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MHD TWO-LAYERED UNSTEADY FLUID FLOW AND HEAT TRANSFER THROUGH A HORIZONTAL CHANNEL BETWEEN PARALLEL PLATES IN A ROTATING SYSTEM

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An unsteady magnetohydrodynamic (MHD) two-layered fluids flow and heat transfer in a horizontal channel between two parallel plates in the presence of an applied magnetic and electric field is investigated, when the whole system is rotated about an axis perpendicular to the flow. The flow is driven by a constant uniform pressure gradient in the channel bounded by two parallel insulating plates, when both fluids are considered as electrically conducting, incompressible with variable properties, viz. different viscosities, thermal and electrical conductivities. The transport properties of the two fluids are taken to be constant and the bounding plates are maintained at constant and equal temperatures. The governing partial differential equations are then reduced to the ordinary linear differential equations using two-term series. Closed form solutions for primary and secondary velocity, also temperature distributions are obtained in both the fluid regions of the channel. Profiles of these solutions are plotted to discuss the effects of the flow and heat transfer characteristics, and their dependence on the governing parameters involved, such as the Hartmann number, rotation parameter, ratios of the viscosities, heights, electrical and thermal conductivities.

Key words: magnetohydrodynamics, immiscible fluids, rotating fluids, unsteady flow, heat transfer.

1. Introduction

The problems of fluid motion in parallel plate channels and rectangular channels have been studied by several authors due to their importance in engineering and technological fields. Subsequently, considerable attention has been also given to the study of magnetohydrodynamic flow of viscous fluids in a rotating system in connection with theories of fluid motion of two-phase/two-layered flows, flow of immiscible fluids, stratified flows, flows in the presence of heat source/heat flux, flow through channels of different geometries with varied constraints and so on. New and emerging ideas have been added to the literature to possible applications in geophysics, astrophysics, engineering problems, geothermal energy, stem stimulation of oil field, food drying and heat pipes etc.

The viscous fluid flow in a rotating frame of reference is of considerable importance due to the occurrence of various natural phenomena and for its application in various technological situations, which are governed by the actions of Coriolis forces. The broad subjects of oceanography, meteorology, atmospheric science and astronomy involve some important and essential features of rotating fluids. The rotating flow of an electrically conducting fluid in the presence of a magnetic field is encountered in cosmological and geophysical fluid dynamics. Many important observations on the viscous fluid flow problems in a rotating system under different conditions and configurations have come out from the analytical studies of many investigators, namely, Greenspan and Howard (1963), Holton (1965), Vidyanidhi (1969), Walin (1969), Siegman (1971), Jana and Datta (1977), Seth *et al*. (1982), Mazumder (1991), Ganapathy (1994), Hayat *et al.* (2001), Hayat and Hutter (2004) and Das *et al.* (2008). The investigation on an oscillatory flow in a rotating channel is important from a practical point of view, because fluid oscillations may be expected in many MHD devices and natural phenomena where the fluid flow is generated due to the oscillating pressure gradient or due to vibrating plates/walls. In view of these facts, Mukherjee and Debnath (1977), Seth and Jana (1980), Singh (2000), Ghosh (1993), Ghosh and Pop (2003), Hayat *et al*. (2004) and Guria and Jena (2007) investigated an oscillatory flow of a viscous incompressible electrically conducting fluid in a rotating channel under different conditions to analyze various aspects of the problem. Rahman and Sattar (1999) studied an MHD free convection and mass transfer flow with an oscillating plate velocity and constant heat source in a rotating frame of reference.

All the above investigations have been carried out in a fluid system having single fluid flows. But many problems relating to astrophysics, geophysical fluid dynamics, aeronautics, and in petroleum industry, also in industrial applications, etc; involve multi layered-fluid flow situations. In the petroleum industry as well as in other engineering and technological fields, a stratified two-phase/two-layered fluid flow often occurs. For example, in geophysics, it is so important to study the interaction of the geomagnetic field with the hot springs/fluids in geothermal regions, in which, once the interaction of the geomagnetic field with the flow field is known, then one can easily find the temperature distribution from the well known energy equation. Moreover, the temperature distribution plays an important role in MHD generators, plasma physics, turbines, etc. Also, it is a known fact that, to generate electricity, the temperature is used to run the turbine across a magnetic field. Transportation and extraction of the products of oil are other obvious applications using a two-phase system to obtain the increased flow rates in an electromagnetic pump from the possibility of reducing the power required to pump oil in a pipe line by a suitable addition of water (Shail, 1973). There are several investigations with regards to both experimental and theoretical aspects of magnetohydrodynamic two-phase/two-layered fluids flow problems, which are available in the literature [viz., Packham and Shail (1971), Lielausis (1975), Michiyoshi *et al*. (1977), Chan (1979), Chao *et al*. (1979), Dunn (1980), Gherson (1984), Lohrasbi and Sahai (1987; 1989), Alireza and Sahai (1990), Serizawa *et al*. (1990), Malashetty and Leela (1992), Malashetty and Umavathi (1997), Ramadan and Chamkha (1999), Chamkha (2000), Raju and Murty (2006), Tsuyoshi Inoue and Shu-Ichiro Inutsuka (2008) etc.]. Also, recent studies show that magnetohydrodynamic (MHD) flows can also be a viable option for transporting conducting fluids in microscale systems, such as a flow inside the micro-channel networks of a lab-on-a-chip device (Haim *et al*., 2003; Hussameddine *et al*., 2008). In micro-fluidic devices, multiple fluids can be transported through a channel for different reasons. For example, an increase in mobility of a fluid may be achieved by stratification of a highly mobile fluid or mixing of two or more fluids in transit may be designed for emulsification or heat and mass transfer applications. In this regard, magnetic field-driven micro-pumps are an increasing demand due to their long-term reliability in generating flow, low power requirement and mixing efficiency (Yi *et al*., 2002 and Weston *et al*., 2010).

Most of the above investigations correspond to the steady flow situations. However, a significant number of practical problems dealing with immiscible fluids are unsteady in nature. In many practical problems, it is also advantageous to consider both immiscible fluids as electrically conducting, one of which is highly electrically conducting compared to the other. The fluid of low electrical conductivity compared to the other is helpful to reduce the power required to pump the fluid in MHD pumps and flow meters. In view of these facts, Heavy and Young (1970) studied oscillating two-phase channel flows. Debnath and Basu (1975) discussed the unsteady slip flow in an electrically conducting two-phase fluid under transvrse magnetic fields. Chamkha (2004) studied the unsteady MHD convective heat and mass transfer past a semi-infinite vertical permeable moving plate with heat absorption. Umavathi *et al*. (2006) investigated an oscillatory Hartmann two-fluid flow and heat transfer in a horizontal channel. Linga Raju and Sreedhar (2009) discussed an unsteady two-fluid flow and heat transfer of conducting fluids in channels under transverse magnetic field.

On the other hand, the simultaneous influence of rotation and an external magnetic field on electrically conducting two-layered/two-phase fluid systems seem to be dynamically important and physically useful etc. So, in view of the wide range applications in geophysics and MHD generators, in this paper an unsteady magnetohydrodynamic (MHD) two-layered fluids flow in a horizontal channel between two parallel plates in the presence of an applied magnetic and electric field is investigated, when the whole system is rotated about an axis perpendicular to the flow. The flow is driven by a constant uniform pressure gradient in the channel bounded by two parallel insulating plates, when both fluids are considered as electrically conducting. The two fluids are assumed to be incompressible with variable properties, namely, different viscosities, thermal and electrical conductivities. Also, the transport properties of the two fluids are taken to be constant and the bounding plates are maintained at constant and equal temperatures. The governing partial differential equations are then reduced to the ordinary linear differential equations by using two-term series. Exact solutions for primary and secondary velocity distributions, also the temperatures are obtained in both fluid regions of the channel. Profiles of these solutions are plotted to discuss the effect on the flow and heat transfer characteristics, and their dependence on the governing parameters involved, such as the Hartmann number, Taylor number (rotation parameter), ratios of the viscosities, heights, electrical and thermal conductivities. Moreover, an observation is made how the velocity and temperature distributions vary with hydromagnetic interaction in the case of steady and unsteady motions in the presence of rigid rotation.

The structure of the paper is as follows. Introduction of the problem is given in § 1. The formulation and mathematical analysis of the problem for equations of motion, energy, the boundary and interface conditions are given in § 2. Closed form solutions of the problem are given in § 3. While § 4 gives the results and discussion based on the velocity and temperature profiles, which are displayed in Figs 2 to 17.

2. Formulation and mathematical analysis of the governing equations of motion, energy, boundary and interface conditions

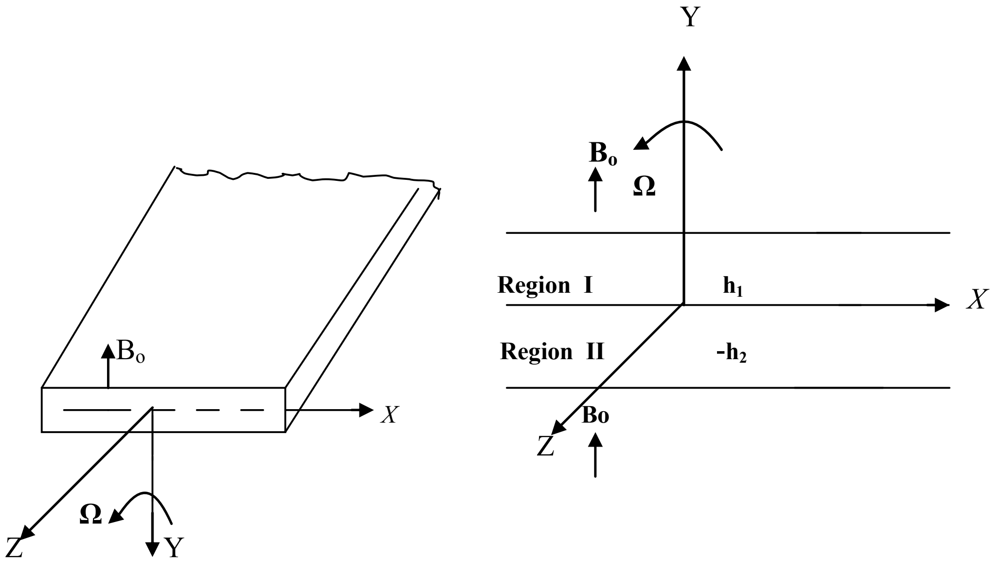

Fig.1. Physical model and co-ordinate system.

We consider an unsteady magnetohydrodynamics (MHD) two layered-fluid flow in a horizontal channel consisting of two infinite parallel plates extending along the x- and z- directions by the planes $y = h_1$ and $y = -h_2$, when both the fluid and plates are in a state of rigid rotation with uniform angular velocity $\overline{\Omega}$ about the y-axis normal to the plates. The fluids in the upper and lower regions, i.e., $0 \le y \le h_1$ and $-h_2 \le y \le 0$ are designated as Region -I and Region-II, respectively. Figure 1 represents the physical model and co-ordinate system choosing the origin midway between the two plates. The flow in both upper and lower regions is driven by a common constant pressure gradient $\left(-\frac{\partial p}{\partial x}\right)$. Both the regions are occupied by two immiscible electrically conducting, incompressible fluids with different viscosities μ_1, μ_2, thermal conductivities K_1, K_2 and electrical conductivities σ_1, σ_2. A constant magnetic field of strength B_0 is applied transverse to the flow direction, that is, along the y-direction. There is also applied a constant electric field E_0 in the z- direction. The induced magnetic field is being neglected by assuming that, it is small when compared with the applied field. The two bounding plates are maintained at constant temperature T_w. With these assumptions, the governing equations of motion, current and energy and the corresponding boundary and interface conditions (as in Lohrasbi and Shahai, 1989; Raju and Murty, 2005) for both fluid regions in a rotating frame of reference are obtained.

Also, we introduce the following non-dimensional variables

$$u^{\bullet}_1=\frac{u_1}{u_p},\quad u^{\bullet}_2=\frac{u_2}{u_p},\quad w^{\bullet}_1=\frac{w_1}{u_p},\quad w^{\bullet}_2=\frac{w_2}{u_p},\quad y^{\bullet}_i=\left(\frac{y_i}{h_i}\right)(i=1,2),\quad u_p=\left(-\frac{\partial p}{\partial x}\right)\frac{h_1^2}{\mu_1},\quad t^*=\frac{\upsilon_1 t}{h_1^2},$$

$\omega^*=\dfrac{\omega h_1^2\rho_1}{\mu_1}$, M^2 (Hartmann number) = $B_0^2h_1^2\left(\dfrac{\sigma_1}{\mu_1}\right)$, T^2 (Taylor number or rotation parameter) $=h_1^2\dfrac{\Omega}{\upsilon_1}$, which is the reciprocal of the Ekman number. α (ratio of the viscosities) $=\dfrac{\mu_1}{\mu_2}$, h (ratio of the heights) = $\dfrac{h_2}{h_1}$, σ (ratio of the electrical conductivities) = $\dfrac{\sigma_1}{\sigma_2}$, β (ratio of thermal conductivities) = $\dfrac{K_1}{K_2}$, $\theta_i=\dfrac{T_i-T_w}{\mu_p^2\mu_2/K_i}$, R_e (electric load parameter) $=E_0/B_0u_p$.

