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Brief note

FORMING OF THE MOST CONVENIENT BENT CONSTRUCTIONAL ELEMENTS WITH A PERMISSIBLE STRENGTH GIVEN

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In the present study, the limiting values are determined of the criteria quantities of optimal forming of the most convenient bent supporting structure for the case of static loads in the range of the Hooke's law applicability. As the criterion of the most convenient constructional element, the following were accepted: the smallest length of the activity of internal forces as well as the equal potential and the gradient of the potential energy of elastic deformation at each point of the constructional element.

Key words: the most optimal structure, supporting construction, the length of the activity of the internal forces.

1. The most convenient supporting structure at bending

In the construction of technical machinery and devices, the material consumption index and rigidity are among the most important criteria that determine the quality of a machine. With an average and mass production of a machine and its assemblies, the relation between the mass and rigidity is of a particular importance due to economic reasons as well as constructional and technological considerations. In technological machines, the relation between the mass and rigidity has an impact on the static and dynamic strength of machines, the dynamics of the kinematic system, the forms of mechanical vibrations and the frequencies of free vibrations and, above all, on the quality of machining.

In devices with an open or closed kinematic system, e.g. in the structures of industrial robots, the relation between the mass and rigidity has an influence on the accuracy of the movement trajectory of movable elements, the accuracy of positioning as well as on the time required to obtain the stationary position of the coordinates of the robot's grip. From a given quantity of the constructional material, the supporting construction can be formed in indefinitely many ways, also external active forces can be transferred to the points of the constructional support of a part or the entirety of the machine. As it is evident from the conditions of the criteria of the formation of the most convenient construction, the machine or its constructional element needs to be formed in such a manner that the transfer of active external loads (the generalized forces of external and technological interactions) and passive loads (supports) could occur at the smallest length of the activity of the internal forces of the supporting construction, and that the elastic strain energy potential could be the same at each point of the construction. Traditional optimization is oriented towards an equalization of stresses (Steven *et al*., 2002; Xie and Steven, 1997). During bending, the elastic strain energy potential in the volume of the element bent is not the same. With such a load, the rigidity criterion will be accepted as the total potential deformation energy of the beam and the rate of a change to the gradient of the relative potential volume deformation energy. A construction that fulfills the abovementioned criteria will be the most convenient construction made from a given quantity of material with specific functional properties.

The criterion of the length of the activity of the internal forces of a constructional element or the entire construction will be quantitatively determined from the integral of the function of bending stresses, for volume $V_0=const$ (Fligiel, 2012; 2013; 2002)

$$Q = \int_{v_0} \sigma_g dV = \min . \tag{2.1}$$

In the present study, the limiting values are determined of the criteria quantities of an optimal formation of the most convenient bendable supporting structure. In order to determine the limiting criteria values of the most convenient supporting construction, permissible bending stresses k_g and volume were accepted $V_0=const$. In the study (Fligiel, 2013), aspects were considered of an optimal formation of the most convenient constructional element for the case of straight bending. It was found based on the analysis carried out that with straight bending of a rectilinear beam, one that is slightly curved and with a constant section, the criteria values are always minimal; therefore, such an element is the most convenient for constructional considerations, while bendable beams with a variable section do not fulfill the criteria of the most convenient supporting construction.

2. The most convenient beam with permissible strength and constant section

2.1. Length of the activity of the internal forces of fixed beam

Let us assume that the element to be bent is a fixed beam with constant length $l_1=l=const$ and rectangular section $b_1 \times h_1$ and volume $V_0=V_{01}=const$. The vector of force F acts vertically downwards as in Fig.1. The unsafe section of the beam is located in a fixing. Considering a small share of the deformation energy of shearing forces in the balance of energy accumulated, the deformation energy of shearing forces is neglected. For the presented load, the type of support in the unsafe section, the height of the beam determined from permissible stress k_g is $h_1 = \sqrt{\dfrac{6Fl}{b_1 k_g}} = const$, and the volume of the entire beam for $b_1=const$, $h_1=const$ and $l=const$ is $V_{01} = V_0 = \sqrt{\dfrac{6Fb_1 l^3}{k_g}}$.

