decentralized quantizer design. It is observed from the proof of Theorem 1 that *the decentralized control strategy of updating the quantizers' parameters is to choose μ_i 's satisfying*

$$\frac{1}{M_i} |y_i| \leq \mu_i < \frac{1}{2\Delta_i} \frac{\lambda_m(R_i)}{\|\tilde{P}_i \tilde{B}_{2i} K_i\| \cdot \|C_{2i}\|} |y_i| \quad (36)$$

for any nonzero y_i , $i = 1, 2, \dots, N$. Since this inequality establishes an interval for choosing the value μ_i , it has robustness to small external disturbances and rounding errors.

Remark 1. It is seen that the condition (25) and the quantizer updating strategy (32) (or (36)) take almost the same form as that in the state feedback case (Zhai *et al.*, 2005) (when $C_{2i} = I$) and the static output feedback case (Zhai *et al.*, 2010). In fact, by using some routine calculation, it can be affirmed that (25) and (32) (or (36)) include the corresponding ones given by Zhai *et al.* (2005; 2010) as special cases. Thus, Theorem 1 is an extension to the discussion for decentralized and quantized state feedback and static output feedback.

Remark 2. In the existing references (e.g., Liberzon, 2003; Zhai *et al.*, 2004), the value of μ is updated in a time-controlled manner, i.e., when to change the value of μ is dependent only on time. This is not possible for the present situation since we do not know the value of $x(t)$, $w(t)$, and thus we cannot drive $x(t)$ into a specified invariant region, as done by Liberzon (2003)

and Zhai *et al.* (2004). To overcome this difficulty, we have proposed an output-dependent strategy (32) or (36) for adjusting the value of μ_i 's. As also pointed out in many other references, such an output-dependent strategy is usually more robust to modeling imperfection than a time-dependent one.

Remark 3. There is an important observation concerning the implementation of the quantizer proposed in this paper. We assume that the function $q_i(\cdot)$, which may be very complicated, has been designed and we implement $\mu_i q_i(y_i/\mu_i)$ (*not* $q(y_i/\mu_i)$ *only*) as a parameter-dependent quantizer. Since the variable of the function $q_i(\cdot)$ is y_i/μ_i , the quantizer can flexibly deal with large or small output y_i by adjusting the value of μ_i , so that the condition (32) is satisfied. This is very important in \mathcal{H}_∞ control problems since the measurement output y_i may be very large temporarily due to unexpected disturbance input. In the case where only $q_i(y_i/\mu_i)$ is viewed as a quantizer, the output of the quantizer has to be scaled by μ_i before it is passed to the controller. The function $q_i(\cdot)$ in this paper is a general concept for signal quantization, and thus careful consideration is required in real implementation.

Remark 4. Although the \mathcal{H}_∞ disturbance attenuation level γ is fixed in this paper, the same discussion is applicable for any positive $\gamma > \gamma_{opt}$, where γ_{opt} is the optimal \mathcal{H}_∞ norm that the system of (5) can reach via decentralized dynamic output feedback.

Remark 5. The condition (25) in Theorem 1 is flexible in the sense that we can choose the matrices \tilde{P}_i , R_i and K_i so that the condition is satisfied. These matrices are not independent and they must satisfy the matrix inequality (24), but we still have much design freedom since it is a strict inequality and we can incorporate some optimization requirement when solving (23) or (24).

5. Design example

In this section, we present a simple example. The interconnected system (5) we consider is composed of two subsystems, whose matrices are

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & B_{21} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ B_{11} &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, & C_{21} &= [1 \quad 2], \\ C_{11} &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, & & \\ D_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & & \\ A_2 &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 B_{12} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, & B_{22} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
 C_{12} &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, & C_{22} &= [-1 \quad 1], \\
 D_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}
 \end{aligned} \tag{37}$$

and whose interconnection matrices are

$$\begin{aligned}
 A_{12} &= \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.2 \end{bmatrix}, \\
 A_{21} &= \begin{bmatrix} 0.1 & -0.1 \\ 0.2 & 0.2 \end{bmatrix}.
 \end{aligned} \tag{38}$$

Set the disturbance attenuation level as $\gamma = 1.5$. Using the existing \mathcal{H}_∞ decentralized controller design for the system, we obtain the following controller coefficient matrices:

$$\begin{aligned}
 K_1 &= \left[\begin{array}{c|cc} 0 & -1.15 & -10.81 \\ \hline 0.53 & -2.14 & -9.67 \\ 5.92 & -5.96 & -11.49 \end{array} \right], \\
 K_2 &= \left[\begin{array}{c|cc} 0 & -2.67 & -22.40 \\ \hline 0.94 & -4.14 & -19.67 \\ -11.51 & 15.51 & -11.59 \end{array} \right].
 \end{aligned} \tag{39}$$

