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NUMERICAL SOLUTION OF NON-NEWTONIAN FLUIDS FLOW PAST AN ACCELERATED VERTICAL INFINITE PLATE IN THE PRESENCE OF FREE CONVECTION CURRENTS

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A similarity analysis of non-Newtonian fluid flow past an accelerated vertical infinite plate in the presence of free convection current is carried out. A group theoretic generalized dimensional analysis is employed to achieve the governing non-linear ordinary differential equations in the most general form. Numerical solutions of these equations are given with the plot of their velocity profiles with the effects of Pr-Prandtl number and Gr-Grashof number.

Key words: non-Newtonian fluids, convection currents**,** similarity analysis.

1. Introduction

For a long time, there has been considerable interest in non-Newtonian fluids (Skelland, 1967; Wilkinson, 1960; Bird *et al.*, 1960; Dunn, 1999; Kapur, 1963; Metzner, 1965; Nakayama and Koyama, 1988; Hansen and Na, 1968). The complex rheology of biological fluids has motivated investigations involving different non-Newtonian fluids. In recent years, non-Newtonian fluids have become more and more important industrially. Polymer solutions, polymer melts, blood, paints and slurries, shampoo, toothpaste, clay coating and suspensions, grease, cosmetic products, custard, are the most common examples of non-Newtonian fluids. Academic curiosity and practical applications have generated considerable interest in finding the solutions of differential equations governing the motion of non-Newtonian fluids. The property of these fluids is that the stress tensor is related to the rate of deformation tensor by some non-linear relationship. These fluids present some interesting challenges to researchers in engineering, applied mathematics and computer science. Thus wide usages of these fluids have prompted modern researchers to explore extensively; the fields of non-Newtonian fluids (Surati and Timol, 2010; Neosi Nguetchue *et al.*, 2009; Patel and Timol, 2004; 2005; 2009a; 2008; 2009b; 2010).

Soundalgekar and Pop (1980) studied a flow past an accelerated vertical infinite plate in the presence of free convection currents. His problem is limited to the Newtonian fluids only. A steady laminar free convection flow of an electrically conducting fluid along a porous hot vertical plate in the presence of a heat source is investigated by Sharma and Pankaj, (1995). The velocity and temperature distributions are shown graphically for two cases Gr >0 and Gr<0. An exact analysis of rotation effects on an unsteady flow of an incompressible and electrically conducting fluid past a uniformly accelerated infinite vertical plate, under the

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action of a transversely applied magnetic field was presented by Muthucumaraswamy *et al*. (2011). Theoretical solution of an unsteady radiative flow past a uniformly accelerated isothermal infinite vertical plate with uniform mass diffusion is presented by Muthucumaraswamy and Shankar (2011), taking into account homogeneous chemical reaction of first order.

The present work further extends the recent work of Soundalgekar and Pop (1980) and also generalized dimensional analysis criteria of Morgan (1952) for non-Newtonian fluids. In the present paper we have carried out a complete similarity analysis of all non-Newtonian fluids flow past an accelerated vertical infinite plate in the presence of free convection currents. A group theoretic generalized dimensional analysis is employed to achieve the governing non-linear ordinary differential equations. A numerical solution is obtained for viscous Newtonian fluids, power-law non-Newtonian fluids and Powell-Eyring non-Newtonian fluids for different flow indices and different values of parameters and Prandtl as well Grashof numbers by spline collocation method. The numerical solution of the power-law fluid for the case $k=1$ and $n=1$ is in good agreement with Soundalgekar and Pop (1980).

The Powell-Eyring model is mathematically more complex and deserves our attention because it has certain advantages over the power-law model. Firstly, it is deduced from the kinetic theory of liquid rather than the empirical relation as in the case of the power-law model. Secondly, it correctly reduces to Newtonian behavior for low and high shear rate.

2. Generalised dimensional analysis method

Groups with the form

$$\overline{z}_i = A_1^{p_{i1}},\ldots A_r^{p_{ir}}\, Z_i\,;\quad (i=1\ldots,n), \tag{2.1}$$

bear a close relationship to traditional approaches to dimensional analysis. As a concrete example, consider a flat plate which is immersed in an incompressible viscous fluid, and which is accelerated from rest to a constant plate velocity $U > 0$, mathematically

$$u_t - \vartheta\, u_{yy} = 0 \quad \text{(momentum)}. \tag{2.2}$$

Subject to

$$\begin{aligned} &u \to 0, && \text{as} && t \to 0 && \text{when} && y \geq 0, \\ &u \to 0, && \text{as} && y \to \infty && \text{when} && t \geq 0, \\ &u \to U, && \text{as} && y \to 0 && \text{when} && t > 0. \end{aligned} \tag{2.3}$$

The conventional dimensional approach to this problem would be to associate with the significant quantities. Thus

$$\begin{aligned} &[u],\, [U] : L^{+1}\ \ \tau^{-1}, \\ &[y] : L^{+1}\ \ \tau^{0}, \\ &[t]\ : L^{0}\ \ \tau^{+1}, \\ &[\vartheta] : L^{+2}\ \ \tau^{-1} \end{aligned} \tag{2.4}$$

where in the brackets [ ] mean "the dimension of". The formulas (2.4) may be regarded as shorthand expressions for the scale change equations

$$\bar{u} = L^{+1}\,\tau^{-1}\,u\,; \qquad \bar{y} = L^{+1}\,\tau^{0}\,y,$$

$$\bar{t} = L^{0}\,\tau^{+1}\,t; \qquad \bar{\vartheta} = L^{+2}\,\tau^{-1}\,\vartheta, \tag{2.5}$$

$$\bar{U} = L^{+1}\,\tau^{-1}\,U.$$

Equation (2.5) has a greater significance, however, than merely for changing scale; that is Eq.(2.5) constitutes a two-parameter group, with the scale factors L and τ playing the role of group parameters.

Each of the variables in any set of the governing equations under consideration is regarded as being in one of three distinct categories: (i) dependent (ii) independent (iii) physical. Thus, the variables appearing in Eqs (2.2) – (2.3), u, y, t, ϑ, U may be identified as follows: the fluid velocity u may be identified as the dependent variable; the position and time coordinates y, t may be identified as independent variables; and the quantities U, ϑ may be identified as physical variables.

