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EFFECT OF CAPILLARITY ON FOURTH ORDER NONLINEAR EVOLUTION EQUATION FOR TWO STOKES WAVE TRAINS IN DEEP WATER IN THE PRESENCE OF AIR FLOWING OVER WATER

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Fourth order nonlinear evolution equations, which are a good starting point for the study of nonlinear water waves, are derived for deep water surface capillary gravity waves in the presence of second waves in which air is blowing over water. Here it is assumed that the space variation of the amplitude takes place only in a direction along which the group velocity projection of the two waves overlap. A stability analysis is made for a uniform wave train in the presence of a second wave train. Graphs are plotted for the maximum growth rate of instability wave number at marginal stability and wave number separation of fastest growing sideband component against wave steepness. Significant improvements are noticed from the results obtained from the two coupled third order nonlinear Schrödinger equations.

Key words: nonlinear evolution equation, surface capillary gravity wave, stability.

1. Introduction

A reasonable approach to the study of the stability of finite amplitude surface gravity waves in deep water is through the application of the lowest order nonlinear evolution equation, which is a nonlinear Schrödinger equation. This analysis is suitable for small wave steepness and for long wavelength perturbations. But, for wave steepness greater than *0.15*, predictions from nonlinear Schrödinger equations do not agree with the exact results of Longuet-Higgins (1978a; b).

Dysthe (1979) showed that stability analysis made from fourth-order nonlinear evolution equation which is one order higher than nonlinear Schrödinger equation, gives results consistent with the exact results of Longuet-Higgins (1978a; b) and with the experimental results of Benjamin and Feir (1967) for wave steepness up to *0.25*.The fourth-order effects give a surprising improvement compared to ordinary nonlinear Schrödinger effects in many respects, and some of these points were elaborated by Janssen (1983). The dominant new effect that comes in the fourth order is the influence of wave-induced mean flow and this produces a significant deviation in the stability character. So it can be concluded that a fourth-order evolution equation is a good starting point for studying nonlinear effects of surface waves in deep water. Fourth-order nonlinear evolution equations for deep-water surface gravity waves in different contexts and stability

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analysis made from them were derived by several authors Debsharma and Das (2002; 2005; 2007), Dhar and Das (1990; 1991; 1994), Hogan (1985), Majumder and Dhar (2009; 2011).

All these analyses made by the said authors are for a single wave. A stability analysis of a surface gravity wave in deep water in the presence of a second wave was made by Roskes (1976) based on the lowest order nonlinear evolution equation, which consists of two coupled nonlinear Schrödinger equations. In his investigation modulational perturbation is restricted to a direction along which group velocity projections of the two waves overlap and it is argued that the modulation will grow at a faster rate along this direction when $0 \le \theta < 70.5^0$, where θ is the angle between the two propagation directions of two waves.

In the present paper we extend the analysis performed by Dhar and Das (1991) for the case of capillary waves in the presence of air blowing over water. Two coupled fourth order nonlinear evolution equations are derived for a surface capillary gravity waves in deep water that propagate in the presence of a second wave. It is supposed that the space variation of the amplitude takes place in a direction along which group velocity projection of the two waves overlap. A stability analysis is made for the case when the amplitude of the two waves is the same. The maximum growth rate of instability and wave number at marginal stability and wave number separation of fastest growing sideband component are derived and graphs are ploted for these expressions against wave steepness for different values of wind velocity v and θ . Significant derivations are noticed from the results obtained from the third-order evolution equations which consist of two coupled nonlinear Schrödinger equations.

2. Basic equations

A common horizontal interface between water and air in the undisturbed state is taken as the $z = 0$ plane. In the undisturbed state air flows over water with a velocity U in a direction that is taken as the x- axis. We consider that the two waves move in the x-y plane with wave numbers k_1 and k_2 respectively. We take the x- axis in a direction along which group velocity projection of the two waves overlap and consider the modulations only along this line. We take $z=\zeta(x, y, t)$ as the equation of the common interface at any time t in the perturbed state. We introduce the dimensionless quantities $\phi^*, \phi'^* \zeta^*, \left(x^*, y^*, z^*\right)$, $\left(x^*, y^*, z^*\right), t^*, v^*, s^*$ and γ^* which are, respectively, the perturbed velocity potential in water, perturbed velocity potential in air, surface elevation of the air water interface, space coordinates, time, air flow velocity, surface tension and the ratio of densities ρ', ρ of air to water. These dimensionless quantities are related to the corresponding dimensional quantities by the following relations

$$\phi^* = \sqrt{k_0^3/g}\phi, \qquad \phi'^* = \sqrt{k_0^3/g}\phi', \qquad \zeta^* = k_0\zeta, \qquad \left(x^*, y^*, z^*\right) = \left(k_0x, k_0y, k_0z\right),$$

$$t^* = \omega t, \qquad v^* = \sqrt{k_0/g}U, \qquad s^* = \frac{Tk_0^2}{\rho g}, \qquad \gamma^* = \frac{\rho'}{\rho}$$

where k_0 is some characteristic wave number, T is the surface tension and g is the acceleration due to gravity. In the future, all the quantities will be written in their dimensionless form with their asterisks ($*$) dropped.

The perturbed velocity potentials ϕ, ϕ' for water and air respectively satisfy the following Laplace equations for a irrotational flow of incompressible, inviscous fluids

$$\nabla^2\phi = 0 \qquad \text{in} \qquad -\infty < z < \zeta, \tag{2.1}$$

$$\nabla^2\phi' = 0 \qquad \text{in} \qquad \zeta < z < \infty\ . \tag{2.2}$$

The kinematic boundary conditions to be satisfied at the interface are as follows.

$$\frac{\partial\phi}{\partial z} - \frac{\partial\zeta}{\partial t} = \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}, \qquad \text{when} \qquad z = \zeta\ , \tag{2.3}$$

$$\frac{\partial\phi'}{\partial z} - \frac{\partial\zeta}{\partial t} - v\frac{\partial\zeta}{\partial x} = \frac{\partial\phi'}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi'}{\partial y}\frac{\partial\zeta}{\partial y}, \qquad \text{when} \qquad z = \zeta\ . \tag{2.4}$$

The dynamical condition of continuity of pressure at the interface gives

$$\begin{aligned}
&\frac{\partial\phi}{\partial t} - \gamma\frac{\partial\phi'}{\partial t} + (1-\gamma)\rho - \gamma v\frac{\partial\phi'}{\partial x} = \\
&= -\frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\} + \frac{\gamma}{2}\left\{\left(\frac{\partial\phi'}{\partial x}\right)^2 + \left(\frac{\partial\phi'}{\partial y}\right)^2 + \left(\frac{\partial\phi'}{\partial z}\right)^2\right\} + \\
&+ s\left\{1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right\}^{-\frac{3}{2}} \left\{\begin{array}{l} \left(\frac{\partial\zeta}{\partial x}\right)^2 \frac{\partial^2\zeta}{\partial y^2} + \left(\frac{\partial\zeta}{\partial y}\right)^2 \frac{\partial^2\zeta}{\partial x^2} + \\ -2\left(\frac{\partial\zeta}{\partial x}\right)\left(\frac{\partial\zeta}{\partial y}\right)\frac{\partial^2\zeta}{\partial x\partial y} + \frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2} \end{array}\right\},
\end{aligned}$$

when $\qquad z = \zeta\ .$ (2.5)