And for simplicity neglecting the asterisks, the non-dimensional forms of equations for both the fluid regions are simplified as

Region-I

$$\frac{du_1}{dt}-\frac{d^2u_1}{dy^2}+\mathrm{M}^2\left(R_e+u_1\right)-1=-2\mathrm{T}^2w_1, \tag{2.1}$$

$$\frac{dw_1}{dt}-\frac{d^2w_1}{dy^2}+\mathrm{M}^2w_1=2\mathrm{T}^2u_1, \tag{2.2}$$

$$\frac{d\theta_1}{dt}-\frac{1}{\mathrm{Pr}}\frac{d^2\theta_1}{dy^2}-\left[\left(\frac{du_1}{dy}\right)^2+\left(\frac{dw_1}{dy}\right)^2\right]-\mathrm{M}^2\left[\left(R_e+u_1\right)^2+w_1^{\,2}\right]=0. \tag{2.3}$$

Region-II

$$\frac{du_2}{dt}-\frac{d^2u_2}{dy^2}+\mathrm{M}^2h^2\alpha\sigma\left(R_e+u_2\right)-\alpha h^2=-2\rho\alpha h^2\mathrm{T}^2w_2, \tag{2.4}$$

$$\frac{dw_2}{dt}-\frac{d^2w_2}{dy^2}+\mathrm{M}^2h^2\alpha\sigma w_2=2\rho\alpha h^2\mathrm{T}^2u_2, \tag{2.5}$$

$$\frac{d\theta_2}{dt}-\frac{1}{\Pr}\frac{d^2\theta_2}{dy^2}-\frac{\beta}{\alpha}\left[\left(\frac{du_2}{dy}\right)^2+\left(\frac{dw_2}{dy}\right)^2\right]-\beta\sigma h^2\mathrm{M}^2\left[\left(R_e+u_2\right)^2+w_2{}^2\right]=0\,. \tag{2.6}$$

Here subscripts Eqs (2.10) and (2.2) represent the values for Region-I and Region-II respectively, where u_1, u_2 and w_1, w_2 are the x- and z-components of fluid velocities; which are known as the primary and secondary velocity distributions in the two regions, respectively. Ω is the angular velocity, where $\overline{\Omega}=(0,\Omega,$ $0)$; T_1, T_2 are the fluid temperatures in the two regions respectively and 't' is the time. The boundary conditions on velocity are the no-slip boundary condition at the lower plate and an oscillatory one at the upper plate. The boundary conditions on temperature are isothermal conditions. We also assume the continuity of velocity, shear stress, temperature and heat flux at the interface between the two fluid layers at $y = 0$.

The non-dimensional forms of the velocity, temperature and interface boundary conditions become

$$\begin{aligned} &u_1(+1) \quad \text{and} \quad w_1(+1)=0 \quad \text{for} \quad t\le 0,\\ &=\mathrm{Re}\left(\varepsilon e^{i\omega t}\right), \quad \text{for} \quad t>0, \end{aligned} \tag{2.7}$$

$$u_2(-1)=0\,, \qquad w_2(-1)=0, \tag{2.8}$$

$$u_1(0)=u_2(0)\,, \qquad w_1(0)=w_2(0), \tag{2.9}$$

$$\frac{du_1}{dy}=\left(1/\alpha h\right)\frac{du_2}{dy} \quad \text{and} \quad \frac{dw_1}{dy}=\left(1/\alpha h\right)\frac{dw_2}{dy} \quad \text{at} \quad y=0, \tag{2.10}$$

$$\theta_1(+1)=0, \tag{2.11}$$

$$\theta_2(-1)=0, \tag{2.12}$$

$$\theta_1(0)=\theta_2(0), \tag{2.13}$$

$$\frac{d\theta_1}{dy}=\left(1/\beta h\right)\left(d\theta_2/dy\right) \quad \text{at} \quad y=0. \tag{2.14}$$

Equations (2.8) represent the no-slip conditions at the lower plate and the conditions Eq.(2.7) are due to oscillation of the upper plate for any time t. Conditions Eqs (2.9) and (2.10) represent the continuity of velocities and shear stress at the interface $y = 0$. The conditions Eqs (2.11) and (2.12) represent the isothermal conditions, while the conditions Eqs (2.13) and (2.14) denote the continuity of temperatures and heat flux at the interface $y = 0$.

3. Solutions of the problem

The governing momentum Eqs (2.1), (2.2) and (2.4), (2.5) along with the energy Eqs (2.3) and (2.6) are to be solved subject to the boundary and interface conditions Eqs (2.7) - (2.14) for the velocity and temperature distributions in both regions. These equations are coupled partial differential equations, which cannot be solved in a closed form. But they can be solved analytically by reducing to the ordinary linear differential equations with the assumption of the following two term series

$$u_1(y,t)=u_{01}(y)+\varepsilon\cos\omega t\cdot u_{11}(y), \tag{3.1}$$

$$w_1(y,t)=w_{01}(y)+\varepsilon\cos\omega t\cdot w_{11}(y), \tag{3.2}$$

$$u_2(y,t)=u_{02}(y)+\varepsilon\cos\omega t\cdot u_{12}(y), \tag{3.3}$$

$$w_2(y,t)=w_{02}(y)+\varepsilon\cos\omega t\cdot w_{12}(y), \tag{3.4}$$

$$\theta_1(y,t)=\theta_{01}(y)+\varepsilon\cos\omega t\cdot\theta_{11}(y), \tag{3.5}$$

$$\theta_2(y,t)=\theta_{02}(y)+\varepsilon\cos\omega t\cdot\theta_{12}(y) \tag{3.6}$$

where, $u_{01}(y), u_{02}(y)$ and $\theta_{01}(y), \theta_{02}(y)$ are velocity and temperature distributions in the basic steady state case in the two regions, while, $u_{11}(y), u_{12}(y)$ and $\theta_{11}(y), \theta_{12}(y)$ are the corresponding time dependent components of the solutions, which are the factors of real $\left(\varepsilon e^{i\omega t}\right)$ to be determined with the help of Eqs (2.1) to (2.6).

Using the expressions given in Eqs (3.1) - (3.6) into Eqs (2.1) - (2.6) and separating the steady state and transient time varying components, the following ordinary linear differential equations for $u_{01}(y), u_{02}(y)$ and $\theta_{01}(y)$, $\theta_{02}(y)$; also, $u_{11}(y)$, $u_{12}(y)$ and $\theta_{11}(y)$, $\theta_{12}(y)$ in terms of the complex notations $q_{01}=u_{01}+iw_{01}$, $q_{11}=u_{11}+iw_{11}$, $q_{02}=u_{02}+iw_{02}$, $q_{12}=u_{12}+iw_{12}$ are obtained in both fluid regions as:

Region-I

For the steady-state part

$$\frac{d^2q_{01}}{dy^2}-f_1^2q_{01}=f_2, \tag{3.7}$$

$$\begin{aligned}\frac{1}{\Pr}\frac{d^2\theta_{01}}{dy^2}&=-g_{22}e^{g_5y}-g_{23}e^{g_6y}-g_{24}e^{g_7y}-g_{25}e^{g_8y}+\\&-g_{26}e^{f_1y}-g_{27}e^{-f_1y}-g_{28}e^{\overline{f}_1y}-g_{29}e^{\overline{-f}_1y}-g_{21}.\end{aligned} \tag{3.8}$$

For the transient time dependent part

$$\frac{d^2q_{11}}{dy^2}-f_3^2q_{11}=0, \tag{3.9}$$

$$\begin{aligned}\frac{1}{\Pr}\frac{d^2\theta_{11}}{dy^2}-g_{136}^2\theta_{11}&=-g_{137}e^{g_{112}y}-g_{138}e^{g_{113}y}-g_{139}e^{g_{114}y}+\\&-g_{140}e^{g_{115}y}-g_{141}e^{f_3y}-g_{142}e^{-f_3y}-g_{143}e^{\overline{f}_3y}-g_{144}e^{\overline{-f_3}y}.\end{aligned} \tag{3.10}$$

Region – II

For the steady-state part

$$\frac{d^2 q_{02}}{dy^2} - f_4^{\,2} q_{02} = f_5, \tag{3.11}$$

$$\begin{aligned}\frac{1}{\Pr}\frac{d^2\theta_{02}}{dy^2} = &-g_{53}e^{g_{36}y} - g_{54}e^{g_{37}y} - g_{55}e^{g_{38}y} - g_{56}e^{g_{39}y} + \\ &-g_{57}e^{f_4 y} - g_{58}e^{-f_4 y} - g_{59}e^{\overline{f}_4 y} - g_{60}e^{\overline{-f}_4 y} - g_{52}.\end{aligned} \tag{3.12}$$

For the transient time dependent part

$$\frac{d^2 q_{12}}{dy^2} - f_6^{\,2} q_{12} = 0, \tag{3.13}$$

$$\begin{aligned}\frac{1}{\Pr}\frac{d^2\theta_{12}}{dy^2} - g_{136}^{\;2}\theta_{12} = &-g_{167}^{\;g_{155}y} - g_{168}e^{g_{156}y} - g_{169}e^{g_{157}y} + \\ &-g_{170}e^{g_{158}y} - g_{163}e^{f_6 y} - g_{164}e^{-f_6 y} - g_{165}e^{\overline{f_6}y} - g_{166}e^{\overline{-f_6}y}.\end{aligned} \tag{3.14}$$

The corresponding boundary and interface conditions on velocity and temperature become:

For the steady-state part

$$q_{01}(+1) = 0, \tag{3.15}$$

$$q_{02}(-1) = 0, \tag{3.16}$$

$$q_{01}(0) = q_{02}(0), \tag{3.17}$$

$$\frac{dq_{01}}{dy} = \frac{1}{\alpha h}\frac{dq_{02}}{dy} \qquad \text{at} \qquad y = 0. \tag{3.18}$$

For the transient time dependent part

$$q_{11}(+1) = 1, \tag{3.19}$$

$$q_{12}(-1) = 0, \tag{3.20}$$

$$q_{11}(0) = q_{12}(0), \tag{3.21}$$

$$\frac{dq_{11}}{dy} = \frac{1}{\alpha h}\frac{dq_{12}}{dy} \qquad \text{at} \qquad y = 0. \tag{3.22}$$

The differential equations given in Eqs (3.7) - (3.14) along with the boundary and interface conditions from Eqs (3.15) to (3.22) represent a system of ordinary linear differential equations and conditions. These equations are solved in a closed form separately in two parts for both the steady state and transient time dependent components of the solutions. Hence, the final solutions for velocity and temperature distributions of the unsteady flow problem become:

Region-I

$$q_1(y,t) = q_{01}(y) + \text{Real}\left(\varepsilon e^{i\omega t}\right) q_{11}(y),$$

$$= f_{30}e^{f_1 y} + f_{31}e^{-f_1 y} - \frac{f_2}{f_1^2} + \text{Real}\left(\varepsilon\, e^{i\omega t}\right) f_{34}e^{f_3 y} + f_{35}e^{-f_3 y}, \tag{3.23}$$

$$\begin{aligned} \theta_1(y,t) &= \theta_{01}(y) + \text{Real}\left(\varepsilon e^{i\omega t}\right) \theta_{11}(y), \\ &= g_{69}e^{g_5 y} + g_{70}e^{g_6 y} + g_{71}e^{g_7 y} + g_{72}e^{g_8 y} + g_{73}e^{f_1 y} + g_{74}e^{-f_1 y} + \\ &+ g_{75}e^{\overline{f_1} y} + g_{76}e^{\overline{-f_1} y} + g_{77}y^2 + g_{221}\, y + g_{222} + \text{Real}\left(\varepsilon\, e^{i\omega t}\right) g_{225}e^{g_{136} y} + \\ &+ g_{226}e^{-g_{136} y} + g_{171}e^{g_{112}\cdot y} + g_{172}e^{g_{113} y} + g_{173}e^{g_{114} y} + \\ &+ g_{174}e^{g_{115} y} + g_{175}e^{f_3 y} + g_{176}e^{-f_3 y} + g_{177}e^{\overline{f_3} y} + g_{178}e^{\overline{-f_3} y}. \end{aligned} \tag{3.24}$$

Region-II

$$q_2(y,t) = q_{02}(y) + \text{Real}\left(\varepsilon e^{i\omega t}\right) q_{12}(y)$$

$$= f_{32}e^{f_4 y} + f_{33}e^{-f_4 y} - \frac{f_5}{f_4^2} + \text{Real}\left(\varepsilon e^{i\omega t}\right) f_{36}e^{f_6 y} + f_{37}e^{-f_6 y}, \tag{3.25}$$

$$\begin{aligned} \theta_2(y,t) &= \theta_{02}(y) + \text{Real}\left(\varepsilon e^{i\omega t}\right) \theta_{12}(y) \\ &= g_{86}e^{g_{36} y} + g_{87}e^{g_{37} y} + g_{88}e^{g_{38} y} + g_{89}e^{g_{39} y} + g_{90}e^{f_4 y} + g_{91}e^{-f_4 y} + g_{92}e^{\overline{f_4} y} + \\ &+ g_{93}e^{\overline{-f_4} y} + g_{94}y^2 + g_{223}\, y + g_{224} + \text{Real}\left(\varepsilon\, e^{i\omega t}\right) g_{227}e^{g_{136} y} + g_{228}e^{-g_{136} y} + \\ &+ g_{187}e^{g_{155}\cdot y} + g_{188}e^{g_{156} y} + g_{189}e^{g_{157} y} + g_{190}e^{g_{158} y} + \\ &+ g_{191}e^{f_6 y} + g_{192}e^{-f_6 y} + g_{193}e^{\overline{f_6} y} + g_{194}e^{\overline{-f_6} y}. \end{aligned} \tag{3.26}$$

In the above expressions, the solutions of the non-periodic terms represent the steady-state fluid flow solutions for both regions, and without going into details, the steady-state velocity and temperature profiles are shown in Figs 2 to 17. The solution of the periodic terms gives the transient velocity and temperature distribution in both regions of the channel. The solutions of the unsteady problem given in Eqs (3.23) to (3.26) are evaluated numerically for different non-dimensional governing flow parameters involved in the study. Also, these results are plotted and are shown in Figs 2 to 17. Here the value for ε is fixed at *0.5* and $\Pr = 1$ for all graphs. The constants appearing in the above solutions are given in Appendix.