In recognition of the foregoing three categories for variables, the class of r-parameter groups

$$\bar{z}_i = C_i\left(A_1,...A_r\right)z_i + D_i\left(A_1,...A_r\right)\left(i = 1...,n\right), \tag{2.6}$$

can be written somewhat more explicitly. Thus, we consider the following r-parameter groups of the form,

$$\bar{Z}_j = A_1^{a_{j1}}....A_r^{a_{jr}} \qquad Z_j\ \left(j = 1,...,n \geq 1\right), \tag{2.7}$$

$$\bar{X}_k = A_1^{b_{k1}}....A_r^{b_{kr}} \qquad X_k\ \left(k = 1,...,m \geq 1\right), \tag{2.8}$$

$$\bar{Y}e = A_1^{c_{e1}}....A_r^{c_{er}} \qquad Ye\ \left(e = 1,...,p \geq 1\right) \tag{2.9}$$

where in the Z's are to be associated with the dependent variables of a set of governing equations and the X's are associated with the independent variables, but the Y's are associated with the physical variables.

Subsequent discussions reveal that the dimensional matrix associated with Eqs (2.7) – (2.9) plays an important role. To facilitate the presentation, let B denote the $\left(m\times r\right)$ matrix $\left[b_{k1},\ldots,b_{kr}\right]$; and let C denote the $\left(p\times r\right)$ matrix $\left[C_{e1},\ldots,C_{er}\right]$. Similarly, let BC denote the $\left(\left[m\times p\right]\times r\right)$ matrix,

$$BC:\begin{bmatrix} b_{k1},....b_{kr} \\ C_{e1},....C_{er}\end{bmatrix}.$$

The matrix BC is assumed to have rank r, while the matrix C has rank s, $s \leq r$. Thus the dimensional matrix associated with Eqs (2.7) – (2.9) has rank r.

As an additional means of facilitating the presentation, the rows of BC are assumed to be arranged, so that

(i) when $s = r$, the first r rows of C are linearly independent,

(ii) when $s < r$, the first s rows of C plus the last $[r - s]$ rows of B are linearly independent.

To illustrate the foregoing motions, consider again Eqs (2.5). By inspection, the matrices, B. C and BC are given, respectively, by

$$B:\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad C:\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}; \qquad BC:\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \\ 1 & -1 \end{bmatrix}.$$

Also for Eqs (2.5); $n = 1$, $m = p = r = s = 2$, having defined the class of r-parameter groups (2.7) – (2.9) and having introduced some important aspects of the dimensional matrix associated with such groups, attention now turns to certain features of the generalized dimensional analysis approach.

Theorem 1: If the function Ij is invariant in form under an r-parameter group (2.7) – (2.9), i.e., if $Zj = Ij \left(X_1,\ldots X_m, Y_1,\ldots,Y_p\right)$ transforms to $\bar{Z}j = Ij\left(\bar{X}_1,...,\bar{X}_m,\bar{Y}_1,...,\bar{Y}_p\right)$, then $Zj = Ij$ (---) is equivalent to a relationship in fewer variables,

$$\pi_j\left(Z_j, X_1,...,Xm;\, Y_1,...,Yp\right) = F_j\left(\pi_1\left(X_1,...,Xm;\, Y_1,...,Yp\right),...\pi_\delta(...)\right) \tag{2.10}$$

where in $\delta = [m + p - r] > 0$, and $\left\{\pi_j,...\pi_1,...\pi_\delta\right\}$ are independent absolute invariants of Eqs (2.7) – (2.9). In the present discussion Theorem 1 plays the role of the so-called Pi theorem of the conventional dimensional analysis.

To apply Theorem 1 expressions for the absolute invariants of Eqs (2.7) – (2.9) are required. By definition: $\pi\left(X_1,...,Xm;\, Y_1,...,Yp\right)$ are absolute invariants provided that under the transformations Eqs (2.8) – (2.9)

$$\pi\left(\bar{X}_1,...,\bar{X}m;\, \bar{Y}_1,...,\bar{Y}p\right) = \pi\left(X_1,...,Xm;\, Y_1,...,Yp\right), \tag{2.11}$$

upon differentiation of Eq.(2.11) with respect to each of the parameters in turn,

$$\sum_{k=1}^{m} \frac{\partial\pi}{\partial\bar{X}_k}\frac{\partial\bar{X}_k}{\partial A_\alpha} + \sum_{e=1}^{p} \frac{\partial\pi}{\partial\bar{Y}_e}\frac{\partial\bar{Y}_e}{\partial A_\alpha} = 0, \qquad (\alpha = 1,...r), \tag{2.12}$$

with Eqs (2.8) – (2.9) it follows that

$$\frac{\partial\bar{X}_k}{\partial A_\alpha} = \left[\frac{b_{k\alpha}}{A_\alpha}\right]\bar{X}_k, \qquad \frac{\partial\bar{Y}_e}{\partial A_\alpha} = \left[\frac{C_{e\alpha}}{A_\alpha}\right]\bar{Y}_e, \tag{2.13}$$

combining Eqs (2.12) and (2.13), a system of first order, linear partial differential equations evolves,

$$\sum_{k=1}^{m} b_{k\alpha}\,\bar{X}_k\frac{\partial\pi}{\partial\bar{X}_k} + \sum_{e=1}^{p} C_{e\alpha}\,\bar{Y}_e\frac{\partial\pi}{\partial\bar{Y}_e} = 0, \qquad (\alpha = 1,...r). \tag{2.14}$$

According to the theory of first order linear partial differential equations Eq.(2.14) has $[m+p-r]$ independent solutions. It will now be shown that each of the independent solutions may be determined in the form,

$$\begin{aligned}\pi &= \left[\bar{X}_1\right]^{\Gamma 1} \ldots \left[\bar{X}_m\right]^{\Gamma m} \left[\bar{Y}_1\right]^{\gamma_1} \ldots \left[\bar{Y}_p\right]^{\gamma_p} \\ &= \left[X_1\right]^{\Gamma 1} \ldots \left[X_m\right]^{\Gamma m} \left[Y_1\right]^{\gamma_1} \ldots \left[Y_p\right]^{\gamma_p} .\end{aligned} \tag{2.15}$$

Indeed, upon substitution of Eqs (2.15) into (2.14) and simplification, a linear system of ordinary equations is derived,

$$\sum_{k=1}^{m} \Gamma_k \begin{bmatrix} b_{k1} \\ b_{k2} \\ . \\ . \\ b_{kr} \end{bmatrix} + \sum_{e=1}^{p} \gamma_e \begin{bmatrix} C_{e1} \\ C_{e2} \\ . \\ . \\ C_{er} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}. \tag{2.16}$$