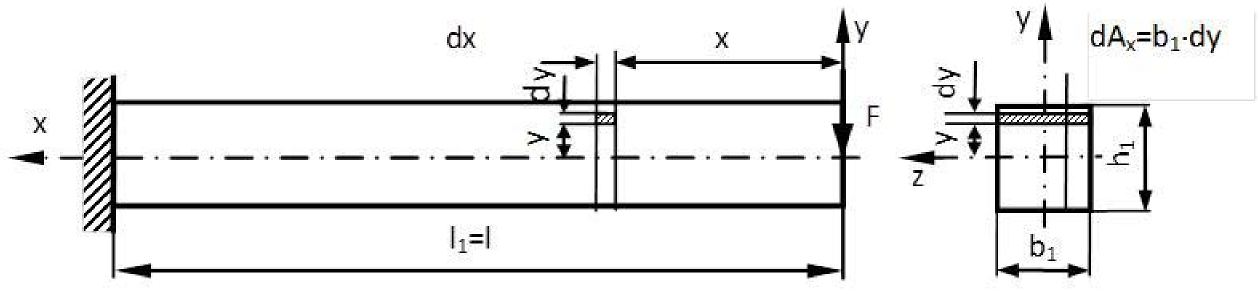

Fig.1. Diagram of the support and load of a beam with a constant cross-section.

The infinitesimal volume of the beam is $dV=b_1 \cdot dy \cdot dx$, and the stresses in the section with coordinates x and y have the value of $\sigma_{gx} = \dfrac{12Fyx}{b_1 h_1^3}$, hence the following integral (1.1)

$$Q_1 = 2\int\limits_{v_0} \left|\sigma_{gx}\right| dV \ . \tag{2.1}$$

The two above the sign of the integral (2.1) is the result of the signs of stresses, the upper fibres are stretched and lower fibres are compressed. Integral (1.1) for such a distribution of stresses and cross symmetry of the section in relation to the main central inertia axes is equal to zero; from here, calculations follow related to the absolute value for one side of the section compressed or stretched.

By putting infinitesimal volume dV and stresses σ_{gx} into Expression (2.1), we will obtain

$$Q_1 = \frac{24F}{h_1^3}\int\limits_0^{\frac{h_1}{2}} ydy \int\limits_0^{l} xdx \, , \tag{2.2}$$

from where, after the calculation of Integral (2.2) and substituting $V_0 = V_{01} = \sqrt{\dfrac{6\,F\,b_1 l^3}{k_g}}$, we will obtain a formula for the length of the activity of internal forces

$$Q_1 = \frac{1}{2}\sqrt{\frac{3\,F\,b_1 l^3 k_g}{2}} \ . \tag{2.3}$$

2.2. Potential energy of the deformation of the beam with permissible strength

The inertia moment of the beam section taking into consideration its height that is determined from permissible stress k_g, is equal to $J_z = \sqrt{\dfrac{3\,F^3 l^3}{2\,b_1 k_g^3}}$, and stresses in the section with coordinates x and y are determined with the following function

$$\sigma_{gx} = \sqrt{\frac{2\,b_1 k_g^3}{3\,F\,l^3}}\ x\,y \, . \tag{2.4}$$

The relative volume potential energy of elastic deformation is

$$u_1 = \frac{\sigma_{gx}^2}{2E} = \frac{b_1\,k_g^3}{3\,E\,F\,l^3}\,x^2 y^2 \, , \tag{2.5}$$

and the total energy of the beam for bending moment $M_g = F \cdot x$

$$U_1 = \frac{1}{2\mathrm{E}}\int\limits_0^{l} \frac{M_g^2}{J_z}\,dx = \frac{1}{3\,E}\sqrt{\frac{F\,l^3 b_1 k_g^3}{6}} \ . \tag{2.6}$$

3. The most convenient beam with equal strength

3.1. Length of the activity of the internal forces of a beam with equal permissible strength

As the second element bent, a beam will be considered with an equal strength, presented in Fig.2. The beam is fixed, with constant length $l_2=l=const$ and rectangular section dimensions $b_2 \times h_2$ and volume $V_{02}=V_0=const$.

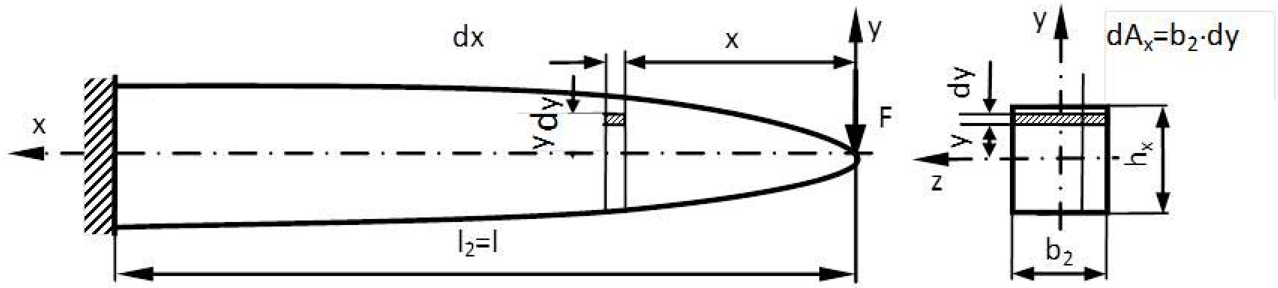

Fig.2. Diagram of beam bent with equal strength.

