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Int. J. of Applied Mechanics and Engineering, 2013, vol.18, No.1, pp.249-257
DOI: 10.2478/ijame-2013-0016

Brief note

THE TWO-PHASE HELL-SHAW FLOW: CONSTRUCTION OF AN EXACT SOLUTION

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We consider a two-phase Hele-Shaw cell whether or not the gap thickness is time-dependent. We construct an exact solution in terms of the Schwarz function of the interface for the two-phase Hele-Shaw flow. The derivation is based upon the single-valued complex velocity potential instead of the multiple-valued complex potential. As a result, the construction is applicable to the case of the time-dependent gap. In addition, there is no need to introduce branch cuts in the computational domain. Furthermore, the interface evolution in a two-phase problem is closely linked to its counterpart in a one-phase problem

Key words: Laplace growth, Hele-Shaw cell, Schwarz function, free boundary problem.

1. Introduction

The Hele-Shaw cell is an apparatus used to study plane flow problems, e.g., the porous medium which is familiar in the oil industry. In particular, two immiscible fluids of different viscosities are confined between a pair of parallel plates which are separated by a narrow gap. Consider the Hele-Shaw cell in which a viscous fluid droplet such as oil, is surrounded by a less viscous fluid (e.g., air or water) of constant physical quantities. The sharp interface between the viscous fluid domain $\Omega(t)$ and its complement moves due to injection (suction) of oil through an inlet in the middle of one of the plates, with a boundary velocity proportional to a gradient of scalar field, i.e., the pressure

$$\nu = -k\nabla p = \frac{d\partial\Omega(t)}{dt} \tag{1.1}$$

where $k = h^2/12\mu$ is the fluid pearmability, h is the gap width and μ is the viscosity. Assume that the fluid is incompressible. Therefore, the pressure is harmonic

$$\Delta p = Q\delta(x,y) \qquad \text{in} \qquad \Omega(t). \tag{1.2}$$

Here, $\delta(x,y)$ is the Dirac distribution supported at a set of points and/or curves and located strictly within $\Omega(t)$ for any time, and Q is a constant strength (Gustafsson *et al*., 2006 and Savina *et al*., 2011). As such, this pattern is classified under the Laplacian growth which governs a variety of physical and mathematical applications such as solidification, viscous fingering and the growth of bacterial colonies (Oust, 2009). In addition to Eq.(1.2), we stress that the boundary $\partial\Omega(t)$ is equipotential, i.e., an interface satisfying the boundary condition (neglecting the surface tension)

$$p=0 \qquad \text{on} \qquad \partial\Omega(t). \tag{1.3}$$

Furthermore, the normal velocity of the boundary coincides with the normal velocity of the fluid at that boundary

$$\mathrm{v}_n = -\frac{\partial p}{\partial n} \qquad \text{on} \qquad \partial\Omega(t)\,. \tag{1.4}$$

The structure of this paper is organized as follows: we have some preliminary results in Section 2. We derive the governing equation for the two-phase Hele-Shaw cell in terms of a single-valued velocity potential in Section 3. We confirm the validity of the derived equation to the case when the gap width is a function of time, but not of position in Section 4. Examples are considered in Section 5, and a conclusion is given in Section 6.

2. Mathematical description

In this section we discuss several useful properties of the Schwarz function of the interface between a fluid domain and its complement (David, 1974). Consider an interface $\Gamma(t)$ with a Schwarz function $S(z,t)$ which is a unique analytic function in the neighborhood of $\Gamma(t)$ and $\bar{z} = S(z,t)$ on $\Gamma(t)$, where $z = x + iy$. First, let $d\tau$ be an element of the curve; we write the unit tangent vector, $dz/d\tau$, in terms of the Schwarz function as follows

$$d\tau = \sqrt{(dx+idy)(dx-idy)} = \sqrt{dz\,\overline{dz}} = \sqrt{dz\,S_z(z)dz}\,.$$

Therefore,

$$\frac{dz}{d\tau} = \sqrt{\frac{1}{S_z(z)}}\,, \qquad \frac{d\bar{z}}{d\tau} = \sqrt{S_z(z)}\,. \tag{2.1}$$

Second, the normal velocity of the boundary is expressed in terms of the Schwarz function (Howison, 1992)

$$\mathrm{v}_n = -i\frac{S_t}{\sqrt{S_z}}\,. \tag{2.2}$$

Third, it follows immediately from Eq.(1.2) that for each instance of time, there is a multiple-valued function $w(z,t)$ defined on $\Omega(t)$ in which $\mathrm{Re}(w) = p$ satisfies the Dirchlet boundary conditions (1.3) and (1.4). The Cauchy-Riemann equations in $(n,\tau)-$ coordinates together with the boundary conditions and (2.2) yield the following theorem.