Therefore, to determine the $[m+p-r]$ independent absolute invariants π needed to apply Theorem 1 requires only that $[m+p-r]$ independent solutions be established by Eq.(2.16). A like procedure can be involved to show that the absolute invariants Π_j of Theorem 1 may be established in the form

$$\begin{aligned}\Pi_j &= \bar{Z}_j \left[\bar{X}_1\right]^{A_{ji}} \ldots \left[\bar{X}_m\right]^{A_{jm}} \left[\bar{Y}_1\right]^{\lambda_{j1}} \ldots \left[\bar{Y}_p\right]^{\lambda_{jp}} , \\ &= \left[\bar{Z}_j\right]\left[\bar{X}_1\right]^{A_{ji}} \ldots \left[\bar{X}_m\right]^{A_{jm}} \left[\bar{Y}_1\right]^{\lambda_{j1}} \ldots \left[\bar{Y}_p\right]^{\lambda_{jp}}\end{aligned} \tag{2.17}$$

wherein,

$$\sum_{k=1}^{m} A_{jk} \begin{bmatrix} b_{k1} \\ b_{k2} \\ . \\ . \\ b_{kr} \end{bmatrix} + \sum_{e=1}^{p} \lambda_{je} \begin{bmatrix} C_{e1} \\ C_{e2} \\ . \\ . \\ C_{er} \end{bmatrix} = \begin{bmatrix} a_{j1} \\ a_{j2} \\ . \\ . \\ a_{jr} \end{bmatrix}. \tag{2.18}$$

With the foregoing preliminaries in hand, the principal results are presented below:

Principal results

The statement of Theorem 1 does not suggest a preferred form for the required absolute invariants. However, experience reveals that for practical applications of the theorem it is frequently good practice to establish the required set of absolute invariants in one of the two forms to be given in Theorem 2 and Theorem 3.

Theorem 2 treats the case where the rank r of the matrix BC associated with r-parameter groups (2.7) – (2.9) equals the rank s of the matrix C, the case $r > s$ is then considered in Theorem 3.

Theorem 2: If, and only if, $r = s$, the set of $[n + m + p - r]$ independent absolute invariants required by Theorem 1 may be obtained in the form,

$$\Pi_j = Z_j \left[Y_1\right]^{\lambda_{j1}} ... \left[Y_r\right]^{\lambda_{jr}} \qquad \left(j = 1,...n\right), \tag{2.19}$$

$$\pi_k = X_k \left[Y_1\right]^{\gamma_{k1}} ... \left[Y_r\right]^{\gamma_{kr}} \qquad \left(k = 1,...m\right), \tag{2.20}$$

$$\tilde{\pi}_p = Y_p \left[Y_1\right]^{\delta_{\rho 1}} ... \left[Y_r\right]^{\delta_{\rho r}} \qquad \left(j = 1,...n\right). \tag{2.21}$$

Equation (2.19) is readily established via Eqs (2.17) – (2.19) upon utilizing the assumed condition that when $s = r$, the first r rows of the matrix C are linearly independent. Thus Eq.(2.18) yields the following system of equations for the exponents $\lambda_{j\alpha}$ of Eq.(2.19),

$$\sum_{\alpha=1}^{r} \lambda_{j\alpha} \begin{bmatrix} C_{\alpha 1} \\ C_{\alpha 2} \\ . \\ . \\ C_{\alpha r} \end{bmatrix} = - \begin{bmatrix} a_{j1} \\ a_{j2} \\ . \\ . \\ a_{jr} \end{bmatrix} \qquad \left(j = 1,...n\right). \tag{2.22}$$

When $r = s$, Eq.(2.21) follows from Eqs (2.15) – (2.16), indeed (2.16) yields the following system of equations for the exponents $\delta_{\rho\alpha}$ of Eq.(2.21)

$$\sum_{\alpha=1}^{r} \delta_{\rho\alpha} \begin{bmatrix} C_{\alpha 1} \\ C_{\alpha 2} \\ . \\ . \\ C_{\alpha r} \end{bmatrix} = - \begin{bmatrix} C_{\rho 1} \\ C_{j\rho} \\ . \\ . \\ C_{\rho r} \end{bmatrix} \quad ; \quad \left(\rho = [r+1],...,p\right). \tag{2.23}$$

In a similar manner (2.16) yields the following system of equations for the exponents $\gamma_{k\alpha}$ of Eq.(2.20)

$$\sum_{\alpha=1}^{r} \gamma_{k\alpha} \begin{bmatrix} C_{\alpha 1} \\ C_{\alpha 2} \\ . \\ . \\ C_{\alpha r} \end{bmatrix} = - \begin{bmatrix} b_{k1} \\ b_{k2} \\ . \\ . \\ b_{kr} \end{bmatrix} \qquad \left(k = 1,...,m\right). \tag{2.24}$$

[It is assumed that $p > s$. For the special case $p = s$, no absolute invariants are determined solely from the physical variables].

Theorem 3: If and only if $r > s$, the set of $[n + m + p - r]$ independent absolute invariants required by Theorem 1 may be obtained in the form,

$$\Pi_j = Z_j \quad [X_\varepsilon]^{A_{j\varepsilon}} ... [X_m]^{A_{jm}} \quad [Y_1]^{\lambda_{j1}} ... [Y_s]^{\lambda_{js}}, \qquad (j = 1,...n), \tag{2.25}$$

$$\tilde{\pi}_\sigma = X_\sigma \quad [X_\varepsilon]^{\Gamma\sigma\varepsilon} ... [X_m]^{\Gamma\sigma m} \quad [Y_1]^{p_{\sigma 1}} ... [Y_s]^{p_{\sigma s}}, \quad (\sigma = 1,...[m+s-r]), \tag{2.26}$$

$$\tilde{\pi}_\rho = Y_\rho \quad [Y_1]^{\delta_{\rho 1}} ... [Y_r]^{\delta_{\rho r}} \quad ;(\rho = (r+1),...p) \tag{2.27}$$

where $\varepsilon \equiv [m+s-r+1] \leq m.$

Equation (2.25) is readily established via Eqs (2.17) – (2.18) upon utilizing the assumed condition that when $r > s$, the last $[r - s]$ rows of the matrix B plus the first s rows of the matrix C are linearly independent. Thus (2.18) yields the following system of equations for the exponents of Eq.(2.25)

$$\sum_{\alpha=\varepsilon}^{m} A_{j\alpha} \begin{bmatrix} b_{\alpha 1} \\ b_{\alpha 2} \\ . \\ . \\ b_{\alpha r} \end{bmatrix} + \sum_{w=1}^{s} \lambda_{jw} \begin{bmatrix} C_{w1} \\ C_{w2} \\ . \\ . \\ C_{wr} \end{bmatrix} = - \begin{bmatrix} a_{j1} \\ a_{j2} \\ . \\ . \\ a_{jr} \end{bmatrix}, \qquad (j = 1,...n), \tag{2.28}$$

when $r > s$, (2.27) follows from Eqs (2.15) – (2.16), indeed (2.16) yields the following system of equations for the exponents of Eq.(2.27),

$$\sum_{w=1}^{s} \delta_{\rho w} \begin{bmatrix} C_{w1} \\ C_{w2} \\ . \\ . \\ C_{wr} \end{bmatrix} = - \begin{bmatrix} C_{\rho 1} \\ C_{\rho 2} \\ . \\ . \\ C_{\rho r} \end{bmatrix} \quad ; \ (\rho = [s+1],...,p). \tag{2.29}$$