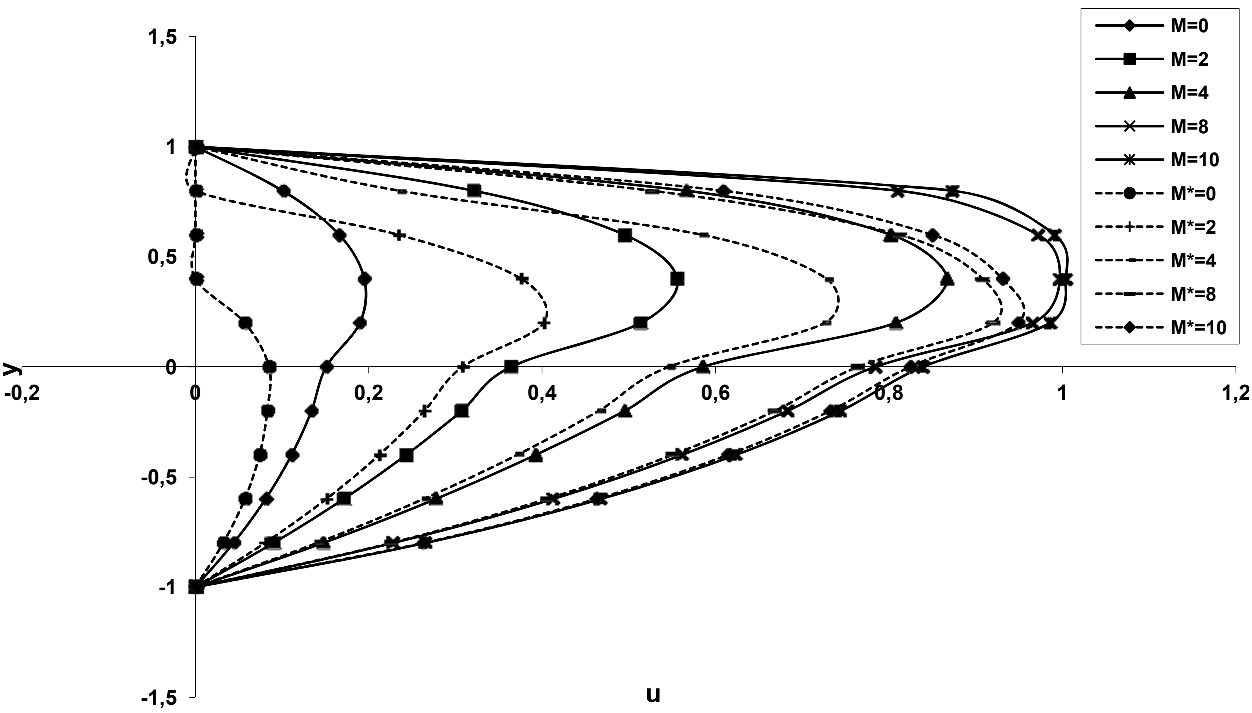

Fig.2. Primary velocity profiles u_1, u_2 (unsteady flow), u^*_1, u^*_2 (steady flow) for different M and T=*1*, $\rho = 1.5$, $\alpha = 0.333, \sigma = 0.1, h = 0.75, R_e = -1, \varepsilon = 0.5, \omega = 1, t = \pi/\omega$.

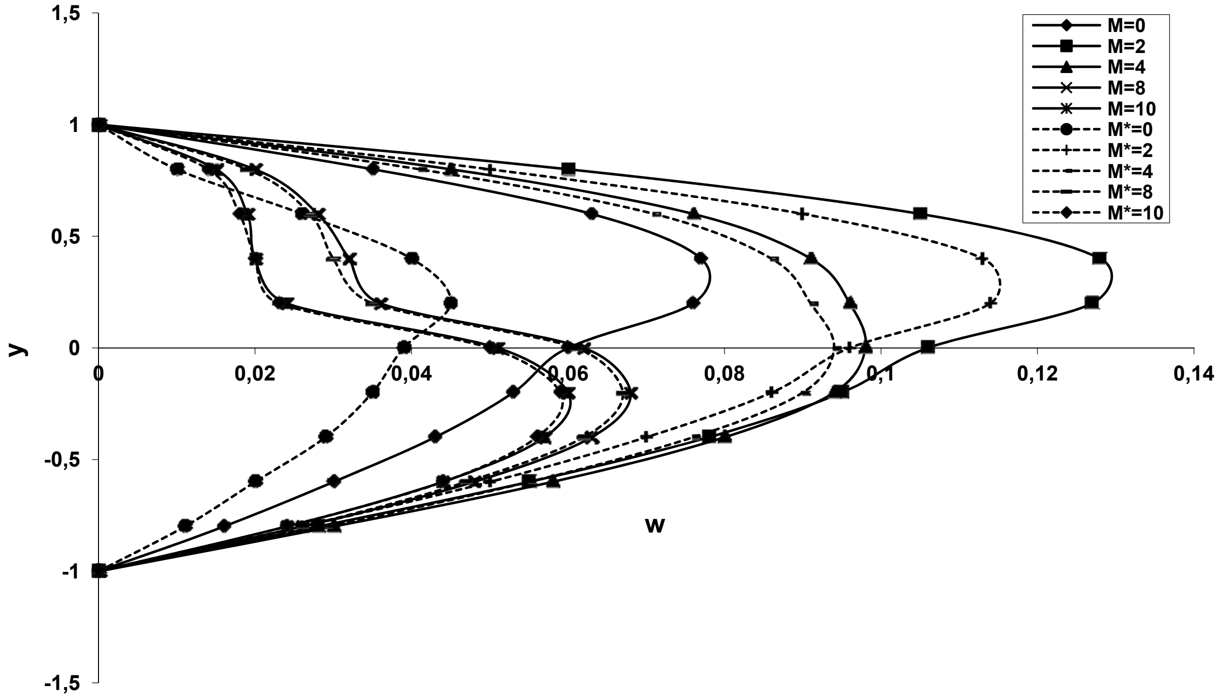

Fig.3. Secondary velocity profiles w_1, w_2 (unsteady flow) w^*_1, w^*_2 (steady flow) for different M and T=*1*, $\rho = 1.5$, $\alpha = 0.333, \sigma = 0.1, h = 0.75, R_e = -1, \varepsilon = 0.5, \omega = 1, t = \pi/\omega$.

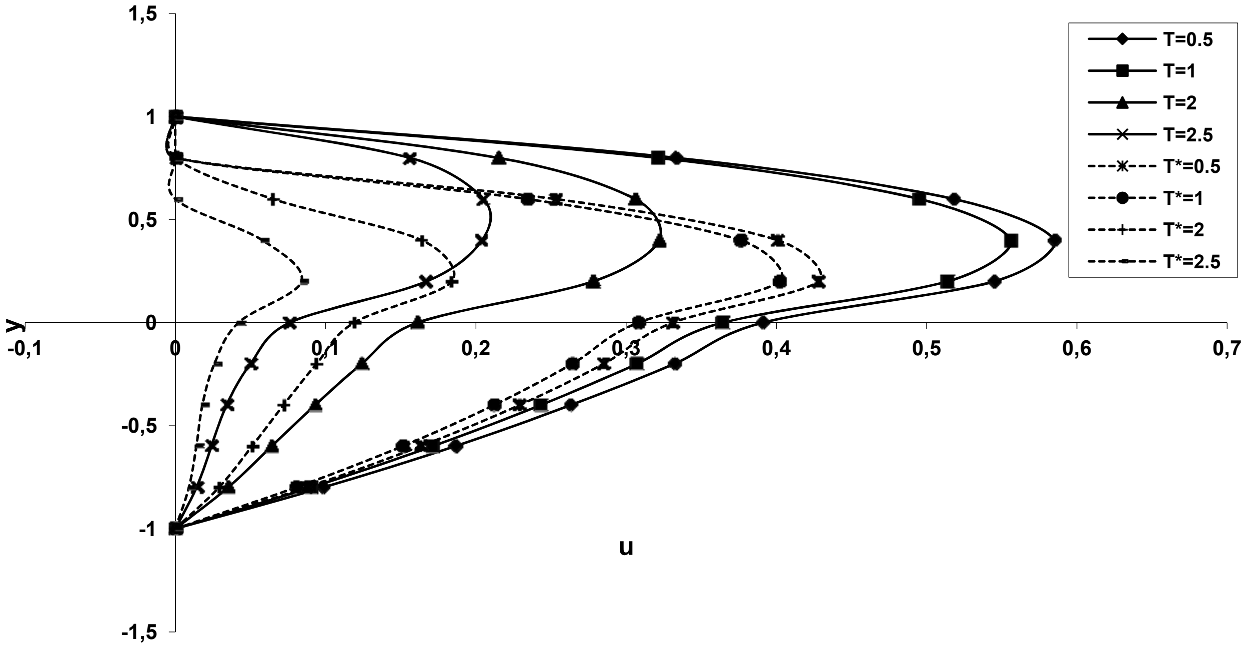

Fig.4. Primary velocity profiles u_1, u_2 (unsteady flow), u^*_1, u^*_2 (steady flow) for different T and $\rho = 1.5$, $M = 2$, $\alpha = 0.333, \sigma = 0.1$, $R_e = -1$, $h = 0.75, \varepsilon = 0.5$, $\omega = 1$, $t = \pi/\omega$.

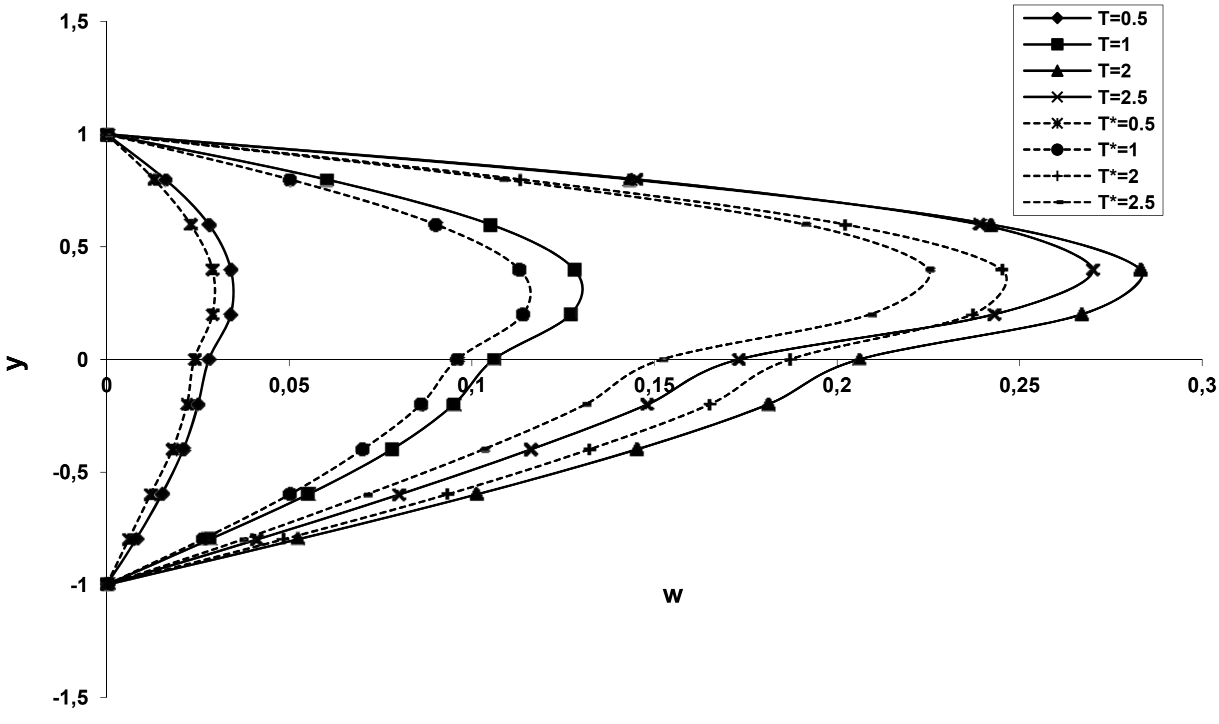

Fig.5. Secondary velocity profiles w_1, w_2 (unsteady flow) w^*_1, w^*_2 (steady flow) for different T and $\rho = 1.5$, $M = 2$, $\alpha = 0.333, \sigma = 0.1$, $R_e = -1$, $h = 0.75$, $\varepsilon = 0.5, \omega = 1, t = \pi/\omega$.

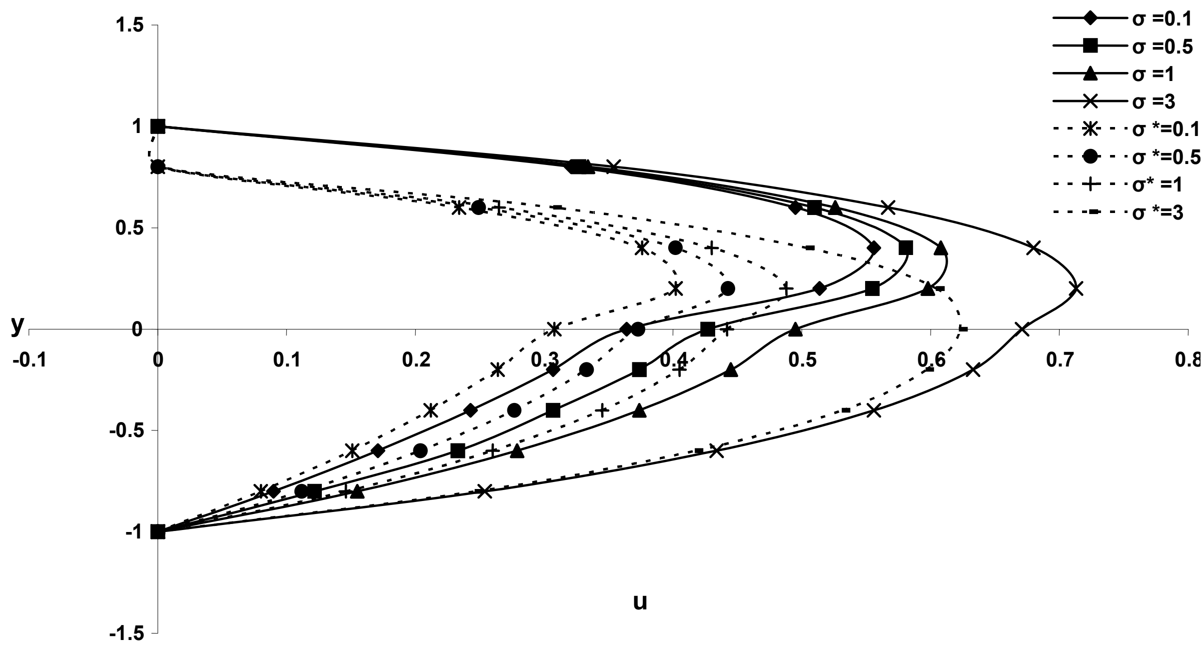

Fig.6. Primary velocity profiles u_1, u_2 (unsteady flow), u^*_1, u^*_2 (steady flow) for different σ and $\rho = 1.5$, $\mathrm{T} = 1$, $\mathrm{M} = 2$, $\alpha = 0.333$, $R_e = -1$, $h = 0.75$, $\varepsilon = 0.5$, $\omega = 1$, $t = \pi/\omega$.