The beam height changes parabolically with coordinate x according to the function: $h_2 = h_x = \sqrt{\dfrac{6Fx}{b_2 k_g}}$. The infinitesimal volume of the beam is $dV=b_2 \cdot dy \cdot dx$, hence the whole volume

$$\mathrm{V}_{02} = \mathrm{b}_2 \int_0^l dx \int_{\frac{h_x}{2}}^{\frac{h_x}{2}} dy\,. \tag{3.1}$$

By substituting to (3.1) the height of the beam section $h_x = \sqrt{\dfrac{6Fx}{b_2 k_g}}$ and integrating, we will obtain an expression for the value of the total volume

$$V_{02} = \sqrt{\frac{8\,F\,b_2 l^3}{3k_g}}\,. \tag{3.2}$$

As we are considering a beam with a constant volume $V_0=V_{01}=V_{02}=const$, in order to determine an equivalent width b_2 of the section, we equate volumes $V_{01}=V_{02}$

$$\sqrt{\frac{6Fb_1l^3}{k_g}} = \sqrt{\frac{8\,F\,b_2 l^3}{3k_g}} \tag{3.3}$$

from there, we obtain the equivalent width of the beam with equal strength: $b_2=9 \cdot b_1/4$.

The inertia moment with coordinate x is equal to $J_z=b_2 h_x^3/12$, from where, after substituting h_x, we obtain the following

$$J_z = \sqrt{\frac{3\,F^3}{2\,b_2 k_g^3}}\, x^{\frac{3}{2}}\,. \tag{3.4}$$

Stresses in the section with coordinates *x* and *y* will be determined from the following dependence

$$\sigma_{gx} = \frac{F\,x\,y}{J_z}\,, \tag{3.5}$$

from where, after substituting to (3.5) the inertia moment (3.4), we obtain the function of change to stresses

$$\sigma_{gx} = \sqrt{\frac{2\,b_2 k_g^3}{3\,F}}\, x^{-\frac{1}{2}}\, y\,. \tag{3.6}$$

Further, by determining on the basis of Integral (1.1) the length of the activity of internal forces, we obtain the following

$$Q_2 = 2\int_{V_0} \left|\sigma_{gx}\right| dV = 2\sqrt{\frac{2\,b_2^3 k_g^3}{3\,F}} \int_0^l x^{-\frac{1}{2}} dx \int_0^{\frac{1}{2}\sqrt{\frac{6F_x}{b_2 k_g}}} y dy\,. \tag{3.7}$$

By integrating Expression (3.7), we obtain the following

$$Q_2 = \sqrt{\frac{2\,F\,b_2\,l^3 k_g}{3}}\;. \tag{3.8}$$

To compare the results of the length of the activity of forces Q_1 and Q_2 from Subsections 2.1 and 3.1, we will put into (3.8) width $b_2=9\cdot b_1/4$ that is an equivalent to constant volume V_0, from where we obtain the length of the activity of internal forces for a beam with an equal strength

$$Q_2 = \sqrt{\frac{3\,F\,b_1 l^3 k_g}{2}}\;. \tag{3.9}$$

By comparing Dependences (2.3) and (3.9), we can state that the length of the activity of the internal forces of the beam with an equal strength is greater, and it is as follows

$$Q_2 = 2\,Q_1\,, \tag{3.10}$$

and so the beam does not fulfill the minimum of the length of the activity of the forces as compared to the beam with a specific permissible strength k_g.