With the above decentralized controller, we solve the linear matrix inequality (23) with respect to block diagonal matrix variables \tilde{P} and R to obtain

$$\begin{aligned}
 \tilde{P}_1 &= \begin{bmatrix} 2.87 & 3.08 & -1.05 & -3.10 \\ 3.08 & 7.89 & -4.74 & -5.50 \\ -1.05 & -4.74 & 3.71 & 2.92 \\ -3.10 & -5.50 & 2.92 & 7.33 \end{bmatrix}, \\
 \tilde{P}_2 &= \begin{bmatrix} 17.81 & -21.84 & 28.13 & -4.55 \\ -21.84 & 32.51 & -39.56 & 2.28 \\ 28.13 & -39.56 & 49.21 & -4.14 \\ -4.55 & 2.28 & -4.14 & 8.06 \end{bmatrix}
 \end{aligned} \tag{40}$$

and

$$\begin{aligned}
 R_1 &= \begin{bmatrix} 3.66 & 7.04 & -3.97 & -8.20 \\ 7.04 & 14.30 & -8.06 & -16.39 \\ -3.97 & -8.06 & 5.02 & 10.17 \\ -8.20 & -16.39 & 10.17 & 23.55 \end{bmatrix}, \\
 R_2 &= \begin{bmatrix} 39.22 & -44.97 & 55.87 & -43.08 \\ -44.97 & 53.46 & -65.66 & 47.16 \\ 55.87 & -65.66 & 81.37 & -58.04 \\ -43.08 & 47.16 & -58.04 & 61.25 \end{bmatrix}.
 \end{aligned} \tag{41}$$

Using the above data, the condition in Theorem 1 turns out to be

$$M_1 > 3470.6\Delta_1, \quad M_2 > 7809.0\Delta_2. \tag{42}$$

Moreover, the decentralized control strategy (36) of updating the quantizers' parameters is given by

$$\begin{aligned}
 \frac{1}{M_1}|y_1| \leq \mu_1 < \frac{1}{3470.6\Delta_1}|y_1|, \\
 \frac{1}{M_2}|y_2| \leq \mu_2 < \frac{1}{7809.0\Delta_2}|y_2|.
 \end{aligned} \tag{43}$$

It is clear that once the ranges M_1, M_2 and the error bounds Δ_1, Δ_2 are determined, the above strategy is very easy to implement in any real application. As observed in Remarks 4 and 5, the computation of $\tilde{P}_1, \tilde{P}_2, R_1, R_2$ can be dealt with in the framework of optimal control problems, if necessary.

6. Conclusion

This paper has complemented and improved the discussion of Zhai et al. (2010) and Chen et al. (2011a) by extending the results to decentralized \mathcal{H}_∞ dynamic output control for interconnected systems with quantized measurement outputs. The situation is that a decentralized \mathcal{H}_∞ dynamic output controller has been designed without considering quantization, but due to physical or environmental reasons the subsystem measurement outputs are quantized before they are passed to the local controllers. For this interconnected system, we have proposed a local-output-dependent strategy for updating the quantizer parameters, so that the overall closed-loop system is asymptotically stable and achieves the same \mathcal{H}_∞ disturbance attenuation level. The main characteristics are that the quantizer updating strategy is expressed by an inequality which has more robustness to small external disturbance and rounding error, and both the pre-designed controllers and the quantizer parameters are constructed in a decentralized manner, depending on local measurement outputs.

There are many open issues in the analysis and design of quantized and interconnected systems. The local quantizers modeled in this paper are static ones (only dependent on the present input). In order to deal with higher control specifications for high speed sampled systems, dynamical quantizers are desired and more practical. Moreover, as also pointed out by Morawski and Zajączkowski (2010) as well as Bushnell (2001), the phenomena of packet dropouts, delays, etc., need to be dealt with in a unified manner, together with the approach proposed in this paper. Actually, the combination of quantizations and dropouts has been dealt with in several existing references (Ling and Lemmon, 2010), but the proposed control strategy there is not applicable directly to the problem formulated in this paper. Furthermore, fault detection (diagnosis) of interconnected systems (Chen et al., 2011b) is another important problem in our future work.

Acknowledgment

This research has been supported in part by the National Natural Science Foundation of China (61074001).