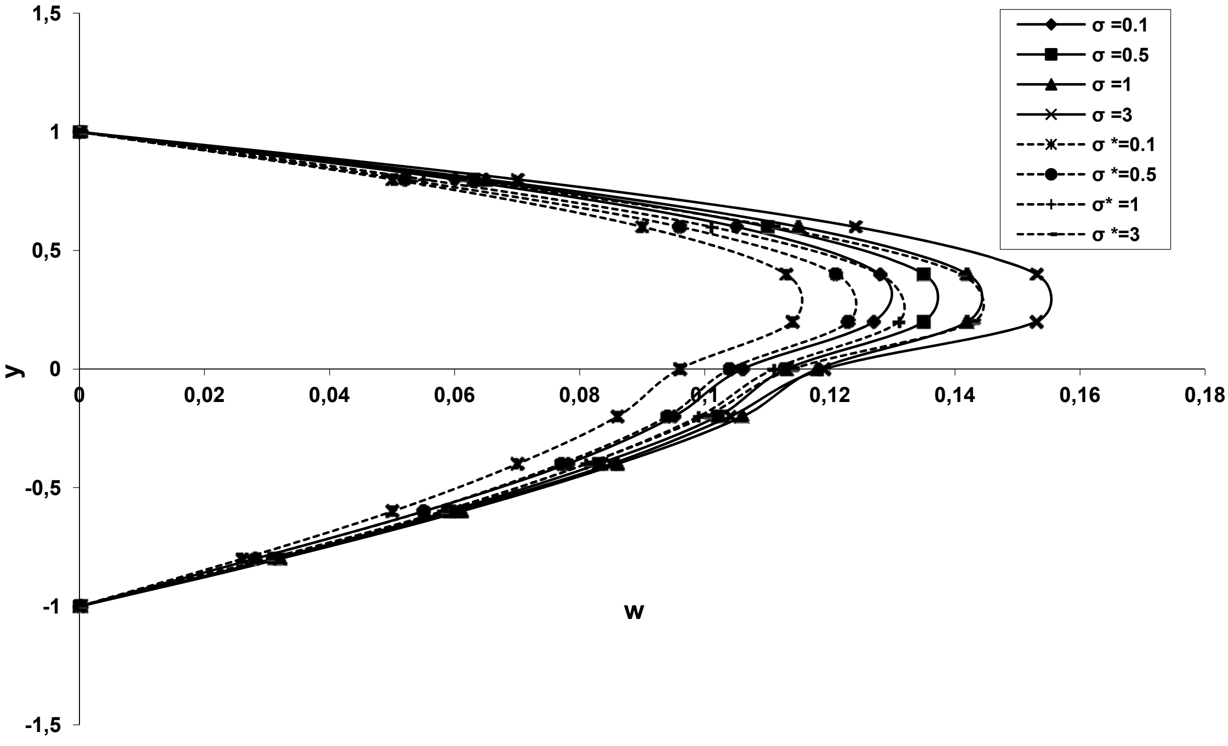

Fig.7. Secondary velocity profiles w_1, w_2 (unsteady flow) w^*_1, w^*_2 (steady flow) for different σ and $\rho = 1.5$, $\mathrm{T} = 1$, $\mathrm{M} = 2$, $\alpha = 0.333$, $R_e = -1$, $h = 0.75$, $\varepsilon = 0.5$, $\omega = 1$, $t = \pi/\omega$.

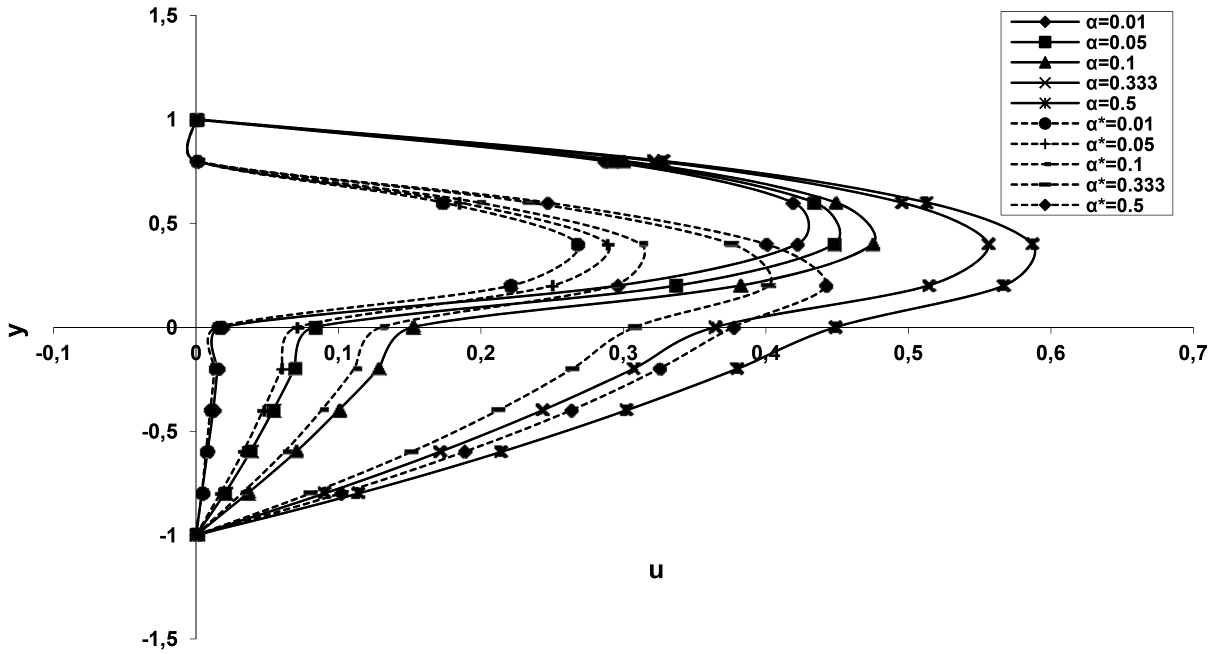

Fig.8. Primary velocity profiles u_1, u_2 (unsteady flow), u^*_1, u^*_2 (steady flow) for different α and $\rho = 1.5$, $\mathrm{T} = 1$, $\mathrm{M} = 2$, $\sigma = 0.1$, $R_e = -1$, $h = 0.75$, $\varepsilon = 0.5$, $\omega = 1$, $t = \pi/\omega$.

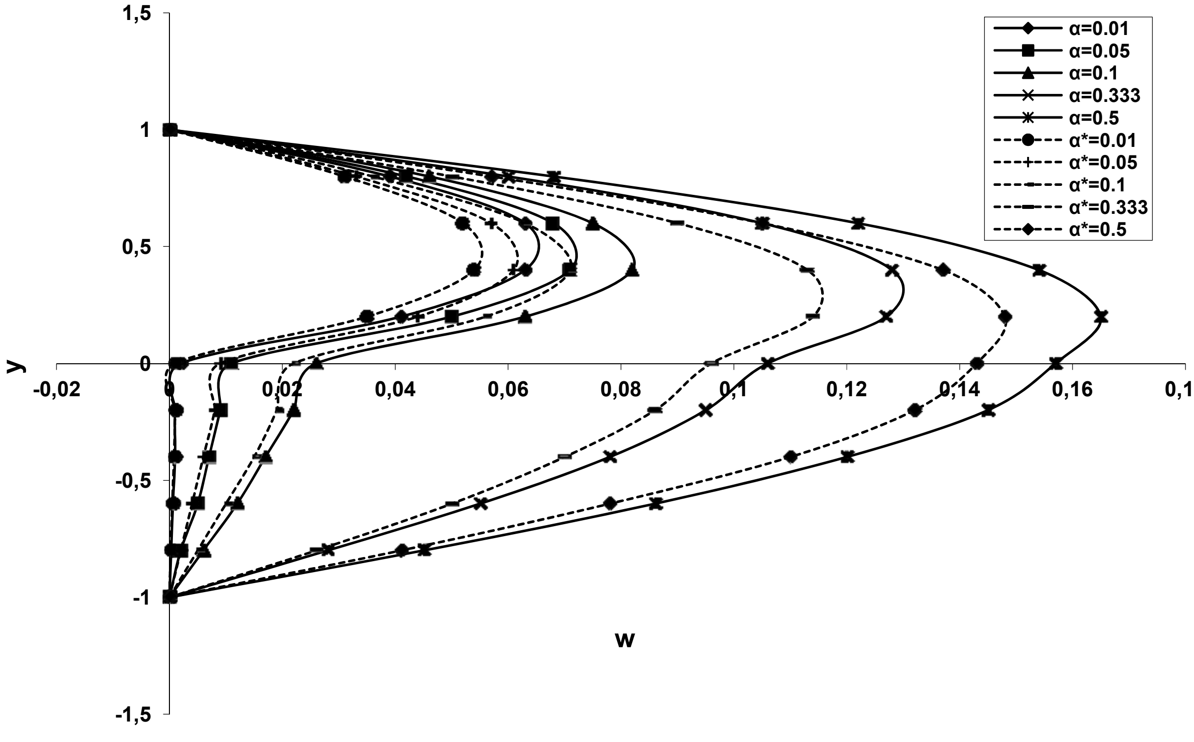

Fig.9. Secondary velocity profiles w_1, w_2 (unsteady flow) w^*_1, w^*_2 (steady flow) for different α and $\rho = 1.5$, $\mathrm{T} = 1$, $\mathrm{M} = 2$, $\sigma = 0.1$, $R_e = -1$, $h = 0.75$, $\varepsilon = 0.5$, $\omega = 1$, $t = \pi/\omega$.

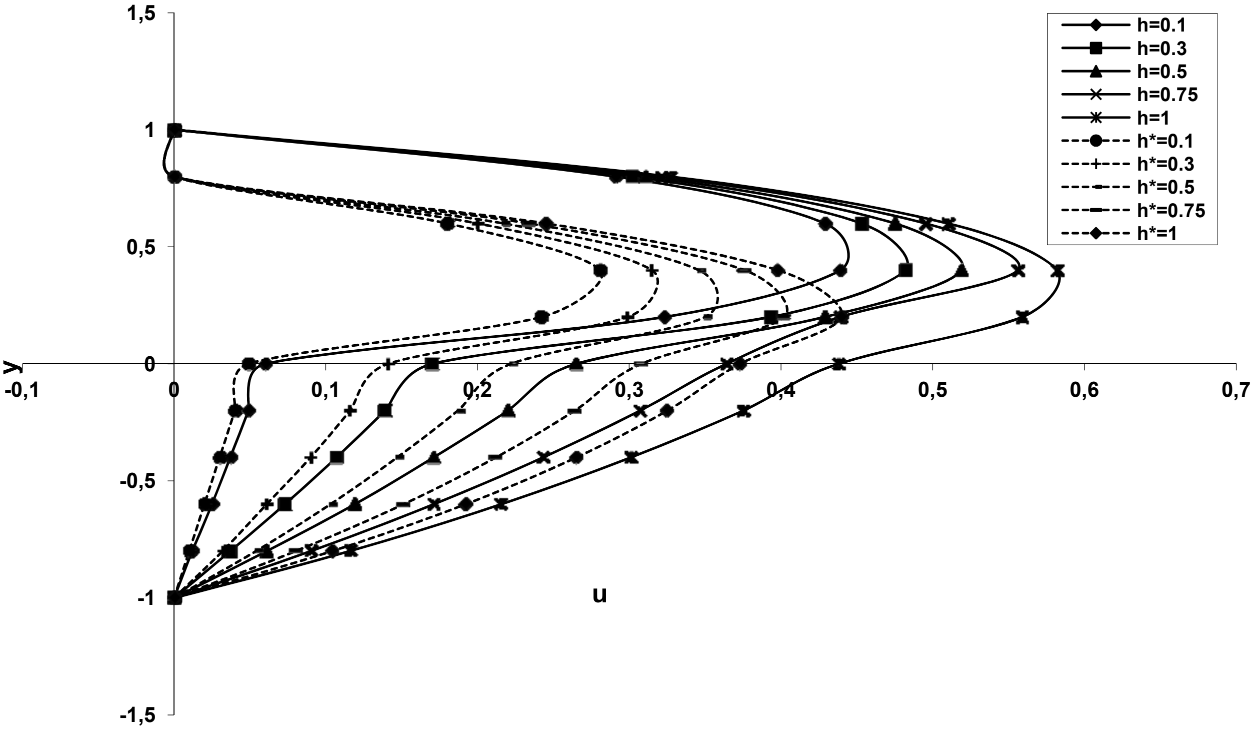

Fig.10. Primary velocity profiles u_1, u_2 (unsteady flow), u^*_1, u^*_2 (steady flow) for different h and $\rho = 1.5$, $\mathrm{T} = 1$, $\mathrm{M} = 2$, $\alpha = 0.333$, $R_e = -1$, $\sigma = 0.1$, $\varepsilon = 0.5$, $\omega = 1$, $t = \pi/\omega$.

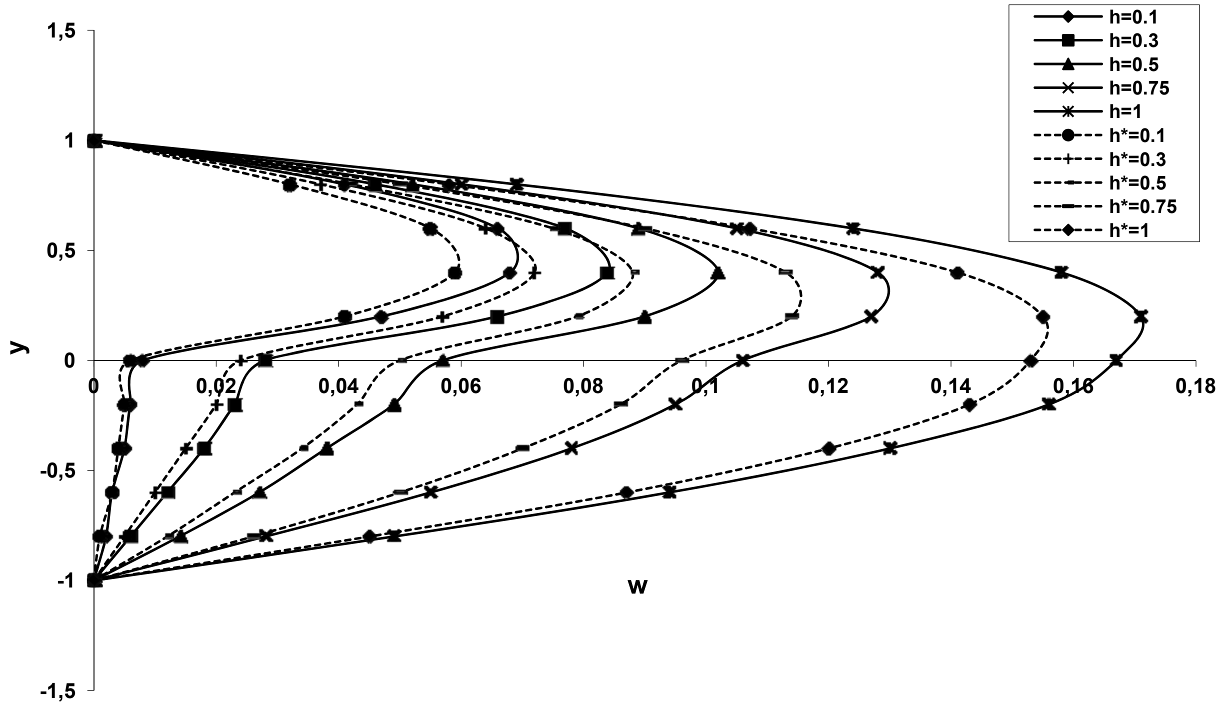

Fig.11. Secondary velocity profiles w_1, w_2 (unsteady flow) w^*_1, w^*_2 (steady flow) for different h and $\rho = 1.5$, $\mathrm{T} = 1$, $\mathrm{M} = 2$, $\alpha = 0.333$, $R_e = -1$, $\sigma = 0.1$, $\varepsilon = 0.5$, $\omega = 1$, $t = \pi/\omega$.