3.2. Potential energy of the deformation of a beam with equal permissible strength

The relative volume energy of an elastic strain is equal to $u_2 = \frac{\sigma_{gx}^2}{2E}$; from there, after taking Function (3.6) into consideration and after transformations, we receive the following

$$u_2 = \frac{b_2 k_g^3}{3EF} \frac{y^2}{x}. \tag{3.11}$$

In order to compare the results of the relative volume potential energy of deformation u_1 and u_2 from Sections 2.2 and 3.2, we will put width $b_2=9 \cdot b_1/4$ that is an equivalent of the constant volume V_0 into (3.11), hence

$$u_2 = \frac{3 b_1 k_g^3}{4EF} \frac{y^2}{x}. \tag{3.12}$$

Taking the energy (2.5) of the relative deformation from Subsection 2.2 into consideration in Eq.(3.12), we obtain the following dependence

$$u_2 = \frac{9 l^3}{4 x^3} u_1. \tag{3.13}$$

It is evident from analysis (3.13) that for $x < \sqrt[3]{\frac{9}{4}}\, l$, $u_2>u_1$; for $x > \sqrt[3]{\frac{9}{4}}\, l$, $u_2<u_1$ and for $x = \sqrt[3]{\frac{9}{4}}\, l = 1,3 \cdot l$, $u_2=u_1$. Because x may accept the values of $0<x \le l$, the relative volume energy of elasticity u_2 will always be greater than u_1.

The total potential energy of the deformation of an elastic beam for the bending moment $M_g=F \cdot x$ and the inertia moment of the section $J_z = \sqrt{\frac{3F^3}{2 b_2 k_g^3}}\, x^{\frac{3}{2}}$ is equal to

$$U_2 = \frac{1}{2E} \int_0^l \frac{M_g^2}{J_z} dx, \tag{3.14}$$

from there, after substituting M_g and J_z to Eq.(3.14) and after integration, we receive the following

$$U_2 = \frac{1}{3E} \sqrt{\frac{2 F l^3 b_2 k_g^3}{3}}. \tag{3.15}$$

By substituting the equivalent of constant volume V_0 with width $b_2=9 \cdot b_1/4$, to Eq.(3.15), we obtain the following

$$U_2 = \frac{1}{E}\sqrt{\frac{F\,l^3 b_1 k_g^3}{6}}\ . \tag{3.16}$$

It is evident from a comparison of Eqs (3.16) and (2.6) that the total potential energy of the elastic deformation of the beam with an equal strength and a beam with the same strength and a constant section is greater, and it is $U_2=3\cdot U_1$. A beam with an equal strength does not fulfill the second criterion of the most convenient structural element.

4. Analysis of the gradients of the relative volume energy of an elastic strain of beams

The gradient of the relative function of the volume potential energy of deformation Eqs (2.5) and (3.12) respectively is equal to

$$\nabla u_1 = \frac{b_1 k_g^3}{E F}\left[\left(\frac{2}{3l^3} x\,y^2\right)\boldsymbol{i} + \left(\frac{2}{3l^3} x^2\,y\right)\boldsymbol{j}\right], \tag{4.1}$$

$$\nabla u_2 = \frac{b_1 k_g^3}{E F}\left[\left(-\frac{3}{4}\frac{y^2}{x^2}\boldsymbol{i} + \frac{3}{2}\frac{y}{x}\right)\boldsymbol{j}\right], \tag{4.2}$$

hence the values of the vectors of gradients

$$\left|\nabla u_1\right| = \frac{b_1 k_g^3}{E F}\left(\frac{2}{3l^3} x\,y\sqrt{x^2+y^2}\right), \tag{4.3}$$

$$\left|\nabla u_2\right| = \frac{b_1 k_g^3}{E F}\left(\frac{3}{2}\frac{x}{y}\sqrt{\frac{x^2}{4y^2}+1}\right). \tag{4.4}$$

It is evident from an analysis of (4.3) and (4.4) that the components of the vectors of gradients are equal to one another respectively, along the x axis when coordinate $x=\sqrt[3]{-\frac{9}{8}}\ l$ and along the y axis when $y=\sqrt[3]{\frac{9}{4}}\ l$. This means that in the range of the permissible changes to the lengths of the beams $0<x\leq l$ and the heights of the section of the beams h_1 and h_2, the directions of the vectors of the gradients are not the same.

Figure 3 presents changes to the values of gradients (4.3) and (4.4) for the section of fixed beams (Fig.3a) and along fibres with coordinate $y=0.03\ m$ and $0\leq x\leq 1.2\ m$ (Fig.3b). On the grounds of Fig.3, it can be found that the rate of the change to the value of the gradient vector of the relative volume potential deformation energy of a beam with a given strength and a constant section is smaller than that of a beam with an equal strength. Hence, a beam with a given strength and a constant section is the beam that fulfills better the criteria condition of the most convenient constructional element.