Theorem 2.1. (Howison,1992) There exists a multiple-valued analytic functions $w(z,t)$ defined in the neighborhood of $\partial\Omega$ satisfying the equation

$$\frac{\partial w}{\partial z} = \frac{\partial \tau}{\partial z}\frac{\partial w}{\partial s} = \sqrt{S_z}\left(\frac{\partial p}{\partial \tau} - i\frac{\partial p}{\partial n}\right) = -\frac{1}{2}S_t\,, \tag{2.3}$$

whose real part $\mathrm{Re}(w) = -p$ is defined by Eqs (1.1)-(1.4).

Here, the function $w(z,t)$ is called the complex potential while its derivative is called the complex velocity potential. Equation (2.3) reduces the elliptic free boundary problem (1.2)-(1.4) to a simple dynamic description of singularities of the Schwarz function (Lundberg, 2010). Next, we derive an analogous equation for the case of the two-phase problem.

3. The Schwarz function equation for the two-phase problem

The two-phase displacement in a Hele-Shaw cell is a motion of an interface $\Gamma(t)$ between two viscous fluids. For each instant of time, the computational domain $\Omega(t)$ is divided into a blob of viscous fluid $\Omega_1(t)$ which is imbedded into another infinite viscous fluid $\Omega_2(t)$, i.e., $\overline{\Omega_1(t)}\cup\overline{\Omega_2(t)}=\overline{\Omega(t)}$, $\overline{\Omega_1(t)}\cap\overline{\Omega_2(t)}=\Gamma(t)$ and $\Omega_1(t)\cap\Omega_2(t)=\varnothing$. The dynamics of the fluid in $\Omega_i(t)$, where *i=1, 2*, is described by the equation of state for the pressure, i.e., Darcy law,

$$\nu_i = -k_i \nabla p_i\,, \tag{3.1}$$

$$\Delta p_i = 0\,, \qquad \text{in} \qquad \Omega_i(t)\,, \tag{3.2}$$

$$p_1 = p_2\,, \qquad \text{on} \qquad \Gamma(t)\,, \tag{3.3}$$

$$\nu_n = -k_1 \frac{\partial p_1}{\partial n} = -k_2 \frac{\partial p_2}{\partial n}\,, \qquad \text{on} \qquad \Gamma(t)\,. \tag{3.4}$$

Equation (3.3) indicates the continuity of pressures at the interface while Eq.(3.4) indicates that a particle of the fluid on the interface remains constantly throughout the period of the motion.

The first exact solutions to unsteady bubble in the two-phase Hele-Shaw cell was constructed by Howison (2000), but it was restricted in its scope. Specifically, the fluids' permeabilities are alike, i.e., the computational domain, $\overline{\Omega(t)}=\overline{\Omega_1(t)}\cup\overline{\Omega_2(t)}$, is occupied by a single fluid with a different color in each subdomain, i.e., a multiply-colored fluid. The general form of this construction was developed by Crowdy (2006) in terms of a multiply-valued complex potential. However, a proper choice of branch cuts was significant toward the later derivation. In the following theorem, we construct an analogous equation in terms of the Schwarz function of the interface based upon the single-valued complex velocity potential.

Theorem 3.1. There exist multiple-valued analytic functions $w_i(z,t)$, where *i=1, 2*, defined in the neighborhood of $\partial\Omega$ satisfying the equation

$$\frac{\partial w_2}{\partial z} = \frac{\mu_1}{\mu_2}\frac{\partial w_1}{\partial z} + \frac{1}{2}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{\partial S}{\partial t}\,, \tag{3.5}$$

whose real part $\mathrm{Re}(w_i) = -k_i p_i$ is defined by Eqs (3.1)-(3.4).