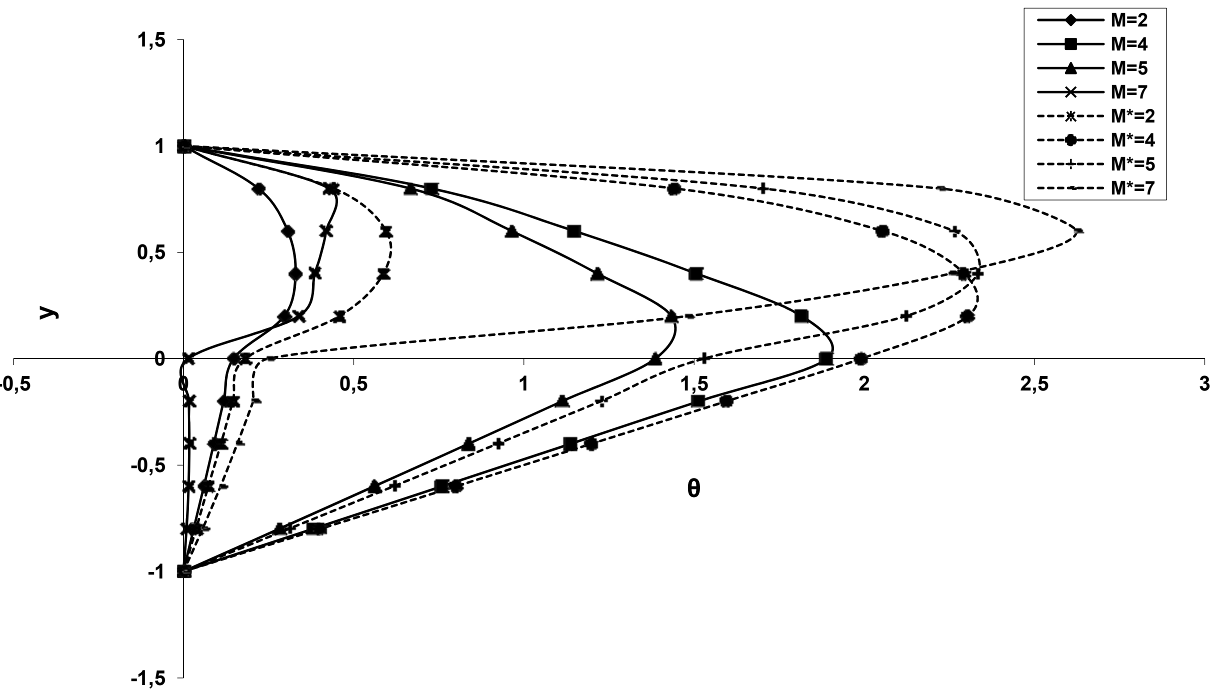

Fig.12. Temperature profiles θ_1,θ_2 (unsteady flow), θ_1,θ_2 (steady flow) for different M and $T=0.75$, $\beta = 0.5, \rho = 1.5, \alpha = 0.333, \sigma = 0.1, R_e = -1, h = 0.1, \varepsilon = 0.5, \omega = 1,\ t = \pi/\omega$.

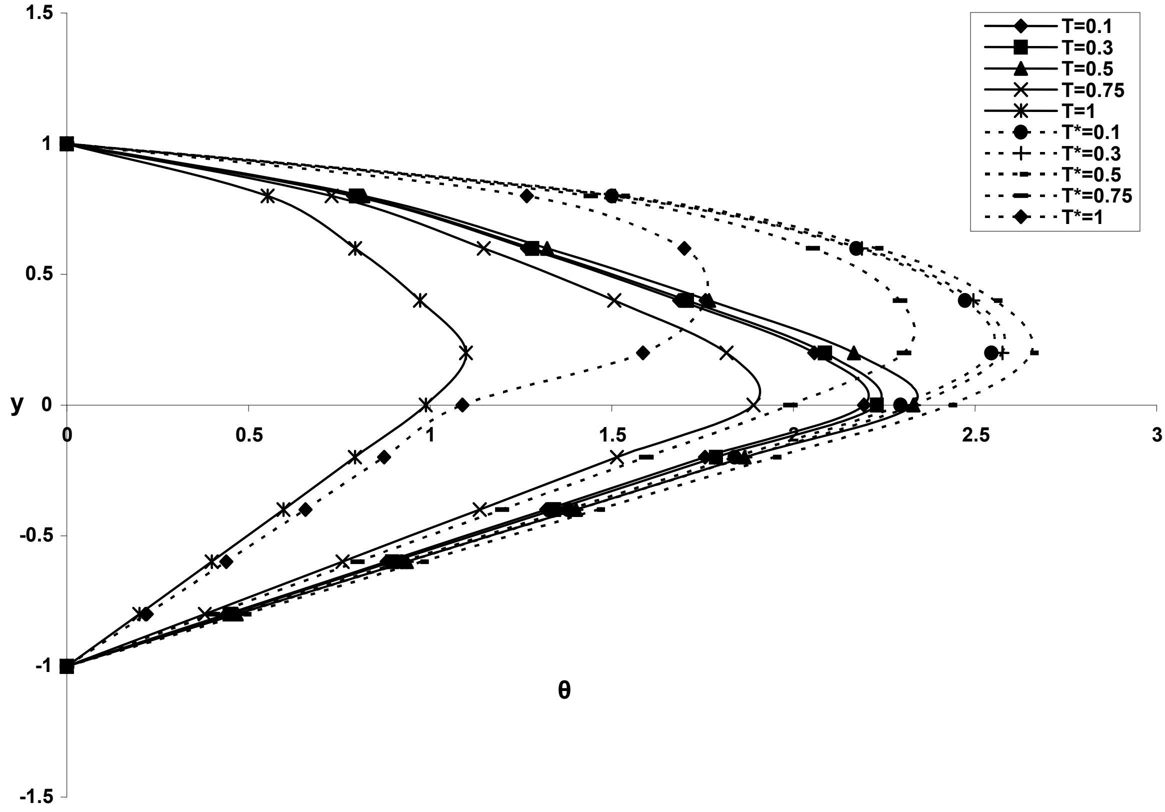

Fig.13. Temperature profiles θ_1,θ_2 (unsteady flow), θ_1,θ_2 (steady flow) for different T and M=4, $\beta = 0.5, \rho = 1.5, \alpha = 0.333, \sigma = 0.1, R_e = -1, h = 0.1, \varepsilon = 0.5, \omega = 1,\ t = \pi/\omega$.

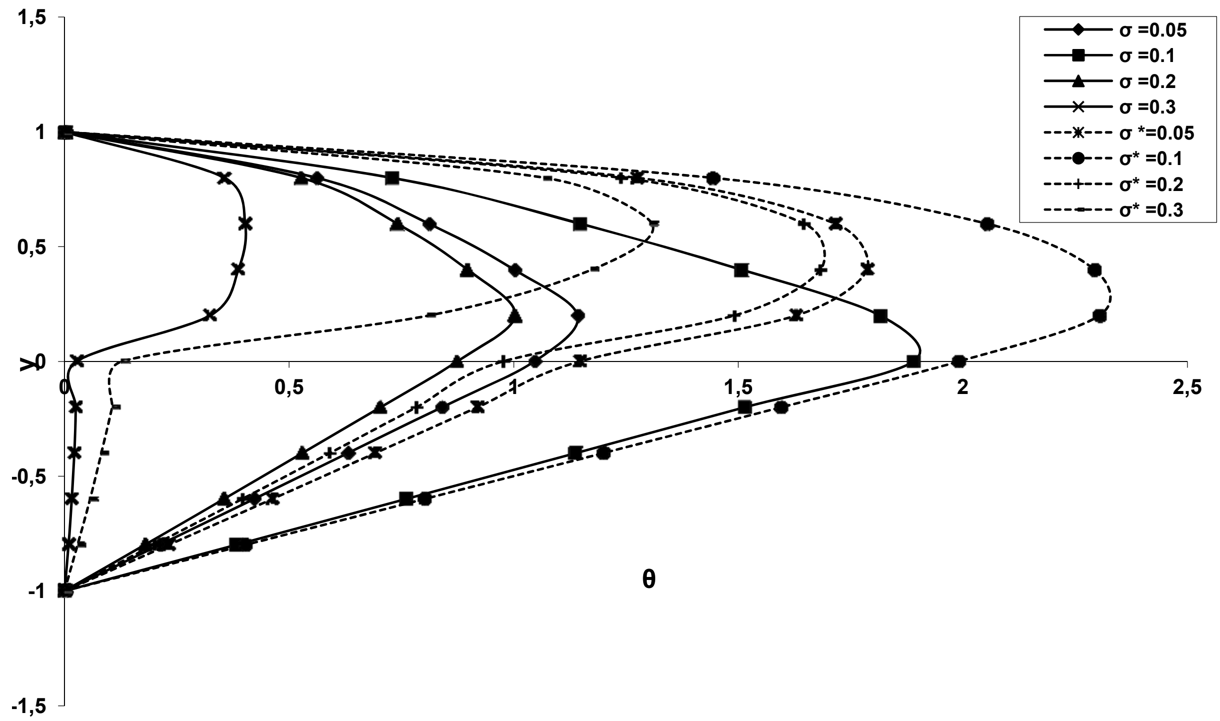

Fig.14. Temperature profiles θ_1, θ_2 (unsteady flow), θ_1, θ_2 (steady flow) for different σ and M=*4*, $\beta = 0.5, \mathrm{T} = 0.75, \rho = 1.5, \alpha = 0.333, R_e = -1, h = 0.1, \varepsilon = 0.5, \omega = 1, t = \pi/\omega$.

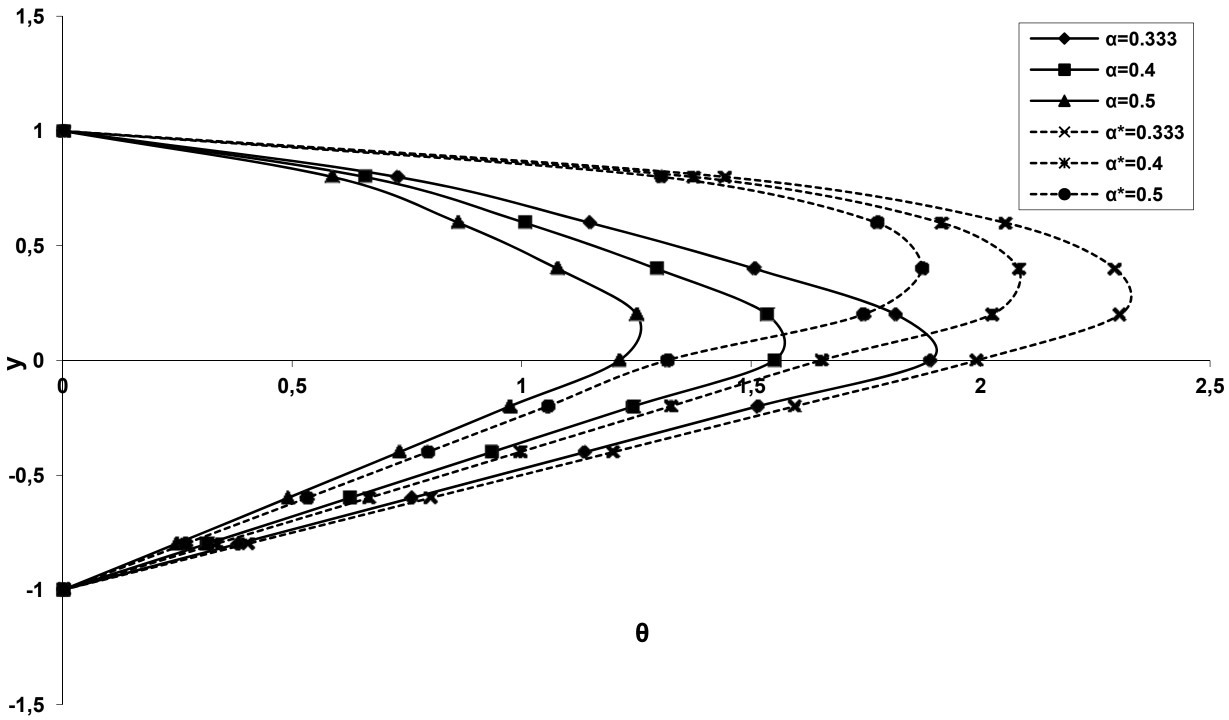

Fig.15. Temperature profiles θ_1, θ_2 (unsteady flow), θ_1, θ_2 (steady flow) for different α and M=*4*, $\beta = 0.5, T = 0.75, \rho = 1.5, \sigma = 0.1, R_e = -1, h = 0.1, \varepsilon = 0.5, \omega = 1, t = \pi/\omega$.