Finally, ϕ and ϕ' should satisfy the following conditions at infinity as

$$\begin{aligned} &\phi \to 0 \qquad \text{as} \qquad z \to \infty, \\ \text{and} \qquad &\phi' \to 0 \qquad \text{as} \qquad z \to -\infty. \end{aligned} \tag{2.6}$$

Since the disturbance is assumed to be a progressive wave we can look for a solution to the above equations in the form

$$G = G_{00} + \sum_{\substack{m=0 \\ m=n\neq 0}}^{\infty} \sum_{n=0}^{\infty}\left[G_{mn}\exp i\left(m\psi_1 + n\psi_2\right) + G^*_{mn}\exp - i\left(m\psi_1 + n\psi_2\right)\right] \tag{2.7}$$

where $\qquad \psi_1 = k_{1x}x + k_{1y}y - \omega_1 t, \qquad \psi_2 = k_{2x}x + k_{2y}y - \omega_2 t\ ,$ (2.8)

and G stands for ϕ, ϕ', ζ and star denotes complex conjugate. Here ϕ_{00}, ϕ'_{00}, ϕ_{mn}, ϕ'_{mn}, ϕ^*_{mn}, ϕ'^*_{mn} are slowly varying functions of z, $x_1 = \varepsilon x, y_1 = \varepsilon y, t_1 = \varepsilon t$, and ζ_{00}, ζ_{mn}, ζ^*_{mn} are the functions of

x_1, y_1, t_1 and ε is a slowness parameter. We consider the simplifying assumption that the wave numbers $|k_1| = \left(k_{1x}^2 + k_{1y}^2\right)^{\frac{1}{2}}$ and $|k_2| = \left(k_{2x}^2 + k_{2y}^2\right)^{\frac{1}{2}}$ are the same and let this common wave number be equal to k_0, the characteristic wave number. Also we have $k_0 = 1$ and obtain the following linear dispersion relation

$$\lambda(\omega) \equiv (1+\gamma)\omega^2 - 2\gamma\omega\ v\cos\frac{\theta}{2} + \gamma v^2 c\text{os}^2\frac{\theta}{2} - (1-\gamma) - s = 0 \tag{2.9}$$

where θ is the angle between the two propagation directions of two waves having wave numbers $|k_1|$ and $|k_2|$, respectively. From Eq.(2.9) we have

$$\omega_{\pm} = \left[\gamma v\cos\frac{\theta}{2} \pm \sqrt{\left(1-\gamma^2\right) - \gamma v^2\cos^2\frac{\theta}{2} + (1+\gamma)s}\right] \Big/ (1+\gamma), \tag{2.10}$$

which corresponds to two modes and we designate this two modes as positive and negative modes.

The positive mode moves in the positive direction of the *x*-axis with a frequency

$$\left[\sqrt{\left(1-\gamma^2\right) - \gamma v^2\cos^2\frac{\theta}{2} + (1+\gamma)s} + \gamma v\cos\frac{\theta}{2}\right] \Big/ (1+\gamma),$$

while the negative mode moves in the negative direction of the *x*-axis with a frequency

$$\left[\sqrt{\left(1-\gamma^2\right) - \gamma v^2\cos^2\frac{\theta}{2} + (1+\gamma)s} - \gamma v\cos\frac{\theta}{2}\right] \Big/ (1+\gamma).$$

If v is replaced by $-v$ the frequency of the positive mode becomes equal to the frequency of the negative mode. So the results for the negative mode can be obtained from those for the positive mode by replacing v by $-v$. Therefore we have made a nonlinear analysis for the positive mode only and then we have obtained the results for the negative mode by replacing v by $-v$. From the expression Eq.(2.10) for $\omega_{\pm}$ we find that for linear stability v should satisfy the following condition

$$|v| \le \frac{1}{\cos\frac{\theta}{2}} \cdot \sqrt{\frac{(1+\gamma)(1+s-\gamma)}{\gamma}}. \tag{2.11}$$

Thus our present analysis will remain valid as long as the dimensionless flow velocity of the air becomes less than the critical value $\frac{1}{\cos\frac{\theta}{2}} \cdot \sqrt{\frac{(1+\gamma)(1+s-\gamma)}{\gamma}}$. For air flowing over water $\gamma = 0.00129$ and for $\theta = 0$, $s = 0.075$, this critical value becomes *28.87*.

3. Derivation of the evolution equations

By a standard procedure as shown in Dhar and Das (1991), we find that $\zeta_1 = \varepsilon\zeta_{101} + \varepsilon^2\zeta_{102}$ and $\zeta_2 = \varepsilon\zeta_{011} + \varepsilon^2\zeta_{012}$, where ζ_1 and ζ_2 are the complex amplitudes of the first and second wave, respectively, satisfying the following fourth order nonlinear evolution equations, where we assume that the complex amplitudes depend on the ξ coordinate only and not on the coordinate perpendicular to the ξ axis

$$\begin{aligned} & i\frac{\partial\zeta_1}{\partial\tau} + \delta_{11}\frac{\partial^2\zeta_1}{\partial\xi^2} + i\delta_{12}\frac{\partial^3\zeta_1}{\partial\xi^3} = \zeta_1\left(\beta_{11}|\zeta_1|^2 + \beta_{12}|\zeta_2|^2\right) + i\sigma_{11}\zeta_1\zeta_1{}^*\frac{\partial\zeta_1}{\partial\xi} + \\ & + i\sigma_{12}\zeta_1{}^2\frac{\partial\zeta^*{}_1}{\partial\xi} + i\sigma_{13}\zeta_2\zeta_2{}^*\frac{\partial\zeta_1}{\partial\xi} + i\sigma_{14}\zeta_1\zeta_2{}^*\frac{\partial\zeta_2}{\partial\xi} + \\ & + i\sigma_{15}\zeta_1\zeta_2\frac{\partial\zeta^*{}_2}{\partial\xi} + \mu\zeta_1 \mathrm{H}\frac{\partial\left(\zeta_1\zeta_1^*\right)}{\partial\xi} + \mu\zeta_1 \mathrm{H}\frac{\partial\left(\zeta_2\zeta_2^*\right)}{\partial\xi} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} & i\frac{\partial\zeta_2}{\partial\tau} + \delta_{11}\frac{\partial^2\zeta_2}{\partial\xi^2} + i\delta_{12}\frac{\partial^3\zeta_2}{\partial\xi^3} = \zeta_2\left(\beta_{11}|\zeta_2|^2 + \beta_{12}|\zeta_1|^2\right) + i\sigma_{11}\zeta_2\zeta_2{}^*\frac{\partial\zeta_2}{\partial\xi} + \\ & + i\sigma_{12}\zeta_2{}^2\frac{\partial\zeta^*{}_2}{\partial\xi} + i\sigma_{13}\zeta_1\zeta_1{}^*\frac{\partial\zeta_2}{\partial\xi} + i\sigma_{14}\zeta_2\zeta_1{}^*\frac{\partial\zeta_1}{\partial\xi} + \\ & + i\sigma_{15}\zeta_2\zeta_1\frac{\partial\zeta^*{}_1}{\partial\xi} + \mu\zeta_2 \mathrm{H}\frac{\partial\left(\zeta_2\zeta_2^*\right)}{\partial\xi} + \mu\zeta_2 \mathrm{H}\frac{\partial\left(\zeta_1\zeta_1^*\right)}{\partial\xi} \end{aligned} \tag{3.2}$$