In a similar manner, Eq.(2.16) yields the following system of equations for the exponents of Eq.(2.26)

$$\sum_{\alpha=\varepsilon}^{m} \Gamma\sigma\alpha \begin{bmatrix} b_{\alpha 1} \\ b_{\alpha 2} \\ . \\ . \\ b_{\alpha r} \end{bmatrix} + \sum_{w=1}^{s} \gamma_{\sigma w} \begin{bmatrix} C_{w1} \\ C_{w2} \\ . \\ . \\ C_{wr} \end{bmatrix} = - \begin{bmatrix} b_{\sigma 1} \\ b_{\sigma 2} \\ . \\ . \\ a_{\sigma r} \end{bmatrix} \qquad (\sigma = 1...,[m+s-r]). \tag{2.30}$$

3. Mathematical analysis

Considering a laminar, two-dimensional incompressible boundary layer equation with a Cartesian co-ordinate system, we take x' axis along the sheet in the vertical direction and y' axis is taken normal to it.

If T'_w and T'_∞ are the temperature of the plate and the fluid far away from the plate, then the boundary layer equations governing the flow and heat transfer in dimension less form are given by,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\left(\tau_{yx}\right) + \mathrm{Gr}\,\theta, \tag{3.1}$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 \theta}{\partial y^2}, \tag{3.2}$$

and under the boundary layer assumption the stress strain relationship will be

$$F\left(\tau_{yx}; \frac{\partial u}{\partial y}\right) = 0, \tag{3.3}$$

the form of which differs for different fluid models of non-Newtonian fluids.

The boundary conditions are

$$y = 0 \Rightarrow u = 0, \qquad T = T_w - T_\infty \neq 0 \qquad \text{for} \qquad t \leq 0, \tag{3.4}$$

$$y = 0 \Rightarrow u = U(t), \qquad T = T_w : \qquad \text{for} \quad t > 0, \tag{3.5a}$$

$$y = \infty \Rightarrow u \to 0; \qquad T \to T_\infty : \qquad \text{for} \quad t > 0 \tag{3.5b}$$

where in ***u*** is a velocity component in the *x* direction, *y* is a rectangular co-ordinate, τ_{yx} is a component of shearing stress, *F* is an arbitrary function, *t* is dimensionless time, Gr is the Grashof number, Pr is Prandtl number, θ is dimensionless temperature, *U*(*t*) is dimensionless free stream velocity.

With the use of dimensionless quantities

$$u = \frac{u'}{U_0}, \qquad y = \frac{y'}{L}, \qquad \tau'_{y'x'} = \frac{\tau_{yx}}{\rho U_0{}^2}, \qquad t = \frac{t' U_0}{L},$$

$$\theta = \frac{\theta'}{T'_w - T'_\infty} = \frac{T' - T'_\infty}{T'_w - T'_\infty}, \qquad U = \frac{U'}{U_0}, \qquad \mathrm{Pr} = \frac{\mu c_p}{K}, \tag{3.6}$$

$$\mathrm{Gr} = \frac{L}{U_0^2}\left(T'_w - T'_\infty\right)$$

where U_0 is constant with the dimension of velocity; ρ is the density of the fluid; *K* is thermal conductivity; Cp is specific heat at constant pressure; μ is viscosity; θ is acceleration due to gravity, we may reduce the system of Eqs (3.1)–(3.3) by the group-theoretic generalized dimensional analysis will cause restrictions to be imposed on the function *U*(*t*) and $\theta(t)$ in such a way that boundary conditions can be transformed into a meaningful form.

So in order to find a possible form of *U*(*t*) and $\theta(t)$. The following group of transformations G_1 are introduced.

$$G_1:\begin{cases}\overline{u}=A_1 u, & \tau_{\overline{y}\,\overline{x}}=A_4\ \tau_{yx}, & \overline{\theta}=A_5\,\theta \text{ dependent variables}\\ \overline{y}=A_3\ y, & \overline{t}=A_2\ t & \text{independent variables}\\ \overline{\rho}=A_6\ \rho, & \overline{U}=A_7\,U & \text{physical variables}\end{cases}$$

where $A_i' S\ (i=1................7)$ in G_1 are positive real parameters.

At the first step of similarity requirement $A_i S$ of G_1 must be inter-related in order for system (3.1) (3.5a) (3.5b) to be an invariant in form. Indeed, this requirement will be met under following three-parameter groups of transformation

$$\Gamma_1:\left|\begin{array}{ll}\overline{y}=A_1^0\,A_2^0\,A_3^1\,y; & \overline{t}=A_1^0\,A_2^1\,A_3^0\,t,\\ \overline{\rho}=A_1^0\,A_2^1\,A_3^{-2}\,\rho; & \overline{U}=A_1^1\,A_2^0\,A_3^0\,U,\\ \tau_{\overline{y}\,\overline{x}}=A_1^1\,A_2^{-1}\,A_3^1\,\tau_{yx}; & \overline{u}=A_1^1\,A_2^0\,A_3^0\,u,\\ & \overline{\theta}=A_1^1\,A_2^{-1}\,A_3^0\,\theta.\end{array}\right.$$

Now in order to apply Theorem 3, a pi theorem stated by Moran and Murshek (1972), (given in section-2) the rank of dimensional matrix associated with independent and physical variables BC and physical variables C of Γ_1 is required to be determined.

The associated dimensional matrix will be

$$BC:\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ \hline 0 & 1 & -2\\ 1 & 0 & 0\end{bmatrix}. \tag{3.7}$$

By inspection, the rank of $BC=r=3$ and rank of $C=S=2$ and since r and s in the light of Theorem – 3 following set of $\hat{\pi}\,s$ can be obtained.

$$\hat{\pi}=y\left(\frac{\rho}{t}\right)^{\frac{1}{2}},\qquad \pi_1=\frac{u}{U},\qquad \pi_2=\frac{\theta\,t}{U},\qquad \pi_3=\frac{\tau_{yx}}{U}\left(\frac{t}{\rho}\right)^{\frac{1}{2}}. \tag{3.8}$$

Clearly, $\hat{\pi}$ is a similarity independent variable whereas π_1, π_2 and π_3 are similarity variables, i.e., in the usual notation.

$$\pi_1=f_1(\hat{\pi}),\qquad \pi_2=f_2(\hat{\pi}),\qquad \pi_3=f_3(\hat{\pi}). \tag{3.9}$$