Proof. Let τ be a parameterization along the interface, the tangent velocity of the interface, $\nu_t = \nu_i \cdot z_\tau$, where ν_i is the velocity vector defined in Eq.(3.1) and $z_\tau = dz/d\tau$ is the unit tangent vector. Consider a point of the interface $z = z(t)$; the complex conjugate of the velocity filed is (see, for example, Currie, 1974)

$$\overline{\nu_\iota} = \frac{d\bar{z}}{dt} = \frac{\partial w_i}{\partial z}\,. \tag{3.6}$$

In addition, we have $\bar{z} = S(z,t)$ on the interface so that

$$\overline{\nu_\iota} = \frac{d\bar{z}}{dt} = S_z \nu_i + S_t\,. \tag{3.7}$$

The scalar product of two complex numbers A and B, regarded as vectors, is $\mathrm{Re}\left(\bar{A}\cdot B\right)$. Thus, from Eq.(3.6) it follows that

$$\nu_\tau = \mathrm{Re}\left(\frac{\partial w_i}{\partial z} z_\tau\right). \tag{3.8}$$

Equation (3.1) indicates that

$$\nu_\tau = -{}_{1}\frac{\partial p_i}{\partial_\tau}\,. \tag{3.9}$$

Therefore,

$$-\frac{1}{2_1}\left[\frac{\partial w_1}{\partial z} z_\tau + \overline{\frac{\partial w_1}{\partial z} z_\tau}\right] = \frac{\partial p_1}{\partial_\tau}\,, \tag{3.10}$$

and

$$-\frac{1}{2_2}\left[\frac{\partial w_2}{\partial z} z_\tau + \overline{\frac{\partial w_2}{\partial z} z_\tau}\right] = \frac{\partial p_2}{\partial_\tau}\,. \tag{3.11}$$

Subtraction yields the relation

$$-\frac{1}{2_1}\left[\frac{\partial w_1}{\partial z} z_\tau + \overline{\frac{\partial w_1}{\partial z} z_\tau}\right] + \frac{1}{2_2}\left[\frac{\partial w_2}{\partial z} z_\tau + \overline{\frac{\partial w_2}{\partial z} z_\tau}\right] = 0$$

where Eqs (3.3) has been used. It follows immediately from Eqs (2. 1) that

$$-\frac{1}{2_1}\left[\frac{\partial w_1}{\partial z} + \overline{\frac{\partial w_1}{\partial z}} S_z\right] + \frac{1}{2_2}\left[\frac{\partial w_2}{\partial z} + \overline{\frac{\partial w_2}{\partial z}} S_z\right] = 0\,. \tag{3.12}$$

Substitution of Eq.(3.7) into Eq.(3.12) finishes the proof.
Equation (3.5) is defined in the neighborhood of the interface, and it might be analytically continued elsewhere. The result relates the potential on one side of the interface to its counterpart on the other side. In other words, having the solution of the interior problem, it is straightforward to calculate the solution on its complement.

4. Time-dependent gap two-phase Hele-Shaw cell

Consider a two-phase Hele-Shaw cell in which the gap width is a function of time, but not of a position. The time-dependence arises when the upper plate is lifted at a uniform rate. This model was studied by Shelley *et al.* (1997) and its generalization to multiply-connected domain was developed by Kang *et al.* (2001). The relevant mathematical formulation is identical to that of the regular Hele-Shaw problem, except for the continuity equation. In particular, averaging the divergent-free three dimension velocity field (incompressible fluid) across the gap, we have

$$0=\frac{1}{h(t)}\int_0^{h(t)} u_x+\nu_y+w_z dz=\overline{u_x}+\overline{\nu_y}+\frac{\left(w\big|_{z=h(t)}-w\big|_{z=0}\right)}{h(t)}. \tag{4.1}$$

Indeed, at the height $z=h(t)$ across the gap, where z is the third component in (x,y,z)-coordinates, the corresponding velocity component is

$$0=d\left(z-h(t)\right)/dt=w-dh(t)/dt\,. \tag{4.2}$$

Due to the no-slip condition, we have $w\big|_{z=0}=0$. Substitution of (4.2) into (4.1) yields the modified continuity equation

$$\nabla\nu_i=-\frac{\dot{h}(t)}{h(t)},\qquad i=1,2 \tag{4.3}$$

where ν_i is the averaged velocity field in the xy−plane. The dynamics of the fluid is described by Darcy law

$$\nu_i=-k_i\nabla p_i\,,\qquad \text{in}\qquad \Omega_i(t), \tag{4.4}$$

thus,

$$\Delta p_i=\frac{1}{k_i(t)}\frac{\dot{h}(t)}{h(t)},\qquad \text{in}\qquad \Omega_i(t). \tag{4.5}$$