where the coefficients $\delta_{11}, \delta_{12}, \beta_{11}, \beta_{12}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \mu$ are given in the Appendix and where

$$\xi = \varepsilon(x - ut), \qquad \tau = \varepsilon^2 t\,. \tag{3.3}$$

Here u is the component of group velocity of any one of the two waves along the x-axis and is given by $u = \cos\left(\frac{\theta}{2}\right)\left(\frac{d\omega}{dk}\right)_{k=1}$, where θ is the angle between the two propagation directions of two waves having wave numbers $|k_1|$ and $|k_2|$, respectively, and H is the Hilbert transform given by

$$\mathrm{H}\left(\psi\right) = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\psi\left(\xi'\right)d\xi'}{\xi' - \xi}\,. \tag{3.4}$$

For $\theta = 0, \gamma = 0, v = 0$ and in the absence of the second wave the coupled evolution equations given by Eqs (3.1) and (3.2) reduce to a single equation which becomes same as Eq.(2.20) of Hogan (1985) for a one dimensional case. The interaction coefficient $\overline{\beta}_{12} = -\frac{\beta_{12}}{4}$ depends on γ, v, s and θ.

4. Stability analysis of a finite amplitude wave trains

The evolution Eqs (3.1) and (3.2) admit the Stokes wave solution

$$\zeta_1 = \frac{\alpha_1}{2}\exp\left(i\Delta\omega_1\tau\right) \qquad \text{and} \qquad \zeta_2 = \frac{\alpha_2}{2}\exp\left(i\Delta\omega_2\tau\right) \tag{4.1}$$

where α_1, α_2 are real constants and $\Delta\omega_1$, $\Delta\omega_2$ are the nonlinear frequency shifts of two waves. As the two waves have the same wave number equal to 1, the change in the phase speeds of the two waves Δc_1 and Δc_2 are given by

$$\Delta c_1 = \Delta\omega_1 / |\boldsymbol{k_1}| = \Delta\omega_1 = -\frac{1}{4}\left(\beta_{11}\alpha_1^2 + \beta_{12}\alpha_2^2\right), \tag{4.2}$$

$$\Delta c_2 = \Delta\omega_2 / |\boldsymbol{k_2}| = \Delta\omega_2 = -\frac{1}{4}\left(\beta_{11}\alpha_2^2 + \beta_{12}\alpha_1^2\right). \tag{4.3}$$

The change in the phase speeds of each wave train is therefore made up of two parts. The first correction to c_1 is given by $-\frac{1}{4}\beta_{11}\alpha_1^2$ which is the well known Stokes correction. This term is due to the nonlinearity of the wave train itself and is present even if the second wave train is absent. The second correction is given by $-\frac{1}{4}\beta_{12}\alpha_2^2$ and is entirely due to the presence of the second wave train. It is of the same order as the usual Stokes correction.

To study modulational instability of these uniform wave trains we introduce the following perturbations in the uniform solutions

$$\zeta_j = \frac{\alpha_j}{2}\left(1 + \zeta_j'\right)\exp i\left(\Delta\omega_j\tau + \theta_j'\right), \qquad (j = 1, 2) \tag{4.4}$$

where ζ'_j, θ'_j, $(j = 1,2)$ are small real perturbations in amplitudes and phases, respectively. Substituting Eq.(4.4) for $j = 1, 2$ into Eqs (3.1) and (3.2) respectively, linearizing and then assuming the space time dependence of ζ'_j, θ'_j, $(j = 1,2)$ to be of the form $\exp i\left(\lambda\xi - \Omega'\tau\right)$, we arrive at the following nonlinear dispersion relation, the details of derivation are given in Dhar and Das (1991).

$$\overline{P_1} = -\left[\frac{\alpha^2}{4}\left(\sigma_{11} + \sigma_{13} + \sigma_{14}\right)\lambda\right] \pm \left[\overline{P_2}\left\{\overline{P_2} + \frac{\alpha^2}{2}\left(\beta_{11} + \beta_{13} - 2\mu\lambda\right)\right\}\right]^{1/2}, \tag{4.5}$$

here

$$\overline{P_1} = \Omega - u\lambda + \delta_{12}\lambda^3, \tag{4.6}$$

$$\overline{P}_2 = \delta_{11}\lambda^2, \tag{4.7}$$

$$\Omega = \Omega' + u\lambda, \tag{4.8}$$

and α is the common amplitude of the two waves.

From Eq.(4.5), it follows that for instability we must have

$$\overline{P}_2\left[\overline{P}_2+\frac{\alpha^2}{2}\left(\beta_{11}+\beta_{12}-2\mu\lambda\right)\right]<0\ , \tag{4.9}$$

and if this condition is satisfied , then the maximum growth rate Γ_M is given by

$$\Gamma_M=\frac{\left(\beta_{11}+\beta_{12}\right)\alpha^2}{4}\left[1-\frac{\mu\alpha}{\sqrt{\delta\left(\beta_{11}+\beta_{12}\right)}}\right]\quad \text{where}\quad \delta=-\delta_{11}>0,\quad \beta_{11}+\beta_{12}>0\ , \tag{4.10}$$

$$\Gamma_M=\frac{\left(\beta_{11}+\beta_{12}\right)\alpha^2}{4}\left[1+\frac{\mu\alpha}{\sqrt{\delta_{11}\beta}}\right]\quad \text{where}\quad \delta_{11}>0,\quad \beta=-\left(\beta_{11}+\beta_{12}\right)>0\ . \tag{4.11}$$

At marginal stability we obtain

$$\overline{P_2}\left[\overline{P_2}+\frac{\alpha^2}{2}\left(\beta_{11}+\beta_{12}-2\mu\lambda\right)\right]=0\ , \tag{4.12}$$

and this gives the following expression for the wave number λ at marginal stability

$$\lambda=\sqrt{\frac{\left(\beta_{11}+\beta_{12}\right)}{2\delta}}\,\alpha\left[1-\frac{\mu\alpha}{\sqrt{2\delta\left(\beta_{11}+\beta_{12}\right)}}\right]\quad \text{where}\quad \delta=-\delta_{11}>0,\quad \beta_{11}+\beta_{12}>0\ , \tag{4.13}$$

$$\lambda=\sqrt{\frac{\beta}{2\delta_{11}}}\,\alpha\left[1+\frac{\mu\alpha}{\sqrt{2\delta_{11}\beta}}\right]\quad \text{where}\quad \delta_{11}>0,\quad \beta=-\left(\beta_{11}+\beta_{12}\right)>0\ . \tag{4.14}$$

If we put $\gamma=0$, $v=0$, $s=0$ then Eqs (4.10), (4.11), (4.13) and (4.14) reduce to the corresponding equations of Dhar and Das (1991).