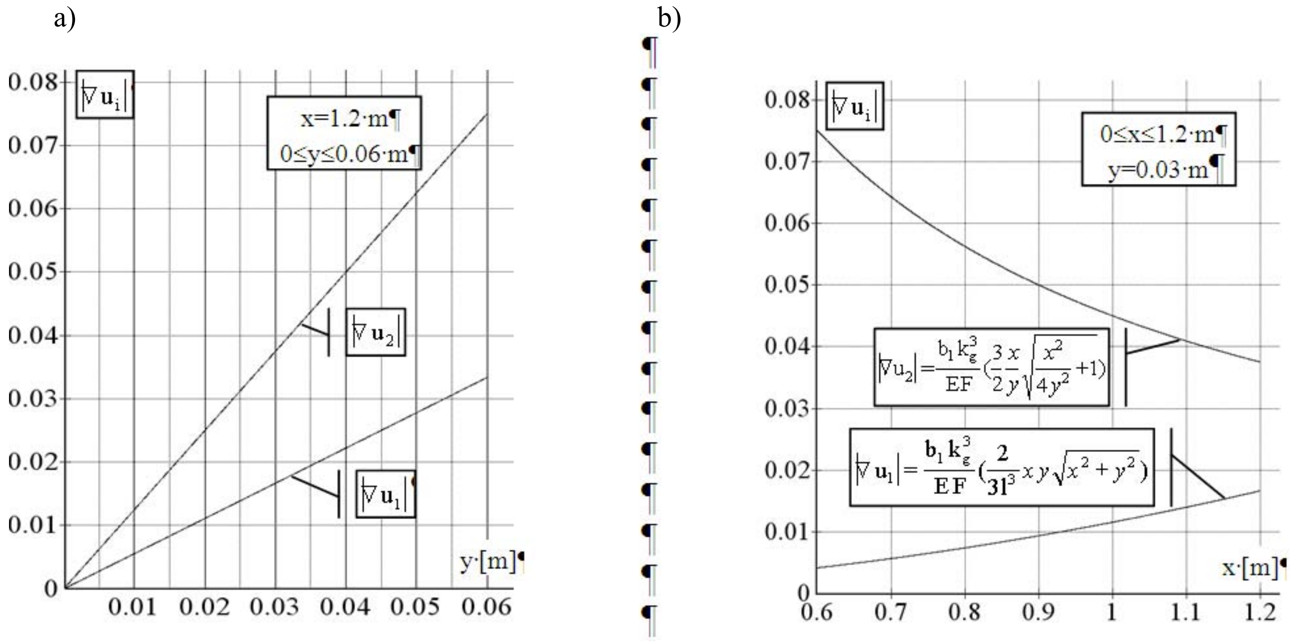

Fig.3. Dependence of the values of gradients ∇u_i: a) for section with coordinate *x=1.2 m*; b) along fibres with coordinate *y=0.03 m.*

5. Conclusions

Based on the theoretical discussions concerning straight beams or slightly curved beams with a rectangular section, a constant volume, and loaded with a force at their ends, it can be found as follows:

1. The length of the activity of the internal forces of a beam with an equal strength is greater, the beam does not fulfill the minimum of the length of the activity of forces as compared with a beam with a specified permissible strength.
2. The relative volume potential energy of the deformation of an elastic beam with an equal strength is greater than the relative volume energy of the deformation of a beam with a specified permissible strength.
3. The total potential energy of the deformation of an elastic beam with an equal strength is greater than the total energy of the deformation of a beam with a specified permissible strength.
4. The directions of the vectors of gradients are not the same, and the rate of a change to the value of the vectors of the gradients of the relative volume potential energy of the deformation of an elastic beam with a strength given and a constant section is smaller than that of a beam with an equal strength.
5. Beams with a strength given and a constant section are those beams that fulfill the criteria conditions of the most convenient constructional element.

Nomenclature

A_x - cross-sectional area of the beam
b_1, b_2 - width of the rectangle
E – first material constant (Young module)
F - bending force
h_1, h_2,h_x - height of the rectangle
J_z - the inertia moment of the beam

k_g - permissible stresses
l, l_1, l_2 - length of the beam
M_g - bending moment
U_1, U_2 – the total elastic energy of the bending moment
u_1, u_2 – the relative volume of potential energy of the elastic deformation
Q, Q_1, Q_2 – the length of the activity of the internal forces
V, V_{01}, V_{02} – volume of construction
x, y, z - coordinates of the construction
σ_g, σ_{gx} – bending stresses
$\nabla u_1 \nabla u_2$ - gradient of relative potential energy of deformation

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