Now for a non uniformly accelerated plate, if we consider $U(t)=t^k$ and put $\hat{\pi}=\eta$ then the set of absolute invariants (3.8) will be

$$\eta = y\left(\frac{\rho}{t}\right)^{\frac{1}{2}}, \qquad f_1(\eta) = \frac{u}{t^k}, \qquad f_2(\eta) = \frac{\theta}{t^{k-1}}, \qquad f_3(\eta) = \frac{\tau_{yx}}{t^{k-\frac{1}{2}}\rho^{\frac{1}{2}}}. \tag{3.10}$$

Under this set of transformations, the system (3.1) – (3.5a,b) will be reduced to the following set of ordinary differential equations.

$$k\, f_1 - \frac{1}{2}\eta\, f_1' = \rho\frac{d}{d\eta}\left(f_3\right) + \mathrm{Gr}\, f_2, \tag{3.11}$$

$$\left(k-1\right) f_2 - \frac{1}{2}\eta\, f_2' = \frac{1}{\mathrm{Pr}} f_2'' \tag{3.12}$$

where prime denotes differentiation with respect to the boundary conditions which will cause the restriction on the wall temperature, that is $\theta = t^{k-1}$ will be

$$\eta = 0 \Rightarrow f_1 = 1, \qquad f_2 = 1, \tag{3.13a}$$

$$\eta = \infty \Rightarrow f_1 = 0, \qquad f_2 = 0, \tag{3.13b}$$

with the stress-strain relationship

$$F\left(\rho^{\frac{1}{2}}\, t^{\left(k-\frac{1}{2}\right)} f_3\, ; \qquad \rho^{\frac{1}{2}}\, t^{\left(k-\frac{1}{2}\right)} f_1'\right) = 0. \tag{3.14}$$

The system prime (3.11) – (3.13a,b) suggests that similarity solutions of general Newtonian fluids exist for a non-uniformly accelerated vertical plate with variable wall temperature. Clearly, for $k = 1$ i.e., $U(t) = t$ and $\theta = 1$, i.e., in the case of a uniformly accelerated plate with constant wall temperature Eqs (3.11) and (3.12) with boundary conditions (3.13a), (3.13b) will be reduced to same equations obtained by Soundalgekarand Pop (1980) for Newtonian fluids.

For $K = \frac{1}{2}$ the strain-stress relationship (3.14) will be free from independent variable. This will lead to an interesting case, i.e., right angle flow geometry. In such a case, similarity solutions for all non-Newtonian fluids will exist.

In this case Eqs (3.11) – (3.12) will be

$$f_1 - \eta\, f_1' = 2\rho\frac{df_3}{d\eta} + 2\,\mathrm{Gr}\, f_2, \tag{3.15}$$

$$f_2 + \eta\, f_2' + \frac{2}{\mathrm{Pr}} f_2'' = 0, \tag{3.16}$$

with the same boundary conditions (3.13a), (3.13b).

4. Analysis for different models

4.1. Newtonian case

In the case of general Newtonian fluids the stress-strain relationship (3.3) will come to

$$\tau_{y'x'} = \mu' \frac{\partial u'}{\partial y'}. \tag{4.1}$$

Using non-dimensional quantities (3.6) and transformations (3.10) in Eq.(4.1) and substituting this in the right hand side of Eq.(3.11) , the system (3.11) – (3.13a,b) will come to ($k=1$)

$$f_1 - \frac{1}{2}\eta\, f_1 = \rho\, f_2'' + \mathrm{Gr}\, f_2, \tag{4.2}$$

$$\frac{1}{2}\eta\, f_2' + \frac{1}{\mathrm{Pr}} f_2'' = 0, \tag{4.3}$$

with the boundary conditions,

$$f_1(0) = 1, \qquad f_2(0) = 1, \tag{4.4a}$$

$$f_1(\infty) = 0, \qquad f_2(\infty) = 0. \tag{4.4b}$$

4.2. Non-Newtonian power-law case

Mathematically, a power-law model can be written as

$$\tau'_{y'x'} = m \left|\frac{\partial u'}{\partial y'}\right|^{n-1} \frac{\partial u'}{\partial y'} \tag{4.5}$$

where m and n are fluid consistency indices. After a non-dimensional process this equation is substituted in Eq.(3.15) which yields

$$\frac{1}{2} f_1 - \frac{1}{2}\eta\, f_1' = \rho^{\frac{n+1}{2}}\, n\left(f_1'\right)^{n-1} f_1'' + \mathrm{Gr}\, f_2, \tag{4.6}$$

$$f_2 + \eta\, f_2' + \frac{2}{\mathrm{Pr}} f_2'' = 0, \tag{4.7}$$

with the same boundary conditions (4.4a,b).

4.3. Non-Newtonian Powell-Eyring case

Mathematically, a Powell-Eyring model can be written as

$$\tau'_{y'x'} = \mu \frac{\partial u'}{\partial y'} + \frac{1}{B}\sinh^{-1}\left(\frac{1}{C}\frac{\partial u'}{\partial y'}\right) \tag{4.8}$$

where μ, B, C are the parameters of fluids. Again, when this model is substituted in Eq.(3.15) we get

$$f_1 - \eta f_1' = \rho f_1'' \left[1 + \frac{1}{\alpha' \left(1 + \beta' f_1'^2\right)^{\frac{1}{2}}} \right] + 2\,\mathrm{Gr}\, f_2, \tag{4.9}$$

$$f_2 + \eta f_2' + \frac{2}{\mathrm{Pr}} f_2'' = 0, \tag{4.10}$$

with the same boundary conditions (4.4a,b) where $\alpha' = \mu BC$; $\beta' = \dfrac{\rho u^3}{3C^2 L\mu}$ are non-dimensional quantities.

5. Numerical solution of the problem

We present the accurate numerical solutions of the system of Eqs (4.2)-(4.4a,b)-(4.6)-(4.7) and (4.9)-(4.10) by the method known as the spline collocation based on cubic B-spline functions which are piecewise polynomials of degree three. The method of cubic B-spline is successfully used by Timol *et al*. (1987) for some specific flow and heat transfer problems. More information about the spline collocation methods is found in the references (Bickley, 1968; Cheng and Minkowycz, 1977; De Boor, 1978; Prenter, 1975; Sun, 1998; Usmani, 1992). Numerical results are presented in a tabular form for a various values of the associated parameters. We believe that these results serve as a reference against which other approximate solutions for the present problem can be compared in the future. In addition, this method can be applied to solve variety of problems in the field of applied mathematics.

Let $S(x)$ be a cubic spline bunching expressed as follows in the interval $\left[x_i, x_{i+1}\right]$

$$S''(x) = Mi \frac{x_{i+1} - x}{h} + Mi_{i+1} \frac{x - x_i}{h} \tag{5.1}$$

where h is the length of the sub-interval $\left[x_i, x_{i+1}\right]$,

and $\qquad M_i = S''(x)$ at $x = x_i$ etc.