Now the maximum growth rate Γ_M occurs for the wave number

$$\Lambda_M=\frac{1}{2}\sqrt{\frac{\left(\beta_{11}+\beta_{12}\right)}{\delta}}\,\alpha-\frac{3\mu}{8\delta}\alpha^2\qquad \text{where}\qquad \delta=-\delta_{11}>0,\qquad \beta_{11}+\beta_{12}>0\ , \tag{4.15}$$

$$\Lambda_M=\frac{1}{2}\sqrt{\frac{\beta}{\delta_{11}}}\,\alpha+\frac{3\mu}{8\delta_{11}}\alpha^2\ ,\qquad \text{when}\qquad \delta_{11}>0,\quad \beta=-\left(\beta_{11}+\beta_{12}\right)>0\ . \tag{4.16}$$

In Figs 1a, 1b, 1c and 1d the maximum growth rate Γ_M of instability from Eqs (4.10), (4.11) as a function of non dimensional wave steepness α have been plotted for $\gamma=0.00129, s=0.075$ and for some different values of v and θ. From these graphs we observe that a significant improvement can be achieved from the results obtained from the third order nonlinear evolution equations. From these graphs, we also observe that for particular values of θ, Γ_M increases up to certain values of v and then decreases with the increase of v.

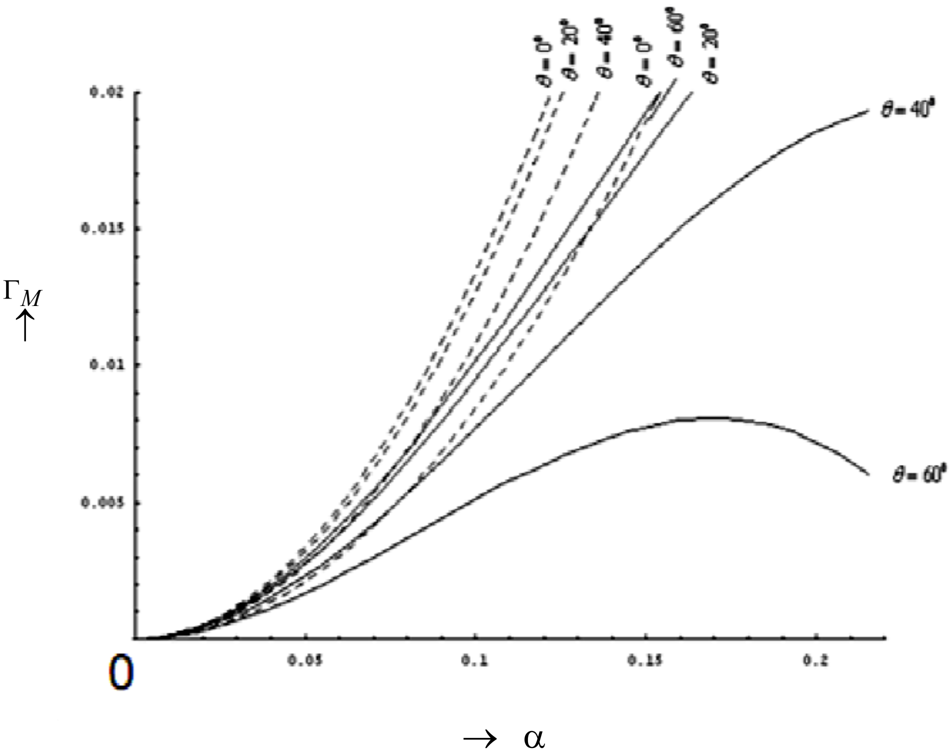

Fig.1a. $v = 0$, $\gamma = 0$, $s = 0.075$.

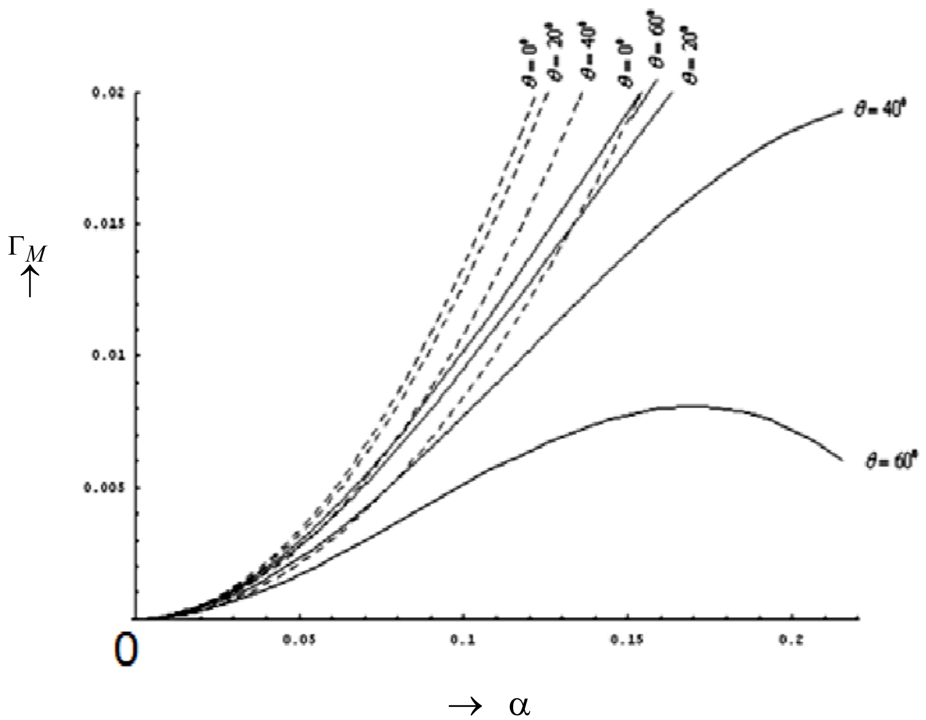

Fig.1b. $v = 10$, $\gamma = 0.00129$, $s = 0.075$.