Equations (4.5) is complement by boundary conditions

$$p_1=p_2\qquad \text{on}\qquad \Gamma(t), \tag{4.6}$$

$$\nu_n=-k_1\frac{\partial p_1}{\partial n}=-k_2\frac{\partial p_2}{\partial n}\qquad \text{on}\qquad \Gamma(t). \tag{4.7}$$

It is evident from Eqs (3.1) and (4.4) that the component of the velocity vector is defined as the gradient of a scalar velocity potential function regardless of whether or not the gap width is time-dependent. In general, the directional velocity is obtained by taking the directional derivative of the velocity potential along that direction. Therefore,

$$\operatorname{Re}\left(\bar{\nu}_\iota \cdot z_\tau\right) = \nu_i \cdot \tau = \nu_\tau = -k_i\left(t\right)\frac{\partial p_i}{\partial \tau} \tag{4.8}$$

where τ is an arclength along the interface and ν_τ is the tangent velocity. Starting from Eqs (4.6) and following the proof of theorem 3.1 yields the next theorem.

Theorem 4.1. Consider an interface $\Gamma(t)$ that moves between two viscous fluids in a narrow and uniform gap at a variable distance. The velocity field of the interior phase is directly related to the exterior as follows

$$\overline{\nu_2} = \frac{\mu_1}{\mu_2}\overline{\nu_1} + \frac{1}{2}\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{\partial S}{\partial t}. \tag{4.9}$$

From Eq.(2.3) it follows that there is a one-phase Hele-Shaw flow whose evolution is identical to that of $\Gamma(t)$ (Cummings *et al*., 1999 and Howison, 2000). As such,

$$\overline{\nu_0} = \frac{\partial w_0}{\partial z} = -\frac{1}{2}S_t, \tag{4.10}$$

here, ν_0 is a velocity component defined as the gradient of an arbitrary harmonic scalar

$$\Delta\varphi_0 = 0, \tag{4.11}$$

with boundary conditions

$$\varphi_0 = 0, \qquad \nu_n = \frac{\partial \varphi_0}{\partial n} \qquad \text{on} \qquad \Gamma(t).$$

Indeed, the one-phase problem is closely linked to the two-phase case. Substitution of Eqs (4.4) and (4.10) into (4.9) indicates that

$$\nabla\left(p_1 - p_2\right) + \left(\frac{1}{_1(t)} - \frac{1}{_2(t)}\right)\nabla\varphi_0 = 0. \tag{4.12}$$

Equation (4.10) holds on the interface. From Eq.(4.11) it follows that φ_0 might be analytically continued so that apart from the singularities and the branch cuts of φ_0 and $p_1 - p_2$, Eq.(4.10) is defined in the entire computational domain. Away from the boundary, $p_1 - p_2$ is the analytic continuation of φ_0, and thus $p_1 - p_2$ is harmonic. Equation (4.5) implies

$$\left(\mu_1 - \mu_2\right)h^{\cdot}\left(t\right) = 0. \tag{4.13}$$

Since our model is concerned with fluids of different viscosities, we have $h^{\cdot}\left(t\right) = 0$ which constitutes the velocity component across the gap generated due to changing the gap between plates. As such, the incompressibility condition remains unchanged when the gap width is a function of time.

5. Examples

5.1. Ellipse injected off the branch cut

Let $\Omega = \left\{ \frac{x^2}{a^2(t)} + \frac{y^2}{b^2(t)} \le 1 \right\}$ be an elliptic domain with the half-axis $a(t)$ and $b(t)$, where $a(0) > b(0)$. The Schwarz function of $\partial\Omega$ is

$$S(z,t) = \frac{a^2(t)+b^2(t)}{d^2(t)} z - \frac{2a(t)b(t)}{d^2(t)} \sqrt{z^2 - d^2(t)} \tag{5.1}$$

where $d = \sqrt{a^2 - b^2}$ is half of the interfocal distance. Thus, $S(z,t)$ has a square root branch along the segment joining $\pm\sqrt{a^2 - b^2}$. Differentiating the Schwarz function with respect to time we have

$$\frac{\partial S}{\partial t} = z \frac{\partial}{\partial t}\left(\frac{a^2 + b^2}{d^2} \right) - \sqrt{z^2 - d^2} \frac{\partial}{\partial t}\left(\frac{2ab}{d^2} \right) + \frac{1}{\sqrt{z^2 - d^2}} \frac{2ab}{d^2} \frac{\partial}{\partial t}\left(d^2 \right). \tag{5.2}$$