Two successive integrations lead to the derivation of $S(x)$ as follows

$$S(x) = M_{i+1} \frac{(x - x_i)}{6h} + M_i \frac{(x_{i+1} - x)}{6h} + \left(y_{i+1} - \frac{h^2}{6} M_{i+1} \right) \frac{(x_{i-1} - x_i)}{h} + \\ + \left(y_i - \frac{h^2}{6} \right) \frac{(x_{i+1} - x)}{h} \quad ; \quad i = 1, 2, 3.....N. \tag{5.2}$$

A similar expression of $S(x)$ in the interval $\left[x_{i-1}, x_i\right]$ is given by

$$S(x) = M_i \frac{\left(x - x_{i-1}\right)}{6h} + M_{i-1} \frac{\left(x_i - x\right)}{6h} + \left(y_i - \frac{h^2}{6} M_i\right) \frac{\left(x - x_{i-1}\right)}{h} +$$
$$+ \left(y_{i-1} - \frac{h^2}{6} M_{i-1}\right) \frac{\left(x_i - x\right)}{h} \quad ; \quad i = 2,3.....N+1. \tag{5.3}$$

The continuity of $S(x)$ at $x = x$ yields a recurrence relation

$$y_{i-1} - 2y_i + y_{i+1} = \frac{h^2}{6}\left(M_{i-1} + 4M_i + M_{i+1}\right) \quad i = 2,3.......N. \tag{5.4}$$

This constitutes the system of $(N+1)$ equations with $(N+1)$ unknowns y, $i = 1,2\ldots(N+1)$. The coefficient matrix of this system is a non-singular matrix and hence a unique solution of the system is guaranteed.

Now in order to obtain a spline solution of the boundary value problem

$$y'' = f\left(x, y, y'\right) \quad 0 \le x \le 1, \tag{5.5}$$

subject to the boundary conditions

$$G_1\left[y(0), y'(0)\right] = 0, \tag{5.6}$$

$$G_2\left[y(1), y'(1)\right] = 0, \tag{5.7}$$

we required to solve the system of Eq.(5.4) where the quantities M, $I = 1,2\ldots\ldots N+1$ can be obtained by evaluations $y''(x)$ at $x = x_i$ from Eq.(5.5). Here maintaining compact computations let us assume that the boundary conditions of the problem are

$$f_1 = f_2 = 1 \quad \text{at} \quad \eta = 0, \tag{5.8}$$

$$f_1 = f_2 = 0 \quad \text{at} \quad \eta = 1.$$

Here we have taken different values of the Prandtl number and Grashof number and observed their effects. Also in the case of the power-law fluid the effects of non-Newtonian behavior index on the flow of the fluid are studied.

6. Discussion of results

Dimensionless velocity distributions for different fluids model are shown in Figs 1, 2 and 3a,b. It is evident from the curves representing velocity profiles that the role of the Prandtl number as well as the Grashof number is quite significant. An increment in the Prandtl number is responsible for the reduction in velocity whereas a reduction in the Grashof number causes a decrees in velocity. Here it will be worthwhile noting that the case Gr >0 corresponds to the cooling of the plate while Gr < 0 to the heating of the plate by free convection currents.

For the power-law fluids dimensionless velocity profiles are presented in Fig.2 for Pr = 7, Gr = 10 or several flow indices n. The velocity profile will be very regular with the parameter n.

Figures 3a and 3b show the velocity of the Powell-Eyring fluids flow as a function of *n* and different values of the Prandtl number and Grashof number as well as $\alpha' = 0.1$ and $\beta' = 10^4$. In this case velocity profiles bear similarity to the Newtonian behavior as shown in Fig.1.

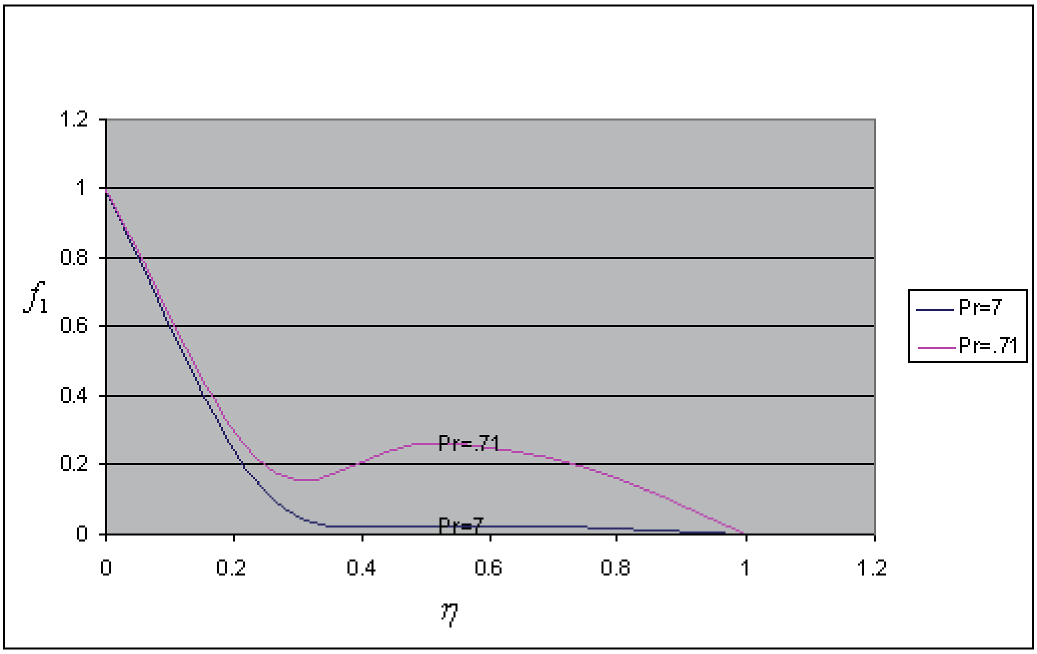

Fig.1. Velocity profiles for Newtonian case.