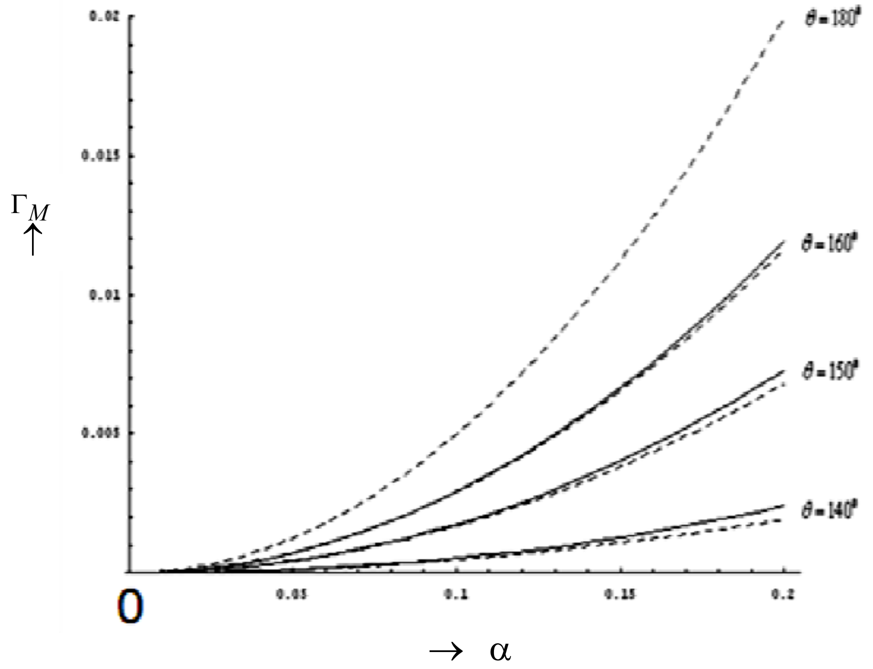

Fig.1c. $v = 0, \gamma = 0, s = 0.075$.

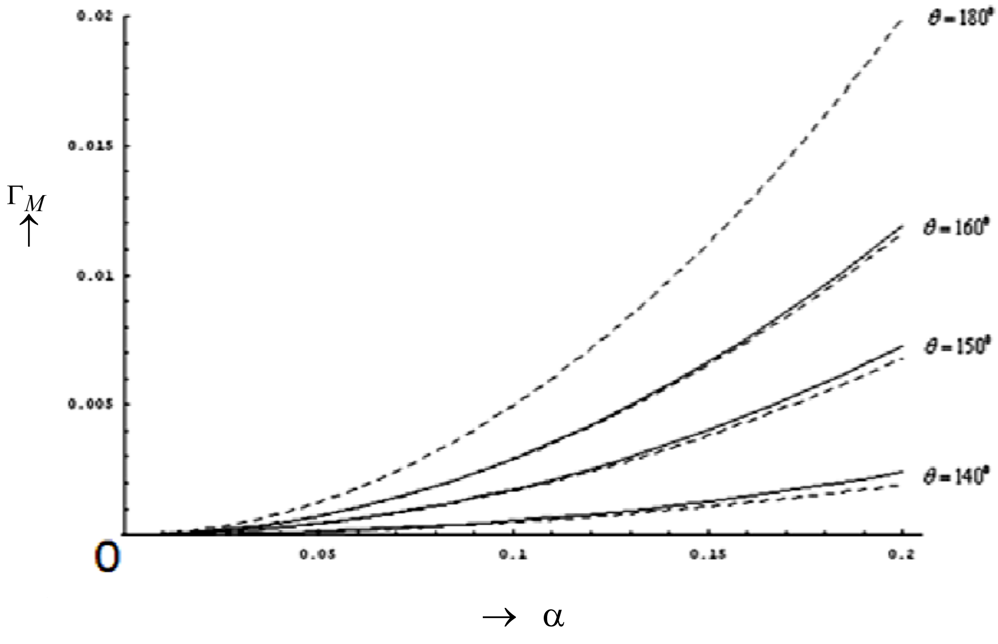

Fig.1d. $v = 6, \gamma = 0.00129, s = 0.075$.

Figs 1a,b,c,d. Maximum growth rate Γ_M as a function of dimensionless wave steepness α . - - - - - - - third-order result; ———————— fourth-order result.

In Figs 2a, 2b, 2c and 2d the wave number λ at marginal stability from Eqs (4.13), (4.14) as a function of non dimensional wave steepness α have been plotted for $\gamma = 0.00129$, $s = 0.075$ and for some different values of v and θ. From these graphs we get the stable-unstable regions.

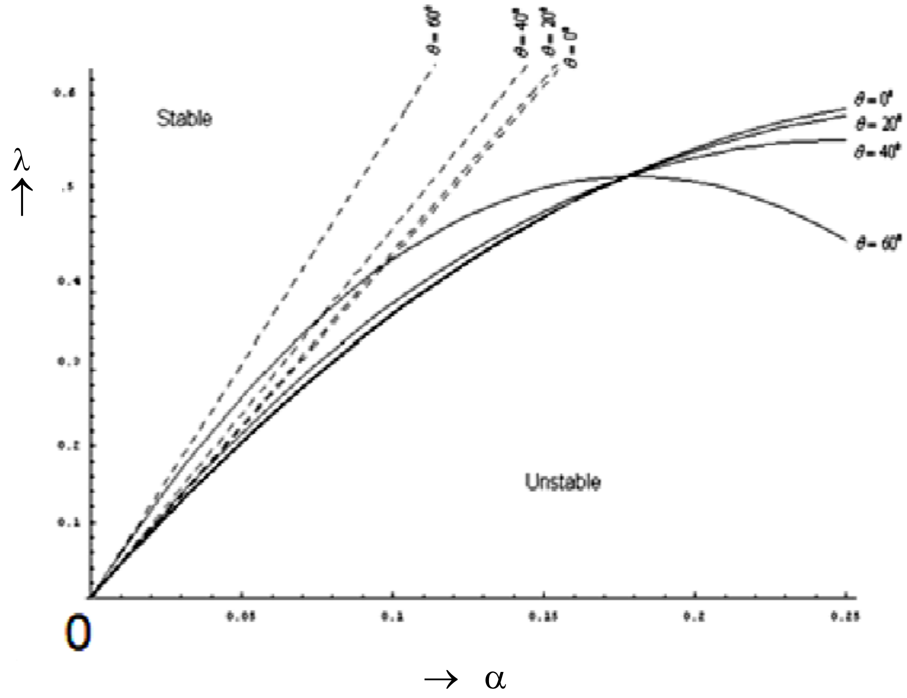

Fig.2a. $v = 0$, $\gamma = 0.00129$, $s = 0.075$.