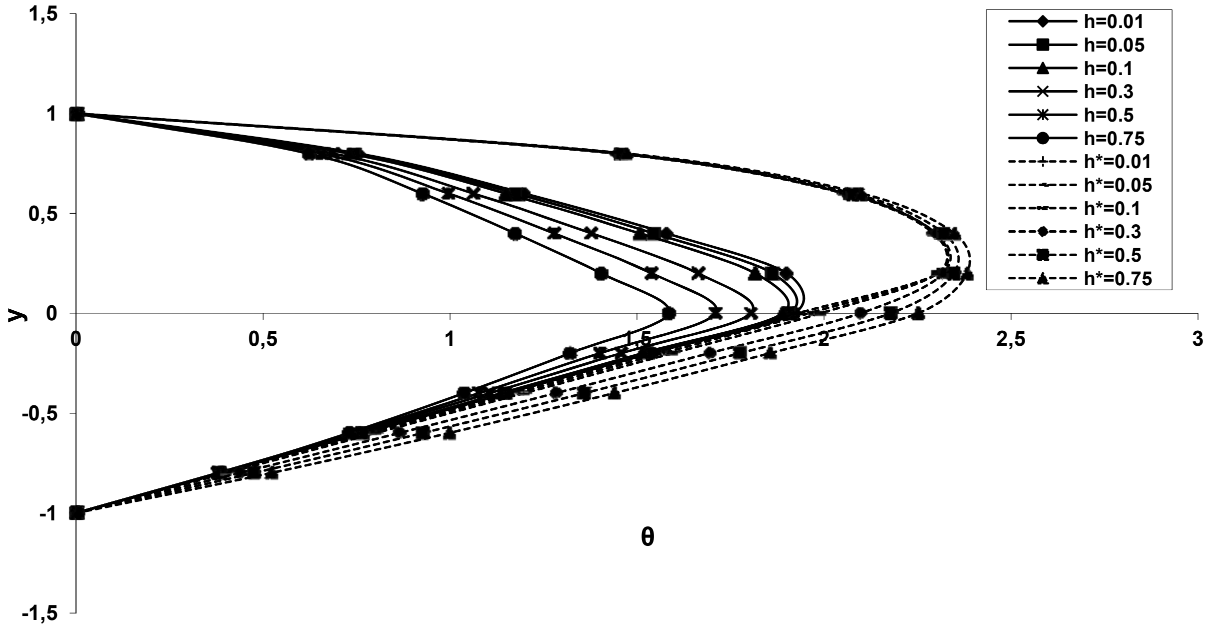

Fig.16. Temperature profiles θ_1, θ_2 (unsteady flow), θ_1, θ_2 (steady flow) for different h and M=4, $\beta = 0.5$, $\mathrm{T} = 0.75$, $\rho = 1.5, \alpha = 0.333$, $R_e = -1, \sigma = 0.1$, $\varepsilon = 0.5, \omega = 1$, $t = \pi/\omega$.

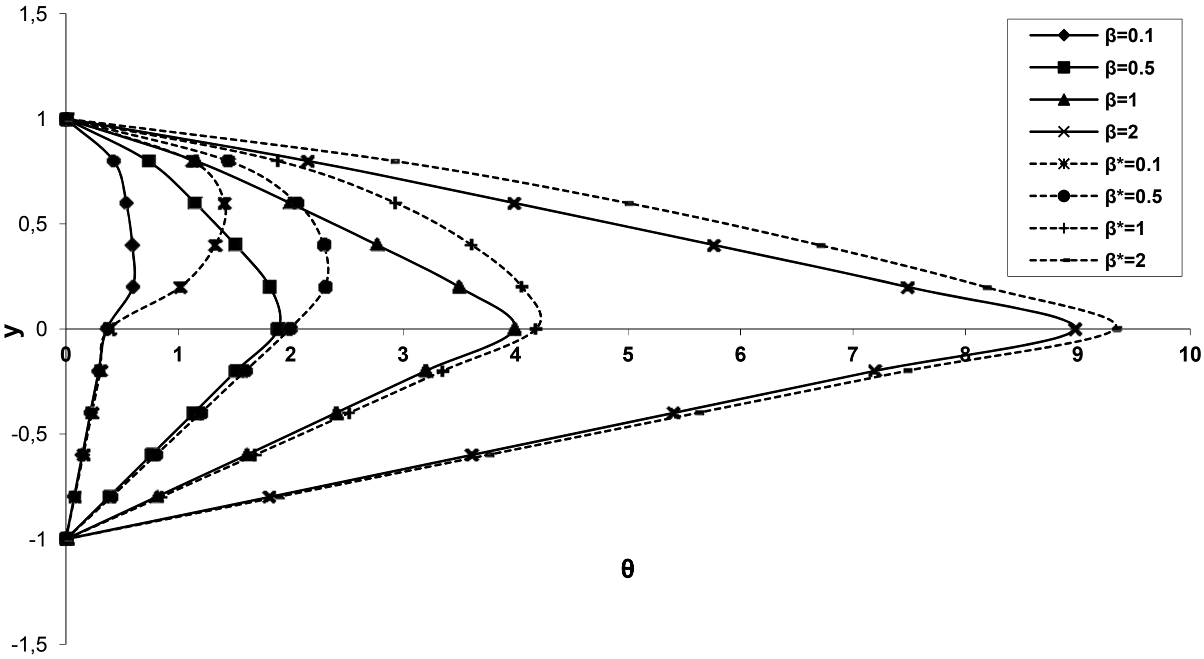

Fig.17. Temperature profiles θ_1, θ_2 (unsteady flow), θ_1, θ_2 (steady flow) for different β and M=4, $\beta = 0.5$, $h = 0.1, \mathrm{T} = 0.75$, $\rho = 1.5, \alpha = 0.333$, $R_e = -1, \sigma = 0.1$, $\varepsilon = 0.5, \omega = 1$, $t = \pi/\omega$.

4. Results and discussion

The closed-form solutions for the velocity distributions, such as primary and secondary velocity distributions, u_1, u_2 and w_1, w_2; also temperature distributions, θ_1, θ_2 in the two fluid regions are reported for small ε □, the coefficient of exponent of the periodic frequency parameter. These solutions are evaluated for various parametric conditions to plot their profiles. The results are depicted graphically in Figs 2– 17 for primary and secondary velocity distributions, also the temperature distributions in both fluid regions to elucidate the interesting features of the magnetohydrodynamic and thermal state of the flow. The solid lines show the profiles for an unsteady and the dash-dot lines for the steady flow motions respectively. We note that when the motion is in a steady state and $T = 0$ (without rigid rotation), these results coincide with those of Malashetty and Leela (1992). Also, when $T = 0$ (i.e., for without rigid rotation) the analysis is in agreement with the solutions of Linga Raju and Sreedhar (2009).

The effect of varying the Hartmann number M on both primary and secondary velocity distributions in the two regions is shown in Figs 2 and 3 respectively. In Fig.2, it is seen that the effect of increasing M is to increase the primary velocity distributions: u_1, u_2 in the two-fluid regions. From Fig.3, it is observed that the secondary velocity distribution increases as M increases and thereafter it decreases in both the regions. Also, the maximum primary velocity in the channel tends to move above the channel centre line towards Region-I (i.e., in the upper fluid region) as M increases, when all the remaining governing parameters are fixed. But, the maximum secondary velocity in the channel tends to move above the channel centre line towards Region-I (i.e., in the upper fluid region) up to M = *3*, thereafter it tends to move below the channel centre line towards Region-II (i.e., in the lower fluid region) as M increases.

The effect of the Taylor number (rotation parameter) T on both primary and secondary velocity distributions is shown in Figs 4 and 5, respectively. From Fig.4, it is noticed that an increase in T decreases the primary velocity distribution in the two regions. From Fig.5, it is observed that an increase in T increases the secondary velocity distribution in both the regions and falls the same when T > *2*. The maximum primary and secondary velocity distributions in the channel tend to move above the channel center line towards Region-I as T increases.

The effect of varying the electrical conductivity ratio σ on both primary and secondary velocity distributions is shown in Figs 6 and 7. It is noticed that both primary and secondary velocity distributions increase as σ increases. The maximum primary and secondary velocity distributions in the channel tend to move above the channel centre line towards Region-I as σ increases.

The effect of the viscosity ratio α on both primary and secondary velocity distributions of the two fluids is shown in Figs 8 and 9. It is observed that an increase in α is to increase both the primary and secondary velocity distributions in the two regions. The maximum primary and secondary velocity distributions in the channel tend to move above the channel centre line towards Region-I as α increases.

The effect of varying the height ratio h on both primary and secondary velocity distributions is shown in Figs 10 and 11 respectively. It is found that an increase in h increases both the primary and secondary velocity distributions in the two regions. The maximum velocity in the channel tends to move above the channel centre line towards Region-I, when *h* increases.

The graphs for temperature distributions are shown in Figs 12 to 17. The effect of varying the Hartmann number M on temperature distribution is exhibited in Fig.12. It is found that an increase in M enhances the temperature distribution in the two regions up to the value of M = *4* and thereafter it decreases in both regions. The maximum temperature in the channel tends to move above the channel centre line towards Region – I, as M increases.

Figure 13 exhibits the effect of the Taylor number (rotation parameter) T on temperature distribution in the two-fluid regions. It is found that an increase in T diminishes the temperature distribution in both the regions. Also, the maximum temperature in the channel tends to move above the channel centre line towards Region–I as Taylor number T increases.

The effect of varying the electrical conductivity ratio σ on temperature distributions is shown in Fig.14. It is seen that an increase in σ decreases the temperature distribution in the two regions. The maximum velocity distribution in the channel tends to move above the channel centre line towards Region-I as σ increases.

The effect of the viscosity ratio α on the temperature distribution is shown in Fig.15. It is observed that an increase in α decreases the temperature distribution in the two regions. And the temperature distribution in the channel tends to move above the channel centre line towards Region-I.

The effect of varying the height ratio h on temperature distribution is shown in Fig.16. It is seen that an increase in h decreases the temperature distribution in the two regions. Also, the maximum temperature in the channel tends to move above the channel centre line towards Region-I as h increases.

The effect of the thermal conductivity ratio β on the temperature distribution is shown in Fig.17. It is observed that an increasing β is to increase the temperature distribution in the two fluids. Also, the maximum temperature in the channel tends to move slightly above the channel centre line towards Region-I as β increases.

5. Conclusion

The heat transfer aspects of an electrically conducting two–layered fluids flow through a horizontal channel bounded by two parallel infinite plates (one being stationary and the other oscillating) in a rotating system with an applied transverse magnetic field are studied analytically. The governing equations of motion and energy are derived, assuming that the two fluids are of different viscosities, electrical and thermal conductivities, which in turn are non-dimensionalised. The resulting partial differential equations are transformed into a set of ordinary linear differential equations using two-term series as a combination of both steady state and transient time dependent parts and solved in closed form. The closed form solutions are evaluated numerically to plot their graphs for the velocity and temperature distributions of both the regions, and are also discussed to demonstrate the combined effect of the magnetic field and Coriolis force on the physical parameters involved in the study. Comparisons with previously published theoretical works are made. It is found that the effect of increasing the Hartmann number M is to rise the primary velocity distributions in the two fluid regions, whereas the secondary velocity distribution increases and then falls the same when $M > 2$. It is found that an increase in M rises the temperature in the two regions and falls down when $M > 4$. Also, as M increases the maximum temperature in the channel tends to move above the channel centre line towards Region – I. It is observed that the temperature decreases in both the regions as T increases. The maximum temperature in the channel tends to move above the channel centre line towards Region–I as the Taylor number T increases. It is seen that an increase in T decreases the primary velocity distribution in the two regions, while an increase in the Taylor number T causes a rise in the secondary velocity distribution of the fluids in both regions and falls when $T > 2$. Also it is noticed that the velocity and temperature in the two regions can be enhanced with the suitable values of the ratios of viscosity, heights, electrical and thermal conductivities. Hopefully the results reported herein will serve as a stimulus for experimental work on this type of problems and will be useful in verifying numerical schemes used to solve more complex/realistic problems of this type. Further, it is concluded that, as expected, these distributions are pronounced more in the unsteady when compared to the steady state problem.

Nomenclature

B_0 – applied uniform transverse magnetic field
E_0 – constant electric field in the z-direction
f_1, f_2, g_1, g_2 . etc. – functions / real constants represented in the equations and solutions
h – ratio of the heights of the two regions
h_1 – height of the channel in the upper region
h_2 – height of the channel in the lower region
K_1, K_2 – thermal conductivities of the two fluids
M – Hartmann number
Pr – Prandtl number
p – pressure

R_e – electric load parameter
T_1, T_2 – temperatures of the fluids in the two regions, respectively
T_{w_1}, T_{w_2} – constant temperatures at both the walls
t – time
$u_p = \left(-\frac{\partial p}{\partial x}\right)\frac{h_1^2}{\mu_1}$ – the characteristic velocity
u_1, u_2 – x-component of velocity distributions in the two fluid regions, known as primary velocity distribution
w_1, w_2 – z-component of velocity distributions in the two fluid regions, called secondary velocity distributions
$u_{01}(y), u_{02}(y)$ – primary velocity distributions in the basic steady state case in two regions
$u_{11}(y), u_{12}(y)$ – time dependent primary velocity components
$w_{01}(y), w_{02}(y)$ – secondary velocity distributions in the basic steady state case in two regions
$w_{11}(y), w_{12}(y)$ – time dependent secondary velocity components
(x, y, z) – space co-ordinates
α – ratio of the viscosities
β – ratio of thermal conductivities
μ_1, μ_2 – viscosities of the two fluids
ρ_1, ρ_2 – densities of the two fluids
σ – ratio of electrical conductivities
σ_1, σ_2 – electrical conductivities of the two fluids
Ω – angular velocity, where $\overline{\Omega} = (0, \Omega, 0)$

Subscripts

1, 2 – refers to the quantities in the upper and lower fluid regions, respectively

References

Alireza S. and Sahai V. (1990): *Heat transfer in developing magnetohydrodynamic poiseuille flow and variable transport properties*. – International Journal of Heat Mass Transfer, vol.33(8), pp.1711-1720.

Chamkha A.J. (2000): *Flow of two-immiscible fluids in porous and non-porous channels*. – ASME Journal of Fluids Engineering, vol.122, pp.117-124.

Chamkha A.J. (2004): *Unsteady MHD convective heat and mass transfer past a semi-infinite vertical permeable moving plate with heat absorption*. – International Journal of Engineering Science, vol.42, pp.217-230.

Chan C.K. (1979): *Finite element formulation and solution of nonlinear heat transfer*. – Journal of Nuclear Engineering and Design, vol.51, p.253.

Chao J., Mikic B.B. and Todreas N.E. (1979): *Radiation streaming in power reactors: proceedings of the special session*. – American Nuclear Society (ANS) Winter Meeting, Washington, D.C. Nuclear Technology, 42, 22.