Consider a point source of constant strength, $p \sim Q \log|z - z_0|$ where $p = \mathrm{Re}(w_1)$, located off the branch cut, the corresponding complex velocity is

$$\frac{\partial w_1}{\partial z} = -\frac{Q}{z - z_0}. \tag{5.3}$$

Substitution of (5.2) and (5.3) into Eq.(3.5) yields the complex velocity potential of the exterior phase

$$\frac{\partial w_2}{\partial z} = -\frac{\mu_1}{\mu_2} \frac{Q}{z - z_0} + \frac{1}{2}\left(1 - \frac{\mu_1}{\mu_2} \right) \left[z \frac{\partial}{\partial t}\left(\frac{a^2 + b^2}{d^2} \right) - \sqrt{z^2 - d^2} \frac{\partial}{\partial t}\left(\frac{2ab}{d^2} \right) + \frac{1}{\sqrt{z^2 - d^2}} \frac{2ab}{d^2} \frac{\partial}{\partial t}\left(d^2 \right) \right].$$

This complex velocity is analytic in the infinite fluid, and the far field flow consists of a stagnation point along with points sink. This example has been studied by Richardson (1992) for the case of a one-phase problem.

5.2. Parabola

Suppose the Γ is the curve described algebraically by

$$x = \frac{-a(t)}{2} y^2 + \frac{1}{2a(t)}, \tag{5.4}$$

assuming that $a(t)$ has real positive value. The boundary Γ is approximately parabolic as $x \to -\infty$. Performing the change of variables $= x + iy, \bar{z} = x - iy$, and then solving for $\bar{z}$, we have the corresponding Schwarz function

$$S(z,t) = \frac{a(t)z + 2 + 2\sqrt{2a(t)z}}{a(t)}. \tag{5.5}$$

Thus $S(z,t)$ has a square root branch cut along the negative real axis. Away from the negative real axis, i.e., the axis of symmetry of the parabola, consider a point source of constant strength. Differentiating the Schwarz function with respect to time we have

$$\frac{\partial S}{\partial t} = \left(2 + 2\sqrt{2a(t)z}\right)\frac{\partial}{\partial t}\left(\frac{1}{a(t)}\right) + \frac{\dot{a}(t)}{2a(t)\sqrt{2a(t)z}}. \tag{5.6}$$

The dynamic of the fluid in the other side of Γ is obtained by Eq.(3.5) as follows

$$\frac{\partial w_2}{\partial z} = -\frac{\mu_1}{\mu_2}\frac{Q}{z - z_0} + \frac{1}{2}\left(1 - \frac{\mu_1}{\mu_2}\right)\left[\left(2 + 2\sqrt{2az}\right)\frac{\partial}{\partial t}\left(\frac{1}{a}\right) + \frac{\dot{a}}{2a\sqrt{2az}}\right].$$

The complex potential is analytic in the relevant domain. This example is considered by Lacey (1982) for the case when Ω_2 is occupied by frictionless fluid.

6. Conclusion

We formulated the Schwarz function equation for the case of a two-phase Hele-shaw flow. The main advantage of the derivation was demonstrated by producing exact solutions to an immiscible two-phase flow in a Hele-Shaw cell with a time-dependent gap. Therefore, the incompressibility condition remains unchanged for the case of time-dependent gap. In addition, there is no need to introduce branch cuts in the computational domain. Furthermore, the derived equation is directly related to the explicit solutions of a one-phase Hele-Shaw flow.

Acknowledgement

I would like to express my gratitude to the Government of Saudi Arabia for providing me financial support and Taibah University for granting me academic leave for higher studies at Ohio University.

Nomenclature

h – gap width of the Hele-Shaw cell
k – fluid permeability
p – scalar field, i.e., pressure
S – Schwarz function
w – complex potential
ν – velocity field
μ – viscosity

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Received: May 1, 2012

Revised: August 2, 2012