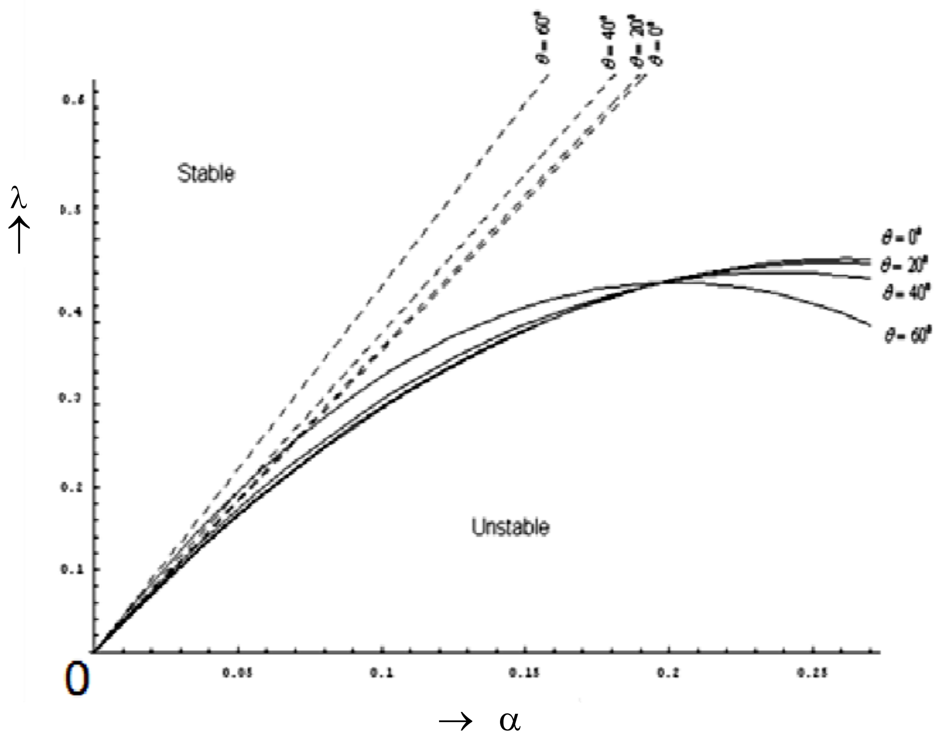

Fig.2b. $v = 10$, $\gamma = 0.00129$, $s = 0.075$.

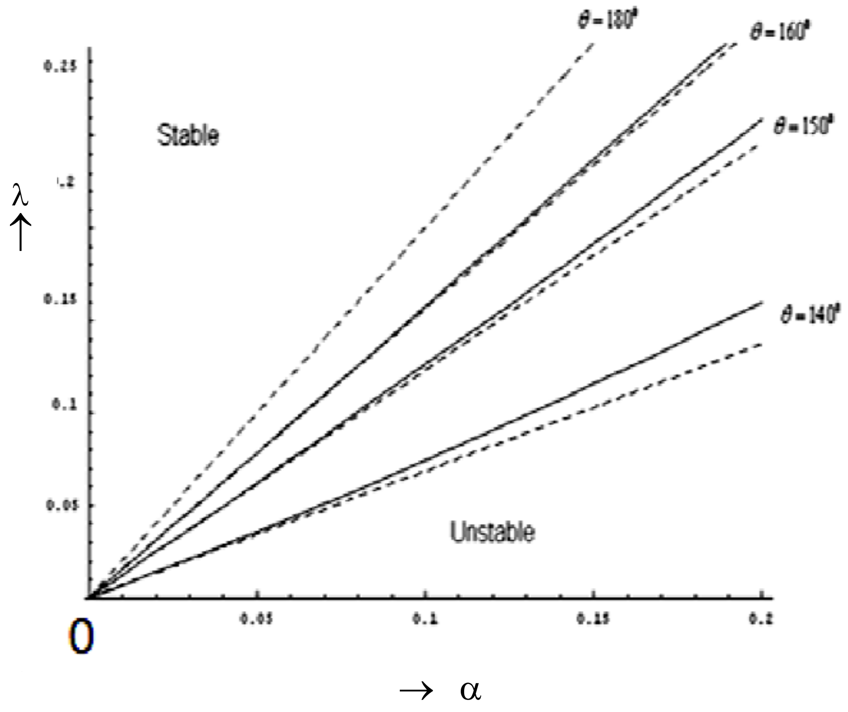

Fig.2c. $v = 0$, $\gamma = 0.00129$, $s = 0.075$.

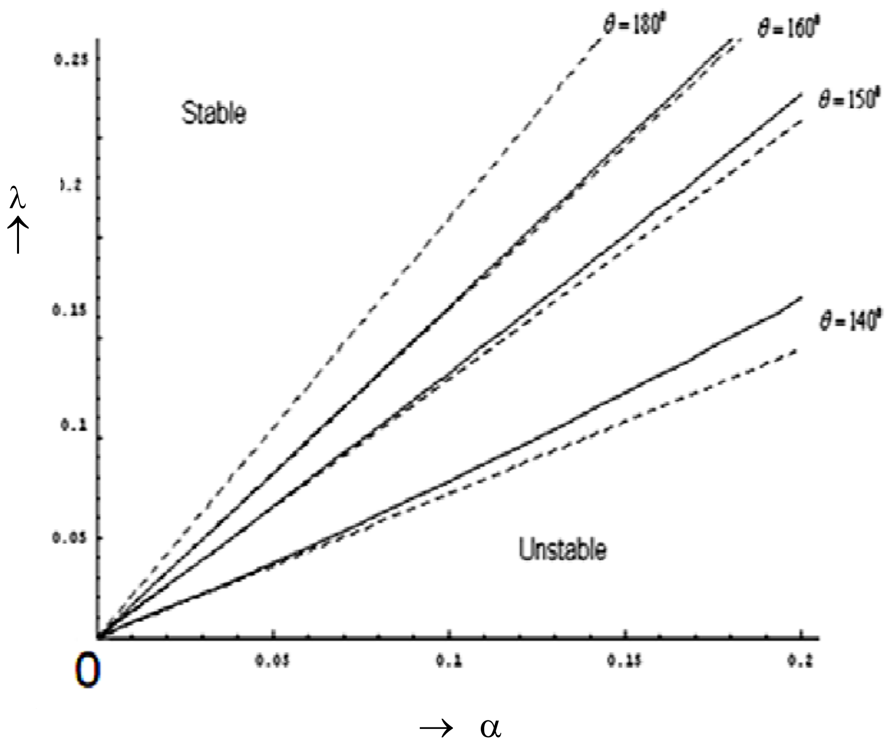

Fig.2d. $v = 6$, $\gamma = 0.00129$, $s = 0.075$.

Figs 2a,b,c,d. Plot of perturbed wave number λ at marginal stability against dimensionless wave steepness α . - - - - - - - third-order result; ———————— fourth-order result.

In Figs 2a, 2b, 2c and 2d we also observe that the fourth order effect produces a decrease in the growth rate giving a stabilising influence and in Figs 2a, 2b, 2c and 2d we get a shrinkage of the instability regions in the $\lambda-\alpha$ plane. In Figs 3a, and 3b the wave number separation Λ_M of fastest growing sideband component from Eqs (4.15), (4.16) as a function of dimensionless wave steepness α has been plotted for $\gamma=0.00129, s=0.075$ and for some different values of v and θ.