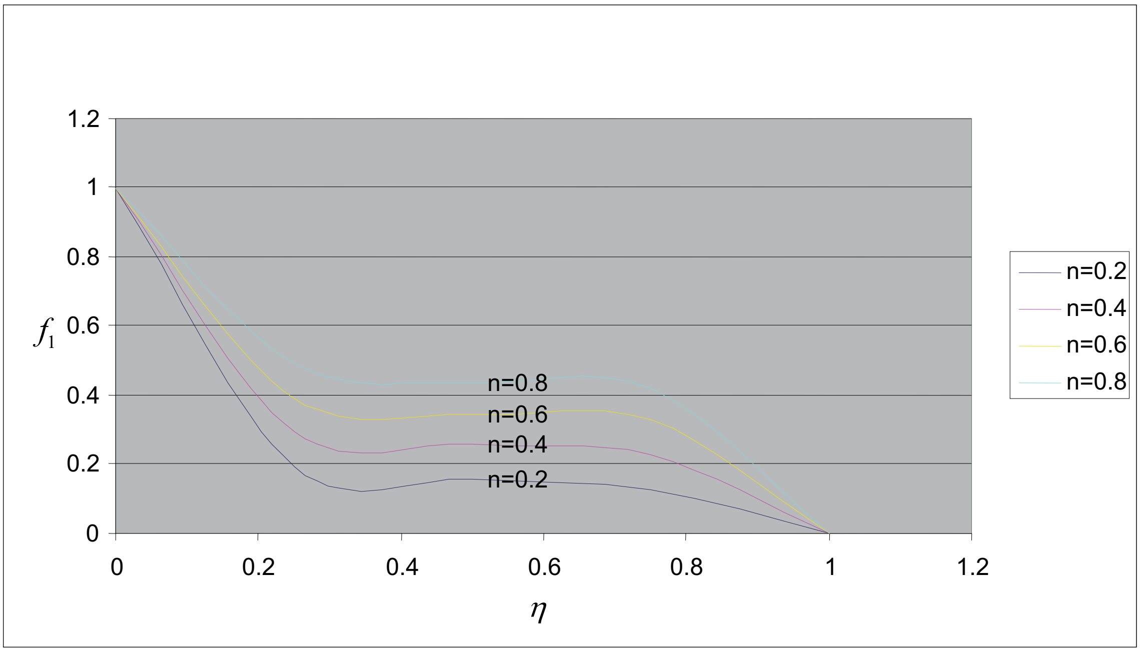

Fig.2. Velocity profiles for power-law case.

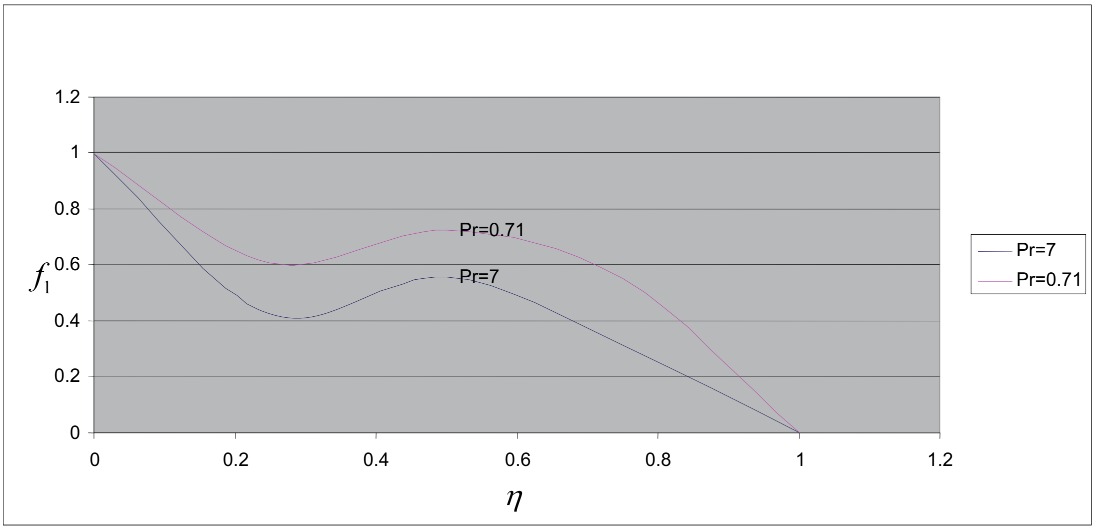

Fig.3a. Velocity profiles for Powell-Eyring case (Gr=*10*).

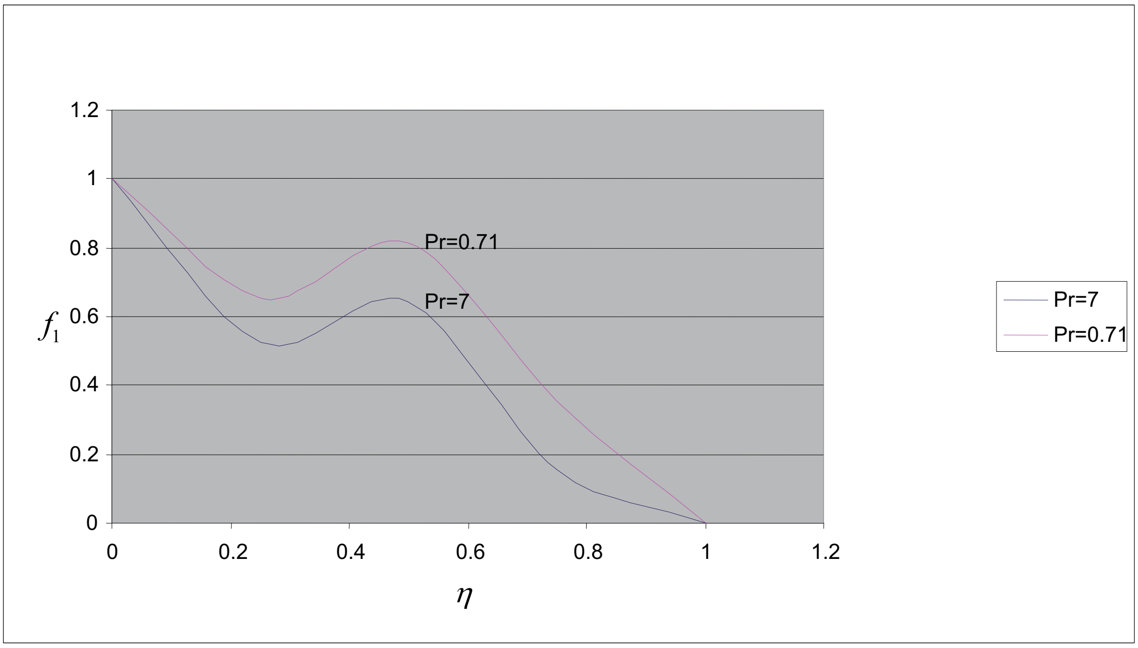

Fig.3b. Velocity profile for Powell-Eyring case (Gr=*5*).

7. Conclusion

The important conclusion drawn from this analysis is that in comparison to other similarity techniques, the new group theoretic generalized dimensional analysis relaxes the restriction on the main stream velocity which finally leads to the class of similarity solutions of the problem.