Das B.K., Guria M. and Jana R.N. (2008): *Unsteady Couette flow in a rotating system*. – Meccanica, vol.43, p.517.

Debnath L. and Basu U. (1975): *Unsteady slip flow in an electrically conducting two-phase fluid under transverse magnetic fields*. – Nuovo Cimento, vol.28B, pp.349-362.

Dunn P.F. (1980): *Single-phase and two-phase magnetohydrodynamic pipe flow*. – International Journal of Heat Mass Transfer, vol.23, p.373.

Ganapathy R. (1994): *A note on oscillatory Couette flow in a rotating system*. – ASME. Journal of Applied Mechanics, vol.61, p.208.

Gherson P. and Lykoudis P.S. (1984): *Local measurements in two-phase liquid-metal magneto-fluid mechanic flow*. – Journal of Fluid Mechanics, vol.147, pp.81-104.

Ghosh S.K. (1993): *Unsteady hydromagnetic flow in a rotating channel with oscillating pressure gradient*. – Journal of the Physical Society of Japan, vol.62, p.3893.

Ghosh S.K. and Pop I. (2003): *Hall effects on unsteady hydromagnetic flow in a rotating system with oscillatory pressure gradient*. – International Journal of Applied Mechanics and Engineering, vol.8, p.43.

Greenspan H.P. and Howard L.N. (1963): *On a time dependent motion of a rotating fluid*. – Journal of Fluid Mechanics, vol.17, p.385.

Guria M. and Jana R.N. (2007): *Hydromagnetic flow in the Ekman layer on an oscillating porous plate*. – Magnetohydrodynamics, vol.43, pp.3-11.

Haim H.B., Jianzhong Z., Shizhi Q. and Yu X. (2003): *A magneto-hydrodynamically controlled fluidic network*. – Sensors and Actuators B, vol.88(2), pp.205–216.

Hayat T. and Hutter K. (2004): *Rotating flow of a second-order fluid on a porous plate*. – International Journal of Non-Linear Mechanics, vol.39, p.767.

Hayat T., Nadeem S. and Asghar S. (2004): *Hydromagnetic Couette flow of an Oldroyd-B fluid in a rotating system*. – International Journal of Engineering Science, vol.42, p.65.

Hayat T., Nadeem S., Asghar S. and Siddiqui A.M. (2001): *Fluctuating flow of a third-grade fluid on a porous plate in a rotating medium*. – International Journal of Non-Linear Mechanics, vol.36, p.901.

Heavy J.V. and Young H.T. (1970): *Oscillating two- phase channel flows*. – Z. Angew. Math. Phys., vol.21, p.454.

Holton J. R. (1965): *The influence of viscous boundary layers on transient motions in a stratified rotating fluid*. – International Journal of Atmospheric Science, vol.22, p.402.

Hussameddine S.K., Martin J.M. and Sang W.J. (2008): *Analytical prediction of flow field in magnetohydrodynamic-based microfluidic devices*. – Journal of Fluids Engineering, vol.130(9), p.6.

Jana R.N. and Datta N. (1977): *Couette flow and heat transfer in a rotating system*. – Acta Mechanica, vol.26, p.301.

Lielausis O. (1975): *Liquid metal magnetohydrodynamics* – Atomic Energy Review, vol.13, p.527.

Linga Raju T. and Murty P.S.R. (2006): *Hydromagnetic two-phase flow and heat transfer through two parallel plates in a rotating system*. – Journal of Indian Academy of Mathematics, Indore, India, vol.28(2), pp.343-360.

Linga Raju T. and Sreedhar S. (2009): *Unsteady two-fluid flow and heat transfer of conducting fluids in channels under transverse magnetic field*. – International Journal of Applied Mechanics and Engineering, vol.14(4), pp.1093-1114.

Lohrasbi J. and Sahai V. (1989): *Magnetohydrodynamic heat transfer in two-phase flow between parallel plates*. – Applied Scientific Research, vol.45, pp.53-66.

Malashetty M.S. and Leela V. (1992): *Magnetohydrodynamic heat transfer in two phase flow*. – International Journal of Engineering Science, vol.30, pp.371-377.

Malashetty M.S. and Umavathi J.C. (1997): Two *phase magnetohydrodynamic flow and heat transfer in an inclined channel*. – International Journal of Multiphase Flow, vol.23(3), pp.545-560.

Mazumder B.S. (1991): *An exact solution of oscillatory Couette flow in a rotating system*. – ASME Journal of Applied Mechanics, vol.58, p.1104.

Michiyoshi Funakawa Kuramoto C., Akita Y. and Takahashi O. (1977): *Instead of the helium-lithium annular-mist flow at high temperature, an air-mercury stratified flow in a horizontal rectangular duct in a vertical magnetic field.* – International Journal of Multiphase Flow, vol.3, p.445.

Mukherjee S. and Debnath L. (1977): *On unsteady rotating boundary layer flows between two porous plates*. – ZAMM, vol.57, p.188.

Packham B.A. and Shail R. (1971): *Stratified laminar flow of two immiscible fluids*. – Proceedings of Cambridge Philosophical Society, vol.69, pp.443-448.

Rahman M.M. and Sattar M.A. (1999): *MHD free convection and mass transfer flow with oscillating plate velocity and constant heat source in a rotating frame of reference*. – The Dhaka University Journal of Science, vol.47, p.63.

Ramadan H.M. and Chamkha A.J. (1999): *Two-phase free convection flow over an infinite permeable inclined plate with non-uniform particle-phase density*. – International Journal of Engineering Science, vol.37, p.1351.

Serizawa A., Ida T., Takahashi O. and Michiyoshi I. (1990): *MHD effect on Nak-nitrogen two-phase flow and heat transfer in a vertical round tube*. – International Journal Multi-Phase Flow, vol.16(5), p.761.

Seth G.S. and Ansari M.S. (2009): *Magnetohydrodynamic convective flow in a rotating channel with Hall effects*. – International Journal of Theoretical and Applied Mechanics, vol.4, p.205.

Seth G.S. and Jana R.N. (1980): *Unsteady hydromagnetic flow in a rotating channel with oscillating pressure gradient*. – Acta Mechanica, vol.37, p.29.

Seth G.S., Jana R.N. and Maiti M.K. (1982): *Unsteady hydromagnetic Couette flow in a rotating system*. – International Journal of Engineering Science, vol.20, p.989.

Shail R. (1973): *On laminar tow-phase flow in magnetohydrodynamics*. – International Journal of Engineering Science, vol.11, p.1103.

Siegmann W.L. (1971): *The spin-down of rotating stratified fluids*. – Journal of Fluid Mechanics, vol.47, p.689.

Singh K.D. (2000): *An oscillatory hydromagnetic Couette flow in a rotating system*. – ZAMM, vol.80, p.429.

Tsuyoshi I. and Shu-Ichiro I. (2008): *Two-fluid magnetohydrodynamic simulation of converging Hi flows in the interstellar medium*. – The Astrophysical Journal, vol.687(1), pp.303-310.

Umavathi J.C., Abdul Mateen, Chamkha A.J. and Al- Mudhaf A. (2006): *Oscillatory Hartmann two-fluid flow and heat transfer in a horizontal channel*. – International Journal of Applied Mechanics and Engineering, vol.11(1), pp.155-178.

Vidyanidhi V. (1969): *Secondary flow of a conducting liquid in a rotating channel*. – Journal of Mathematical and Physical Sciences, vol.3, p.193.

Walin G. (1969): *Some aspects of time dependent motion of a stratified rotating fluid*. – Journal of Fluid Mechanics, vol.36, p.289.

Weston M.C., Gerner M.D. and Fritsch I. (2010): *Magnetic fields for fluid motion* – Analytical Chemistry, vol.82(9), pp.3411–3418.

Yi M., Qian S. and Bau H. (2002): *A magnetohydrodynamic chaotic stirrer*. – Journal of Fluid Mechanics, vol.468, pp.153–177.

Appendix

$$f_1=\sqrt{\mathrm{M}^2+2\mathrm{T}^2 i}\,,\quad f_2=\mathrm{M}^2 R_e-1\,,\quad f_3=\sqrt{f_1-\omega\tan\omega t}\,,\quad f_4=\sqrt{\mathrm{M}^2 h^2\alpha\sigma+i2\rho\alpha h^2 T^2}\,,$$

$$f_5=\mathrm{M}^2 h^2\alpha\sigma R_e-\alpha h^2\,,\qquad f_6=\sqrt{f_4-\omega\tan\omega t}\,,\qquad f_7=e^{-2f_1}\,,\qquad f_8=\frac{f_2}{f_1^{\,2}}e^{-f_1}\,,$$

$$f_9=e^{2f_4}\,,\qquad f_{10}=\frac{f_5}{f_4^{\,2}}e^{f_4}\,,\qquad f_{11}=-f_7+1\,,\qquad f_{12}=-f_9+1\,,$$

$$f_{13}=f_{10}-\frac{f_5}{f_4^2}-f_8+\frac{f_2}{f_1^2}\,,\qquad f_{14}=\frac{1}{\alpha h}\,,\qquad f_{15}=-f_1 f_7-f_1\,,\qquad f_{16}=-f_{14}f_9 f_4-f_4 f_{14}\,,$$

$$f_{17}=f_{14}f_{10}f_4-f_1 f_8\,,\qquad f_{18}=\frac{f_{13}f_{15}-f_{17}f_{11}}{-f_{12}f_{15}+f_{16}f_{11}}\,,\qquad f_{19}=\frac{f_{17}+f_{18}f_{16}}{f_{15}}\,,\qquad f_{20}=e^{-2f_3}\,,$$

$$f_{21}=e^{-f_3}\,,\qquad f_{22}=e^{2f_6}\,,\qquad f_{23}=-f_{20}+1\,,\qquad f_{24}=-f_{22}+1\,,\qquad f_{25}=\frac{f_6}{f_3\alpha h}\,,$$

$$f_{26}=-f_{20}-1,\quad f_{27}=-f_{25}f_{22}-f_{25},\quad f_{28}=\frac{-f_{21}f_{26}+f_{21}f_{23}}{-f_{24}f_{26}+f_{27}f_{23}},\quad f_{29}=\frac{-f_{21}+f_{28}f_{24}}{f_{23}},$$

$$f_{30}=-f_{31}f_7+f_8,\quad f_{31}=f_{19},\quad f_{32}=-f_{33}f_9+f_{10},\quad f_{33}=f_{18},\quad f_{35}=f_{29},$$

$$f_{34}=-f_{35}f_{20}+f_{21},\quad f_{36}=-f_{37}f_{22},\quad f_{37}=f_{28},\quad g_1=f_1f_{30}\overline{f_1}\,\overline{f_{30}},$$

$$g_2=f_1f_{30}\overline{f_1}\,\overline{f_{31}},\quad g_3=f_1f_{31}\overline{f_1}\,\overline{f_{30}},\quad g_4=f_1f_{31}\overline{f_1}\,\overline{f_{31}},\quad g_5=f_1+\overline{f_1},$$

$$g_6=f_1-\overline{f_1},\quad g_7=-f_1+\overline{f_1},\quad g_8=-f_1-\overline{f_1},\quad g_9=\mathrm{M}^2f_{30}\overline{f_{30}},$$

$$g_{10}=\mathrm{M}^2f_{30}\overline{f_{31}},\quad g_{11}=\mathrm{M}^2f_{31}\overline{f_{30}},\quad g_{12}=\mathrm{M}^2f_{31}\overline{f_{31}},\quad g_{13}=\mathrm{M}^2f_{30}\frac{\overline{f_2}}{\overline{f_1}^2},$$

$$g_{14}=\mathrm{M}^2f_{31}\frac{\overline{f_2}}{\overline{f_1}^2},\quad g_{15}=\mathrm{M}^2\overline{f_{30}}\frac{f_2}{f_1^2},\quad g_{16}=\mathrm{M}^2\overline{f_{31}}\frac{f_2}{f_1^2},\quad g_{17}=\mathrm{M}^2f_{30}R_e,$$

$$g_{18}=\mathrm{M}^2f_{31}R_e,\quad g_{19}=\mathrm{M}^2\overline{f_{30}}R_e,\quad g_{20}=\mathrm{M}^2\overline{f_{31}}R_e,$$

$$g_{21}=\mathrm{M}^2\frac{f_2}{f_1^2}\frac{\overline{f_2}}{\overline{f_1}^2}-\mathrm{M}^2R_e\frac{f_2}{f_1^2}-\mathrm{M}^2R_e{}^2\frac{\overline{f_2}}{\overline{f_1}^2}+\mathrm{M}^2R_e{}^2,\quad g_{22}=g_1+g_9,\quad g_{23}=-g_2+g_{10},$$

$$g_{24}=-g_3+g_{11},\quad g_{25}=g_4+g_{12},\quad g_{26}=-g_{13}+g_{17},\quad g_{27}=-g_{14}+g_{18},$$

$$g_{28}=-g_{15}+g_{19},\quad g_{29}=-g_{16}+g_{20},\quad g_{30}=\frac{\beta}{\alpha},\quad g_{31}=\mathrm{M}^2h^2\sigma\beta,$$

$$g_{32}=g_{30}f_{32}f_4\overline{f_{32}}\,\overline{f_4},\quad g_{33}=g_{30}f_{32}f_4\overline{f_{33}}\,\overline{f_4},\quad g_{34}=g_{30}f_{33}f_4\overline{f_{32}}\,\overline{f_4},$$

$$g_{35}=g_{30}f_{33}f_4\overline{c_4}\,\overline{f_4},\quad g_{36}=f_4+\overline{f_4},\quad g_{37}=f_4-\overline{f_4},\quad g_{38}=-f_4+\overline{f_4},$$

$$g_{39}=-f_4-\overline{f_4},\quad g_{40}=g_{31}f_{32}\overline{f_{32}},\quad g_{41}=g_{31}f_{32}\overline{f_{33}},\quad g_{42}=g_{31}f_{33}\overline{f_{32}},$$

$$g_{43}=g_{31}f_{33}\overline{f_{33}},\quad g_{44}=g_{31}f_{32}\frac{\overline{f_5}}{\overline{f_4}^2},\quad g_{45}=g_{31}f_{33}\frac{\overline{f_5}}{\overline{f_4}^2},\quad g_{46}=g_{31}\overline{f_{32}}\frac{f_5}{f_4^2},$$

$$g_{47}=g_{31}\overline{f_{33}}\frac{f_5}{f_4^2},\quad g_{48}=g_{31}f_{32}R_e,\quad g_{49}=g_{31}f_{33}R_e,\quad g_{50}=g_{31}\overline{f_{32}}R_e,$$