References

- Brockett, R.W. and Liberzon, D. (2000). Quantized feedback stabilization of linear systems, *IEEE Transactions on Automatic Control* **45**(7): 1279–1289.
- Bushnell, L.G. (2001). Special section on networks & control, *IEEE Control Systems Magazine* **21**(1): 22–99.
- Chen, N., Shen, X. and Gui, W. (2011a). Decentralized H_∞ quantized dynamic output feedback control for uncertain interconnected networked systems, *Proceedings of the 8th Asian Control Conference, Kaohsiung, Taiwan*, pp. 131–136.
- Chen, W., Khan, A.Q., Abid, M. and Ding, S.X. (2011b). Integrated design of observer based fault detection for a class of uncertain nonlinear systems, *International Journal of Applied Mathematics and Computer Science* **21**(3): 423–430, DOI: 10.2478/v10006-011-0031-0.
- Delchamps, D.F. (1990). Stabilizing a linear system with quantized state feedback, *IEEE Transactions on Automatic Control* **35**(8): 916–924.
- Ikeda, M., Zhai, G. and Fujisaki, Y. (1996). Decentralized H_∞ control for large-scale systems: A matrix inequality approach using a homotopy method, *Proceedings of the 35th IEEE Conference on Decision and Control, Kobe, Japan*, pp. 1–6.
- Ishii, H. and Francis, B. (2002). *Limited Data Rate in Control Systems with Networks*, Springer, Berlin.
- Iwasaki, T., Skelton, R.E. and Grigoriadis, K.M. (1998). *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis, London.
- Liberzon, D. (2000). Nonlinear stabilization by hybrid quantized feedback, *Proceedings of the 3rd International Workshop on Hybrid Systems: Computation and Control, Pittsburgh, PA, USA*, pp. 243–257.
- Liberzon, D. (2003). Hybrid feedback stabilization of systems with quantized signals, *Automatica* **39**(9): 1543–1554.
- Ling, Q. and Lemmon, M.D. (2010). A necessary and sufficient feedback dropout condition to stabilize quantized linear control systems with bounded noise, *IEEE Transactions on Automatic Control* **55**(11): 2590–2596.
- Morawski, M. and Zajączkowski, A.M. (2010). Approach to the design of robust networked control systems, *International Journal of Applied Mathematics and Computer Science* **20**(4): 689–698, DOI: 10.2478/v10006-010-0052-0.
- Murao, S., Zhai, G., Ikeda, M. and Tamaoki, K. (2002). Decentralized H_∞ controller design: An LMI approach, *Proceedings of the 41st SICE Annual Conference, Osaka, Japan*, pp. 2734–2739.
- Tatikonda, S. and Mitter, S. (2004). Control under communication constraints, *IEEE Transactions on Automatic Control* **49**(7): 1056–1068.
- Zhai, G., Chen, N. and Gui, W. (2010). Quantizer design for interconnected feedback control systems, *Journal of Control Theory and Applications* **8**(1): 93–98.
- Zhai, G., Ikeda, M. and Fujisaki, Y. (2001). Decentralized H_∞ controller design: A matrix inequality approach using a homotopy method, *Automatica* **37**(4): 565–572.
- Zhai, G., Matsumoto, Y., Chen, X. and Mi, Y. (2004). Hybrid stabilization of linear time-invariant systems with two quantizers, *Proceedings of the 2004 IEEE International Symposium on Intelligent Control, Taipei, Taiwan*, pp. 305–309.
- Zhai, G., Mi, Y., Imae, J. and Kobayashi, T. (2005). Design of H_∞ feedback control systems with quantized signals, *Preprints of the 16th IFAC World Congress, Prague, Czech Republic, Fr–M17–TO/1*.



Guisheng Zhai received the B.Sc. degree from Fudan University, China, in 1988, and the M.Eng. and Ph.D. degrees, both in system science, from Kobe University, Japan, in 1993 and 1996, respectively. After two-year industrial experience, Dr. Zhai moved to Wakayama University, Japan, in 1998, and then to Osaka Prefecture University, Japan, in 2004. In April 2010, he joined the faculty of the Shibaura Institute of Technology, Japan, where he is currently a professor of mathematical sciences. His research interests include large scale and decentralized control systems, robust control, switched systems and switching control, networked control systems, neural networks and signal processing, etc. Dr. Zhai has published more than 85 academic journal papers and 125 peer-reviewed international conference papers. He is on the editorial boards of several academic journals including the *International Journal of Control*, the *International Journal of Applied Mathematics and Computer Science*, and *Frontiers of Mechanical Engineering*. He is a senior member of the IEEE, and a member of the SICE, ISCIE, JSST and JSME.



Ning Chen received the B.Sc. and M.Sc. degrees in industrial electrical automation from the Central South University of Technology in 1992 and 1995, respectively, and the Ph.D. degree in control theory and engineering from the Central South University in 2002. From 2004 to 2005 she was a visiting scholar at Osaka University, Japan. She has been a professor in the School of Information Science and Engineering, Central South University, Changsha, China, since 2008. Her research interests include decentralized control, robust control, networked control system, and digital signal processing.



Weihua Gui received the B.Eng. degree (automatic control engineering) and the M.Eng. degree (control science and engineering) from Central South University, Changsha, China, in 1976 and 1981, respectively. From 1986 to 1988 he was a visiting scholar at Universität-GH-Duisburg, Germany. He has been a full professor in the School of Information Science and Engineering, Central South University, since 1991. His main research interests are in the modeling and optimal control of complex industrial process, decentralized control, distributed robust control, and fault diagnosis.

Received: 24 October 2012

Revised: 18 February 2013