The application of the spline collocation method to the set of non-linear differential equations leads to the solution of linear algebraic equations. Very compact solutions are involved here. The convergence of the method is found to be very fast as the results presented in all the tables are obtained by successive four approximations. This type of simplicity in application justifies a wide use of the method. Another significant feature of this method is that there is no restriction on the domain, which affects the convergence of the method. It is observed that the value of the function can be obtained at any knots of the domain of boundary value problems, which is the advantage of the method. Thus the cubic spline collocation method with quasilinearization is quite versatile and effective to deal with nonlinear boundary value problems in different fields of applied mathematics.

Nomenclature

Cp – specific heat at constant pressure
e_{ij} – strain rate component
F – arbitrary function
Gr – Grashof number
K – thermal conductivity
Pr – Prandtl number
U, W – main stream velocities in X and Z directions
U_0 – constant with the dimension of velocity
$U(t)$ – dimensionless free stream velocity
u, v, w – velocity components in X, Y, Z directions respectively
η – similarity variable
θ – dimensionless temperature
μ – viscosity
ρ – density of the fluid
τ_{ij} – stress component
τ_{yx} – stress tensor in the direction of X-axis perpendicular to Y-axis
τ_{yz} – stress tensor in the direction of Z-axis perpendicular to Y-axis
ψ – stream function

References

Bickley W.G. (1968): *Piecewise cubic interpolation and two-point boundary problems*. – Comput. J., vol.11, pp.206-208.

Bird R.B., Stewart W.E. and Lightfoot F.W. (1960): *Transport Phenomena*. – John Wiley.

Cheng P. and Minkowycz W.J. (1977): *Free convection about a vertical plate embedded in a porous medium with application to heat transfer from a dike*. – J. Geophys. Res, vol.82, pp.2040-2044.

De Boor C. (1978): *A Practical Guide to Splines.* – New York: Springer. Verlag.

Dunn M.W. (1999): *Non-Newtonian fluid flow through fabrics.* – National Textile Center Annual Report: November, M98-P02.

Hansen A.G. and Na T.Y. (1968): *Similarity solutions of laminar, incompressible boundary layer equations of non-Newtonian fluids.* – Journal of Basic Engineering, vol.90, pp.71-74.

Kapur J.N. (1963): *Note on the boundary layer equations for power-law fluids*. – J. Phys. Soc. Japan, vol.18, pp.144.

Metzner A.B. (1965): *Heat transfer in non-Newtonian fluid*. – Adv. Heat Transfers, vol.2, pp.357-397.

Moran M.J. and Murshek K.M. (1972): *Some matrix aspects of generalized dimensional analysis.* – J. Engg. Maths., vol.6, p.291.

Morgan A.J.A. (1952): *The reduction by one of the number of independent variables in some systems of partial differential equations*. – Quart, J. Math, Oxford, vol.2, p.250.

Muthucumaraswamy R. and Shankar M.R. (2011): *First order chemical reaction and thermal radiation effects on unsteady flow past an accelerated isothermal infinite vertical plate*. – Indian Journal of Science and Technology, vol.4, No.5.

Muthucumaraswamy R., Lal T. and Ranganayakulu D. (2011): *Rotation effects on MHD flow past an accelerated vertical plate with variable temperature and uniform mass diffusion. Analysis of faculty engineering hunedoara.* – International Journal of Engineering, vol.9, p.1.

Nakayama A. and Koyama H. (1988): *An analysis for friction and heat transfer characteristics of power-law non-Newtonian fluid flows past bodies of arbitrary geometrical configuration*. – Warme-und Stoffubertragung, vol.22, pp.29-37.

Neossi Nguetchue S.N., Abelman S. and Momoniat E. (2009): *Symmetries and similarity solutions for the axisymmetric spreading under gravity of a thin power-law liquid drop on a horizontal plane*. – Applied Mathematical Modeling, vol.33, pp.4364–4377.

Patel M. and Timol M.G. (2004): *On the class of similarity solutions for three-dimensional boundary layer flows of non-Newtonian fluids*. – Journal of Veer Narmad South Gujarat University, vol.2B, pp.103-109.

Patel M. and Timol M.G. (2005): *Similarity solutions for three-dimensional, laminar, incompressible boundary layer equation of non-Newtonian fluids by generalized dimensional analysis*. – Varahmihir Journal of Mathematical Sciences, vol.5, No.2, pp.387-394.

Patel M. and Timol M.G. (2008): *Similarity solutions of three-dimensional boundary layer Small-cross flows of power law fluid*. – International Journal of Mathematical Sciences and Engineering Application (IJMSEA), vol.2, No.1, pp.167-174.

Patel M. and Timol M.G. (2009): *Numerical solution of the equation for unsteady boundary layer flow of non-Newtonian fluids past semi-infinite plate*. – International Journal of Applied Mechanics and Mathematics (IJAMM), vol.5, No.3, pp.22-29.

Patel M. and Timol M.G. (2009): *Numerical treatment of Powell–Eyring fluid flow using method of satisfaction of asymptotic boundary conditions (MSABC)*. – J. of Applied Numerical Mathematics, Applied Numerical Mathematics, vol.59, pp.2584–2592 (Elsevier).

Patel M. and Timol M.G. (2010): *The general stress–strain relationship for some different visco-inelastic non-Newtonian fluids*. – International Journal of Applied Mechanics and Mathematics (IJAMM), vol.6 (12), pp.79-93.

Prenter P.M. (1975): *Spline and Variational Methods*. – New York: John Wiley and Sons.

Sharma P.R. and Mathur P. (1995): *Steady laminar free convection flow of an electrically conductiong fluid along a porous hot vertical plate in the presence of heta source/sink*. – Indian J. Pure and Appl. Math., vol.26(11), pp.1125-1134.

Skelland A.H.P. (1967): *Non-Newtonian Flow and Heat Transfer*. – John Wiley.

Soundalgekar V.M. and Pop I. (1980): *Flow past an accelerated vertical infinite plate in the presence of free convection currents*. – Reg. J. Energy Heat Mass Trans, vol.2.2, pp.127.

Sun W. (1998): *Fast algorithms for high-order spline collocation systems*. – Numer. Math., vol.81, pp.143-160.

Surati H.C. and Timol M.G. (2010): *Numerical study of force convection wedge flow of some non-Newtonian fluids.* – Int. J. of Appl. Math and Mech., vol.6 (18), pp.50-65.

Timol M.G., Doctor H.D. and Kalthia N.L. (1987): *Spline solution of magneto hydrodynamic flow of non-Newtonian fluids.*-Proc. of 5th. – Int. Conf. Num. Methods in Laminar and Turbulent flows, Pine ridge Press, U.K., p.1217, (Canada conf.).

Usmani R.A. (1992): *The use of quartic splines in the numerical solution of a fourth-order boundary-value problem.* – J. Comput. Appl. Math, vol.44, pp.187-199.

Wilkinson W.L. (1960): *Non-Newtonian Fluids*. – Pergamon.

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