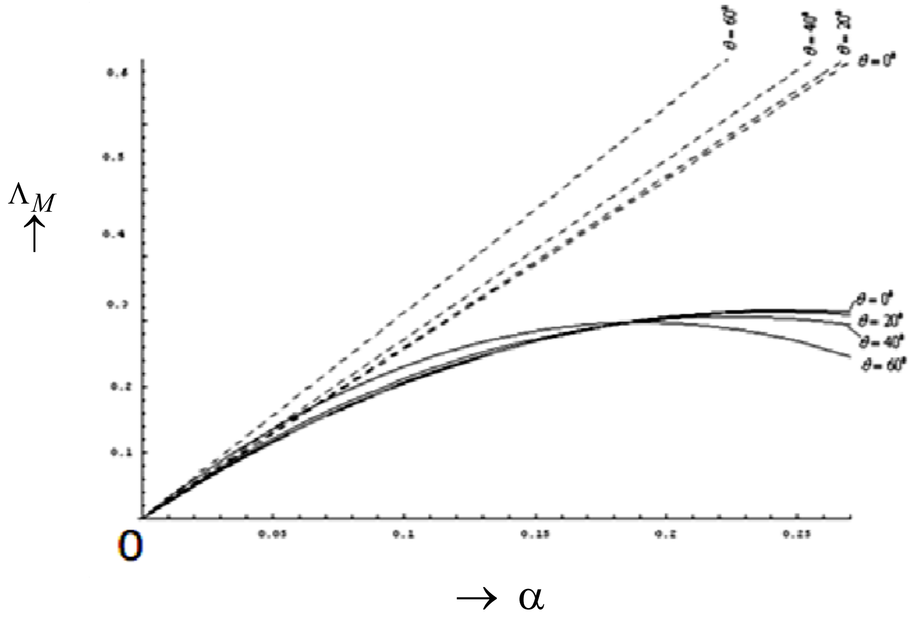

Fig.3a. $v=6$, $\gamma=0.00129$, $s=0.075$.

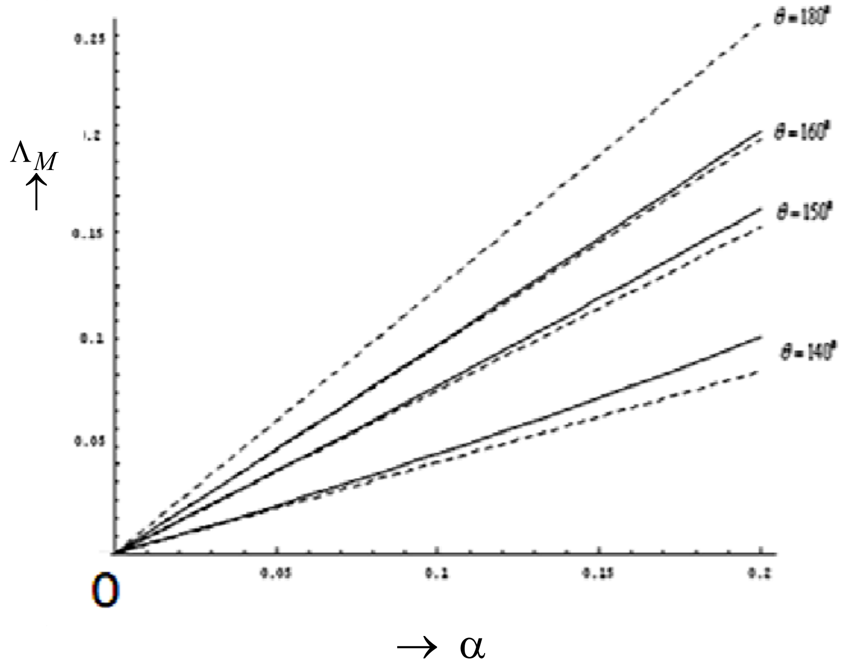

Fig.3b. $v=10$, $\gamma=0.00129$, $s=0.075$.

Figs 3a,b. Wave number separation Λ_M of fastest growing side band against dimensionless wave steepness α . - - - - - - - third-order result; ———————— fourth-order result.

For $\theta = 180^0$, the Hilbert transform terms which only contribute at the fourth order terms vanish identically and so there is no fourth order contribution in the above three expressions for Γ_M, λ and Λ_M.

APPENDIX

The coefficients of Eqs (3.1) and (3.2)

$$\delta_{11} = -\frac{1}{2}\left\{\frac{2(1+\gamma)u^2 - 4\gamma uv + 2\gamma v^2 - (1-\gamma+s)\sin^2\frac{\theta}{2}}{\lambda_\omega}\right\},$$

$$\delta_{12} = \frac{(1-\gamma+s)}{2}\cos\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2} + 2\delta_{11}\left[\frac{(1+\gamma)u - \gamma v}{\lambda_\omega}\right],$$

$$\beta_{11} = 2.\frac{\omega^2 + \gamma\left(\omega - v\cos\frac{\theta}{2}\right)^2 + \left[\omega^2 - \gamma\left(\omega - v\cos\frac{\theta}{2}\right)^2\right]^2 \Big/ (1-\gamma+s)}{\lambda_\omega},$$

$$\beta_{12} = \left[\frac{\left(\cos^3\frac{\theta}{2} + 2\cos^2\frac{\theta}{2} - 2\cos\frac{\theta}{2}\right)\left(3 - \cos\theta - 4\cos\frac{\theta}{2}\right)\left\{\omega^2 - \gamma\left(\omega - v\cos\frac{\theta}{2}\right)^2\right\}^2}{(1-\gamma+s)\left(\cos\frac{\theta}{2} - 2\right)} + \right.$$

$$-2\sin^4\frac{\theta}{2}\left\{\omega^2 - \gamma\left(\omega - v\cos\tfrac{\theta}{2}\right)^2\right\}^2 \Big/ (1-\gamma+s) +$$

$$\left. +2\left(4\cos^2\tfrac{\theta}{2} - 2\cos^3\tfrac{\theta}{2} - 1\right)\left\{\omega^2 + \gamma\left(\omega - v\cos\tfrac{\theta}{2}\right)^2\right\}\right] \Big/ \left\{(1+\gamma+s)\omega - \gamma v\cos\tfrac{\theta}{2}\right\} +$$

$$-\gamma v\left[3\cos\tfrac{\theta}{2}\left(\omega - v\cos\tfrac{\theta}{2}\right)\left\{3\omega^2 + \gamma\left(\omega - v\cos\tfrac{\theta}{2}\right)^2 - \cos\theta\left\{\omega^2 - \gamma\left(\omega - v\cos\tfrac{\theta}{2}\right)^2\right\} + \right.\right.$$

$$\left.\left. -4\cos\tfrac{\theta}{2}\left\{\omega^2 + \gamma\left(\omega - v\cos\tfrac{\theta}{2}\right)^2\right\}\right\}\right\} \Big/ \left\{2\omega^2 + 2\left(\omega - v\cos\tfrac{\theta}{2}\right)^2 - (1-\gamma+s)\cos\tfrac{\theta}{2}\right\} +$$

$$+\cos\tfrac{\theta}{2}\left(\omega - v\cos\tfrac{\theta}{2}\right)\left\{\omega^2 + 3\gamma\left(\omega - v\cos\tfrac{\theta}{2}\right)^2 - \cos\theta\frac{\left\{\omega^2 - \gamma\left(\omega - v\cos\frac{\theta}{2}\right)^2\right\}}{(1-\gamma+s)}\right\} +$$

$$\left. -2\cos\tfrac{\theta}{2}\left(4\cos^2\tfrac{\theta}{2} + 1\right)\left(\omega - v\cos\tfrac{\theta}{2}\right)\right],$$