$$g_{51}=g_{31}\overline{f_{33}}R_e,\quad g_{52}=g_{31}R_e{}^2+g_{31}\frac{f_5}{f_4^2}\frac{\overline{f_5}}{\overline{f_4}^2}-g_{31}R_e\frac{f_5}{f_4^2}-g_{31}R_e\frac{\overline{f_5}}{\overline{f_4}^2},$$

$$g_{53}=g_{32}+g_{40},\quad g_{54}=-g_{33}+g_{41},\quad g_{55}=-g_{34}+g_{42},\quad g_{56}=g_{35}+g_{43},$$

$$g_{57}=-g_{44}+g_{48}\,,\quad g_{58}=-g_{45}+g_{49}\,,\quad g_{59}=-g_{46}+g_{50}\,,\quad g_{60}=-g_{47}+g_{51}\,,$$

$$g_{61}=\frac{-g_{22}}{g_5}\,,\quad g_{62}=\frac{-g_{23}}{g_6}\,,\quad g_{63}=\frac{-g_{24}}{g_7}\,,\quad g_{64}=\frac{-g_{25}}{g_8}\,,\quad g_{65}=\frac{-g_{26}}{f_1}\,,$$

$$g_{66}=\frac{-g_{27}}{-f_1}\,,\quad g_{67}=\frac{-g_{28}}{\overline{f_1}}\,,\quad g_{68}=\frac{-g_{29}}{-\overline{f_1}}\,,\quad g_{69}=\frac{g_{61}}{g_5}\,,\quad g_{70}=\frac{g_{62}}{g_6}\,,$$

$$g_{71}=\frac{g_{63}}{g_7}\,,\quad g_{72}=\frac{g_{64}}{g_8}\,,\quad g_{73}=\frac{g_{65}}{f_1}\,,\quad g_{74}=\frac{g_{66}}{-f_1}\,,\quad g_{75}=\frac{g_{67}}{\overline{f_1}}\,,\quad g_{76}=\frac{g_{68}}{-\overline{f_1}}\,,$$

$$g_{77}=\frac{-g_{21}}{2}\,,\quad g_{78}=\frac{-g_{53}}{g_{36}}\,,\quad g_{79}=\frac{-g_{54}}{g_{37}}\,,\quad g_{80}=\frac{-g_{55}}{g_{38}}\,,\quad g_{81}=\frac{-g_{56}}{g_{39}}\,,$$

$$g_{82}=\frac{-g_{57}}{f_4}\,,\quad g_{83}=\frac{-g_{58}}{-f_4}\,,\quad g_{84}=\frac{-g_{59}}{\overline{f_4}}\,,\quad g_{85}=\frac{-g_{60}}{-\overline{f_4}}\,,\quad g_{86}=\frac{g_{78}}{g_{36}}\,,\quad g_{87}=\frac{g_{79}}{g_{37}}\,,$$

$$g_{88}=\frac{g_{80}}{g_{38}}\,,\quad g_{89}=\frac{g_{81}}{g_{39}}\,,\quad g_{90}=\frac{g_{82}}{f_4}\,,\quad g_{91}=\frac{g_{83}}{-f_4}\,,$$

$$g_{92}=\frac{g_{84}}{\overline{f_4}}\,,\quad g_{93}=\frac{g_{85}}{-\overline{f}_4}\,,\quad g_{94}=\frac{-g_{52}}{2}\,,$$

$$g_{95}=g_{69}e^{g_5}+g_{70}e^{g_6}+g_{71}e^{g_7}+g_{72}e^{g_8}+g_{73}e^{f_1}+g_{74}e^{-f_1}+g_{75}e^{\overline{f_1}}+g_{76}e^{\overline{-f_1}}+g_{77}\,,$$

$$g_{96}=g_{86}e^{-g_{36}}+g_{87}e^{-g_{37}}+g_{88}e^{-g_{38}}+g_{89}e^{-g_{39}}+$$
$$+g_{90}e^{-f_4}+g_{91}e^{f_4}+g_{92}e^{-\overline{f_4}}+g_{93}e^{\overline{f_4}}+g_{94}\,,$$

$$g_{97}=g_{69}+g_{70}+g_{71}+g_{72}+g_{73}+g_{74}+g_{75}+g_{76}\,,$$

$$g_{98}=g_{86}+g_{87}+g_{88}+g_{89}+g_{90}+g_{91}+g_{92}+g_{93}\,,$$

$$g_{99}=g_{61}+g_{62}+g_{63}+g_{64}+g_{65}+g_{66}+g_{67}+g_{68}\,,$$

$$g_{100}=g_{78}+g_{79}+g_{80}+g_{81}+g_{82}+g_{83}+g_{84}+g_{85}\,,$$

$$g_{101}=\frac{1}{\beta h}\,,\quad g_{102}=g_{101}g_{100}+g_{101}g_{96}-g_{99}+g_{95}\,,\quad g_{103}=g_{98}-g_{97}\,,$$

$$g_{104}=g_{101}-1\,,\quad g_{105}=g_{102}+g_{103}\,,\quad g_{106}=\frac{g_{105}}{g_{104}}\,,\quad g_{107}=\varepsilon\cos\omega t\,,$$

$$g_{108}=\varepsilon\omega\sin\omega t\,,\quad g_{109}=\varepsilon^2\cos^2\omega t\,,\quad g_{110}=\mathrm{M}^2\varepsilon^2\cos^2\omega t\,,\quad g_{111}=\mathrm{M}^2\,\mathrm{Re}\,\varepsilon\cos\omega t\,,$$

$g_{112} = f_3 + \overline{f_3}$, $g_{113} = f_3 - \overline{f_3}$, $g_{114} = -f_3 + \overline{f_3}$, $g_{115} = -f_3 - \overline{f_3}$,

$g_{116} = g_{109} f_{34} f_3 \overline{f_{34}}\,\overline{f_3}$, $g_{117} = g_{109} f_{34} f_3 \overline{f_{35}}\,\overline{f_3}$, $g_{118} = g_{109} f_{35} f_3 \overline{f_{34}}\,\overline{f_3}$,

$g_{119} = g_{109} f_{35} f_3 \overline{f_{35}}\,\overline{f_3}$, $g_{120} = g_{110} f_{34} \overline{f_{34}}$, $g_{121} = g_{110} f_{34} \overline{f_{35}}$, $g_{122} = g_{110} f_{35} \overline{f_{34}}$,

$g_{123} = g_{110} f_{35} \overline{f_{35}}$, $g_{124} = g_{111} f_{34}$, $g_{125} = g_{111} f_{35}$, $g_{126} = g_{111} \overline{f_{34}}$, $g_{127} = g_{111} \overline{f_{35}}$,

$g_{128} = g_{116} + g_{120}$, $g_{129} = -g_{117} + g_{121}$, $g_{130} = -g_{118} + g_{122}$, $g_{131} = g_{119} + g_{123}$,

$g_{136} = \sqrt{\dfrac{-g_{108}}{g_{107}}}$, $g_{137} = \dfrac{g_{128}}{g_{107}}$, $g_{138} = \dfrac{g_{129}}{g_{107}}$, $g_{139} = \dfrac{g_{130}}{g_{107}}$, $g_{140} = \dfrac{g_{131}}{g_{107}}$,

$g_{141} = \dfrac{g_{124}}{g_{107}}$, $g_{142} = \dfrac{g_{125}}{g_{107}}$, $g_{143} = \dfrac{g_{126}}{g_{107}}$, $g_{144} = \dfrac{g_{127}}{g_{107}}$, $g_{145} = g_{30} \varepsilon^2 \cos^2 \omega t$,

$g_{146} = g_{31} \varepsilon^2 \cos^2 \omega t$, $g_{147} = g_{31} \operatorname{Re} \varepsilon \cos \omega t$, $g_{148} = \dfrac{g_{145}}{g_{107}}$, $g_{149} = \dfrac{g_{146}}{g_{107}}$,

$g_{150} = \dfrac{g_{147}}{g_{107}}$, $g_{151} = g_{148} f_{36} f_6 \overline{f_{36}}\,\overline{f_6}$, $g_{152} = g_{148} f_{36} f_6 \overline{f_{37}}\,\overline{f_6}$,

$g_{153} = g_{148} f_{37} f_6 \overline{f_{36}}\,\overline{f_6}$, $g_{154} = g_{148} f_{37} f_6 \overline{f_{37}}\,\overline{f_6}$, $g_{155} = f_6 + \overline{f_6}$, $g_{156} = f_6 - \overline{f_6}$,

$g_{157} = -f_6 + \overline{f_6}$, $g_{158} = -f_6 - \overline{f_6}$, $g_{159} = g_{149} f_{36} \overline{f_{36}}$, $g_{160} = g_{149} f_{36} \overline{f_{37}}$,

$g_{161} = g_{149} f_{37} \overline{f_{36}}$, $g_{162} = g_{149} f_{37} \overline{f_{37}}$, $g_{163} = g_{150} f_{36}$, $g_{164} = g_{150} f_{37}$,

$g_{165} = g_{150} \overline{f_{36}}$, $g_{166} = g_{150} \overline{f_{37}}$, $g_{167} = g_{151} + g_{159}$, $g_{168} = -g_{152} + g_{160}$,

$g_{169} = -g_{153} + g_{161}$, $g_{170} = g_{154} + g_{162}$, $g_{171} = \dfrac{-g_{137}}{g_{112}^{\;2} - g_{136}^{\;2}}$, $g_{172} = \dfrac{-g_{138}}{g_{113}^{\;2} - g_{136}^{\;2}}$,

$g_{173} = \dfrac{-g_{139}}{g_{114}^{\;2} - g_{136}^{\;2}}$, $g_{174} = \dfrac{-g_{140}}{g_{115}^{\;2} - g_{136}^{\;2}}$, $g_{175} = \dfrac{-g_{141}}{f_3^{\;2} - g_{136}^{\;2}}$, $g_{176} = \dfrac{-g_{142}}{-f_3^{\;2} - g_{136}^{\;2}}$,

$g_{177} = \dfrac{-g_{143}}{\overline{f_3}^{\;2} - g_{136}^{\;2}}$, $g_{178} = \dfrac{-g_{144}}{-\overline{f_3}^{\;2} - g_{136}^{\;2}}$, $g_{179} = g_{112} g_{171}$, $g_{180} = g_{113} g_{172}$,

$g_{181} = g_{114} g_{173}$, $g_{182} = g_{115} g_{174}$, $g_{183} = g_{175} f_3$, $g_{184} = -f_3 g_{176}$,

$g_{185} = g_{177} \overline{f_3}$, $g_{186} = -g_{178} \overline{f_3}$, $g_{187} = \dfrac{-g_{167}}{g_{155}^{\;2} - g_{136}^{\;2}}$, $g_{188} = \dfrac{-g_{168}}{g_{156}^{\;2} - g_{136}^{\;2}}$,

$$g_{189}=\frac{-g_{169}}{g_{157}^{\ 2}-g_{136}^{\ 2}},\quad g_{190}=\frac{-g_{170}}{g_{158}^{\ 2}-g_{136}^{\ 2}},\quad g_{191}=\frac{-g_{163}}{f_6^{\ 2}-g_{136}^{\ 2}},\quad g_{192}=\frac{-g_{164}}{-f_6^{\ 2}-g_{136}^{\ 2}},$$

$$g_{193}=\frac{-g_{165}}{\overline{f_6}^{\ 2}-g_{136}^{\ 2}},\quad g_{194}=\frac{-g_{166}}{-\overline{f_6}^{\ 2}-g_{136}^{\ 2}},\quad g_{195}=g_{155}g_{187},\quad g_{196}=g_{156}g_{188},$$

$$g_{197}=g_{157}g_{189},\quad g_{198}=g_{158}g_{190},\quad g_{199}=g_{191}f_6,$$

$$g_{200}=-g_{192}f_6,\quad g_{201}=g_{193}\overline{f_6},\quad g_{202}=-g_{194}\overline{f_6},$$

$$g_{203}=g_{171}e^{g_{112}}+g_{172}e^{g_{113}}+g_{173}e^{g_{114}}+g_{174}e^{g_{115}}+$$
$$+g_{175}e^{f_3}+g_{176}e^{-f_3}+g_{177}e^{\overline{f_3}}+g_{178}e^{\overline{-f_3}},$$

$$g_{204}=g_{187}e^{-g_{155}}+g_{188}e^{-g_{156}}+g_{189}e^{-g_{157}}+g_{190}e^{-g_{158}}+$$
$$+g_{191}e^{-f_6}+g_{192}e^{f_6}+g_{193}e^{-\overline{f_6}}+g_{194}e^{\overline{f_6}},$$

$$g_{205}=g_{171}+g_{172}+g_{173}+g_{174}+g_{175}+g_{176}+g_{177}+g_{178},$$

$$g_{206}=g_{187}+g_{188}+g_{189}+g_{190}+g_{191}+g_{192}+g_{193}+g_{194},$$

$$g_{207}=g_{179}+g_{180}+g_{181}+g_{182}+g_{183}+g_{184}+g_{185}+g_{186},$$

$$g_{208}=g_{195}+g_{196}+g_{197}+g_{198}+g_{199}+g_{200}+g_{201}+g_{202},$$

$$g_{209}=e^{2g_{136}},\quad g_{210}=1-g_{209},\quad g_{211}=-g_{209}+1,\quad g_{212}=-g_{204}+g_{206}+g_{203}-g_{205},$$

$$g_{213}=g_{136}+g_{136}g_{209},\quad g_{214}=g_{203}g_{136}+g_{207},\quad g_{215}=-g_{209}g_{136}-g_{136},$$

$$g_{216}=-g_{204}g_{136}+g_{208},\quad g_{217}=g_{101}g_{215},\quad g_{218}=g_{101}g_{216}-g_{214},$$

$$g_{219}=\frac{g_{212}g_{213}-g_{218}g_{210}}{-g_{211}g_{213}+g_{217}g_{210}},\quad g_{220}=\frac{g_{218}+g_{219}g_{217}}{g_{213}},\quad g_{221}=-g_{222}-g_{95},$$

$$g_{222}=g_{103}+g_{224},\quad g_{223}=g_{224}+g_{96},\quad g_{224}=g_{106},\quad g_{225}=g_{220},$$

$$g_{226}=-g_{203}-g_{225}g_{209},\quad g_{227}=-g_{204}-g_{228}g_{209},\quad g_{228}=g_{219}.$$

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