$$\sigma_{11} = -12\cos\tfrac{\theta}{2}\left[\frac{\omega^2 + \gamma\left(\omega - v\cos\frac{\theta}{2}\right)^2}{\lambda_\omega}\right],$$

$$\sigma_{12}=-2\cos\tfrac{\theta}{2}\left[\frac{\omega^2+\gamma\left(\omega-v\cos\tfrac{\theta}{2}\right)^2}{\lambda_\omega}\right],$$

$$\begin{aligned}\sigma_{13}=\frac{\delta_1}{2}\Bigg[&-\left(2\cos^2\tfrac{\theta}{2}+2\cos\tfrac{\theta}{2}-1\right)\delta_2-2\left(2\cos^2\tfrac{\theta}{2}-1\right)\cos\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2}+\\
&-2\delta_1^4\cos\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}+2\right)\left(1+\sin^2\tfrac{\theta}{2}-2\cos\tfrac{\theta}{2}\right)\Big/\left(\cos\tfrac{\theta}{2}-2\right)+\\
&-2\cos\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2}-5\cos\tfrac{\theta}{2}+2\cos^2\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2}+7\cos^2\tfrac{\theta}{2}+\\
&+2\cos^3\tfrac{\theta}{2}\sin\tfrac{\theta}{2}+\tfrac{1}{2}\cos\tfrac{\theta}{2}\Big\{(2\cos^2\tfrac{\theta}{2}+2\cos\tfrac{\theta}{2}-1)\delta_3+\\
&+2\left(2\cos^2\tfrac{\theta}{2}-1\right)\sin^2\tfrac{\theta}{2}-2\delta_1^4\cos\tfrac{\theta}{2}\frac{\left(1+\sin^2\tfrac{\theta}{2}-2\cos\tfrac{\theta}{2}\right)}{\left(\cos\tfrac{\theta}{2}-2\right)}+2\cos^3\tfrac{\theta}{2}-1\Big\}\Bigg],\end{aligned}$$

$$\begin{aligned}\sigma_{14}=\frac{\delta_1}{2}\Bigg[&-\left(2\cos^2\tfrac{\theta}{2}+2\cos\tfrac{\theta}{2}-1\right)\delta_2-2\delta_1^4\cos\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}+2\right)\frac{\left(1+\sin^2\tfrac{\theta}{2}-2\cos\tfrac{\theta}{2}\right)}{\left(\cos\tfrac{\theta}{2}-2\right)}+\\
&-2\cos\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2}\left(3-2\cos^2\tfrac{\theta}{2}\right)-5\cos\tfrac{\theta}{2}+2\cos^2\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2}+\tfrac{1}{2}\cos\tfrac{\theta}{2}\Big\{\left(2\cos^2\tfrac{\theta}{2}+2\cos\tfrac{\theta}{2}-1\right)\delta_3+\\
&-2\delta_1^4\cos^2\tfrac{\theta}{2}\frac{\left(1+\sin^2\tfrac{\theta}{2}-2\cos\tfrac{\theta}{2}\right)}{\left(\cos\tfrac{\theta}{2}-2\right)}-2\sin\tfrac{\theta}{2}\left(2\cos^2\tfrac{\theta}{2}-1\right)+2\cos^3\tfrac{\theta}{2}-2\cos\tfrac{\theta}{2}+1\Big\}-2\cos\tfrac{\theta}{2}\Bigg],\end{aligned}$$

$$\begin{aligned}\sigma_{15}=\frac{\delta_1}{2}\Bigg[&-2\cos\theta\cos\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2}-2\delta_1^4\cos\tfrac{\theta}{2}\left(2\cos^3\tfrac{\theta}{2}-\cos\tfrac{\theta}{2}+2\right)\frac{\left(1+\sin^2\tfrac{\theta}{2}-2\cos\tfrac{\theta}{2}\right)}{\left(\cos\tfrac{\theta}{2}-2\right)}+\\
&-2\cos\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2}-2\cos^3\tfrac{\theta}{2}\sin\tfrac{\theta}{2}+4\cos^2\tfrac{\theta}{2}-2\cos\tfrac{\theta}{2}+\tfrac{1}{2}\cos\tfrac{\theta}{2}\Big\{2\cos\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2}-4\sin^3\tfrac{\theta}{2}+\\
&+2\delta_1^4\cos\tfrac{\theta}{2}\frac{\left(1+\sin^2\tfrac{\theta}{2}-2\cos\tfrac{\theta}{2}\right)}{\left(\cos\tfrac{\theta}{2}-2\right)}+\cos\theta\Big\}-2\cos\tfrac{\theta}{2}\Bigg],\end{aligned}$$

$$\mu=\frac{4\left\{\omega^2+\gamma\left(\omega-v\cos\tfrac{\theta}{2}\right)^2\right\}\cos^2\tfrac{\theta}{2}}{\lambda_\omega}$$

where

$$u=\frac{\left\{2\gamma v\omega\cos\tfrac{\theta}{2}-2\gamma v^2\cos^2\tfrac{\theta}{2}+(1-\gamma+s)\right\}}{\lambda_\omega}\cos\tfrac{\theta}{2},$$

$$\lambda_\omega=2\left\{\left(1+\gamma\right)\omega-\gamma v\cos\tfrac{\theta}{2}\right\},$$

$$\delta_1 = 2\left\{\omega^2 + \gamma\left(\omega - v\cos\tfrac{\theta}{2}\right)^2\right\}\Big/\lambda_\omega\ ,$$

$$\delta_2 = \left[2\cos\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2} + 2\cos\tfrac{\theta}{2} + 2\cos^2\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2} - 1 + \right.$$
$$\left. + \delta_1^4\cos\tfrac{\theta}{2}\frac{\left(1+\sin^2\tfrac{\theta}{2} - 2\cos\tfrac{\theta}{2}\right)}{\left(\cos\tfrac{\theta}{2} - 2\right)}\right]\Big/\left(\cos\tfrac{\theta}{2} - 2\right),$$

$$\delta_3 = \ \delta_3 = 2\delta_1^4\cos\tfrac{\theta}{2}\left(1+\sin^2\tfrac{\theta}{2} - 2\cos\tfrac{\theta}{2}\right)\left\{1 + 2\Big/\left(\cos\tfrac{\theta}{2} - 2\right)\right\}\Big/\left(\cos\tfrac{\theta}{2} - 2\right).$$

Nomenclature

g – acceleration due to gravity
H – Hilbert's transform operator
s – dimensionless surface tension
t – time
v – air flow velocity
(x, y, z) – space coordinates
α – wave steepness
δ_{11}, δ_{12}; β_{11}, β_{12}; $\sigma_{1i} (i = 1,2,3,4,5)$ – coefficients given in the Appendix
ε – slowness parameter
γ – ratio of densities of air to water
ζ – elevation of the air water interface
λ – wave number
Ω – perturbed frequency at marginal stability
ω – frequency
$\Delta\omega$ – frequency shift

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