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Int. J. of Applied Mechanics and Engineering, 2017, vol.22, No.4, pp.1035-1044
DOI: 10.1515/ijame-2017-0066

SIMPLE FLOWS OF PSEUDOPLASTIC FLUIDS BASED ON DEHAVEN MODEL

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In this paper three simple flows of visco-plastic fluids of DeHaven type or fluids similar to them are considered. These flows are: Poiseuille flow in a plane channel, Poiseuille flow through a circular pipe and rotating Couette flow between two coaxial cylinders. After presentation DeHaven model it was presented some models of fluids similar to this model. Next it was given the solutions of equations of motion for three flows mentioned above.

Key words: DeHaven fluids, similar fluids, simple flows.

1. Introduction

In recent years, rheologists have done a great deal of work on pseudo-plastic fluid flows; the viscosity of these kinds of fluids displays a non-linear relationship between the shear stress and the shear strain rate. To be more precise: in constitutive equations of these fluids the shear strain rate is a non-linear function of the shear stress. There are many known formulae to model this relationship. One of the first was a model presented by Miss S.B. Ellis in 1927 [1]. The next was power-series development and in consequence polynomials were suggested. The polynomial given by Kraemer and Williamson [2] was later independently proposed by Weissenberg's student, Rabinowitsch [3]. In the end of the fifties of the past century DeHaven [4, 5] proposed his own model very similar to the model of Miss Ellis (probably he did not know her model; note that the same model as that proposed by Miss Ellis was formulated forty years later by three other researchers, namely by Mr Ellis *et al.* [6]). A bit later, at the beginning of the sixties of the past century, Rotem and Shinnar [7, 8] returned to the polynomial representation proposing their own model

$$\mu\dot{\gamma} = \tau\left(1 + \sum_{i=1}^{n} k_i \tau^{2i}\right). \tag{1.1}$$

Similar relations were proposed by Whorlow [9]

$$\mu\dot{\gamma} = \sum_{i=0}^{n} k_i \tau^{2i+1} \tag{1.2}$$

or

$$\tau = \sum_{i=0}^{n} k_i \dot{\gamma}^{2i+1} \tag{1.3}$$

known as power-series models.

Each of these models, by suitable choice of material coefficients reduces to the DeHaven model or to the Rabinowitsch model, respectively [10].

Most popular models of fluids which are similar to the DeHaven fluid model are presented in Tab.1.

Table 1. Models of fluids similar to the DeHaven fluid model [11].

Author(s)	Original model	Model taken into account	κ_i	Comments
$n_i = n$				"$n+1$" power models
DeHaven	$\mu_0\dot{\gamma} = \tau\left(1+\kappa\lvert\tau\rvert^n\right)$	–	κ	
Meter	$\tau = \left[\mu_\infty + \dfrac{\mu_0 - \mu_\infty}{1+(\kappa\tau)^n}\right]\dot{\gamma}$	$\mu_0\dot{\gamma} = \tau\left(1+\dfrac{\mu\kappa^n}{\mu_0}\tau^n\right)$	$\dfrac{\mu\kappa^n}{\mu_0}$	$\mu = \mu_0 - \mu_\infty$
$n_i = n-1$				"n" power model
Ellis	$\tau = \dfrac{\mu_0\dot{\gamma}}{1+\kappa\lvert\tau\rvert^{n-1}}$	$\mu_0\dot{\gamma} = \tau\left(1+\kappa\lvert\tau\rvert^{n-1}\right)$	κ	
$n_i = 2$				"Cubic" models
Rotem-Shinnar	$\tau = \dfrac{\mu_0\dot{\gamma}}{1+\sum_i^n \kappa_i\tau^{2i}}$	$\mu_0\dot{\gamma} = \tau\left(1+\kappa\tau^2\right)$	κ	The model has a practical meaning for $i = 1$
Ree-Eyring	$\tau = \mu_0\dot{\gamma}\left[\dfrac{\sinh(\kappa\tau)}{(\kappa\tau)}\right]^{-1}$	$\mu_0\dot{\gamma} = \tau\left(1+\dfrac{\kappa^2}{6}\tau^2\right)$,	$\dfrac{\kappa^2}{6}$	
Rabinowitsch	$\tau = \dfrac{\mu_0\dot{\gamma}}{1+\kappa\tau^2}$	$\mu_0\dot{\gamma} = \tau\left(1+\kappa\tau^2\right)$	κ	
Reiner-Philippoff	$\tau = \left[\mu_\infty + \dfrac{\mu_0 - \mu_\infty}{1+(\kappa\tau)^2}\right]\dot{\gamma}$	$\mu_0\dot{\gamma} = \tau\left(1+\dfrac{\mu\kappa^2}{\mu_0}\tau^2\right)$	$\dfrac{\mu\kappa^2}{\mu_0}$	$\mu = \mu_0 - \mu_\infty$
$n_i = 1$				"Quadratic" models
Peek-McLean	$\tau = \left[\mu_\infty + \dfrac{\mu_0 - \mu_\infty}{1+(\kappa\tau)}\right]\dot{\gamma}$	$\mu_0\dot{\gamma} = \tau\left(1+\dfrac{\mu\kappa}{\mu_0}\tau\right)$	$\dfrac{\mu\kappa}{\mu_0}$	$\mu = \mu_0 - \mu_\infty$
Seely	$\tau = \left[\mu_\infty + \dfrac{\mu_0 - \mu_\infty}{e^{(\kappa\tau)}}\right]\dot{\gamma}$	$\mu_0\dot{\gamma} = \tau\left(1-\dfrac{\mu\kappa}{\mu_0}\tau\right)$	$-\dfrac{\mu\kappa}{\mu_0}$	$\mu = \mu_0 - \mu_\infty$

One-dimensional form of the DeHaven model may be written as

$$\mu_0\dot{\gamma} = \tau\left(1+k_i\lvert\tau\rvert^{n_i}\right) \tag{1.4}$$

whereas its three-dimensional form is as follows

$$\mu_0 \boldsymbol{A}_1 = \Lambda\left(1 + k_i \left|\Lambda\right|^{n_i}\right); \tag{1.5}$$

this form will be used in the next Section to illustrate some flows.
The equations of motion are as follows

$$\operatorname{div} \mathbf{v} = 0, \tag{1.6}$$

$$\rho \frac{d\mathbf{v}}{dt} = \operatorname{div} \boldsymbol{T}, \tag{1.7}$$

$$\boldsymbol{T} = -p\boldsymbol{1} + \Lambda; \tag{1.8}$$

it is necessary to include the constitutive Eq.(1.5) in these equations. Note that the body forces are neglected in Eq.(1.7). The flows of pseudoplastic fluids of DeHaven type were studied first by Matsuhisa and Bird [12], Wadhwa [13] and thereafter by many rheologists; lately, these flows have been studied by Keyfets and Kieweg [14] and Walicka *et al.* [15, 16].

In that follows we will consider three simple flows of the DeHaven type which may be frequently used in practical applications.

2. Poiseuille flow in a plane channel

Let consider the steady laminar one-dimensional flow of a DeHaven fluid, due to a pressure gradient, in a plane channel shown in Fig.1.

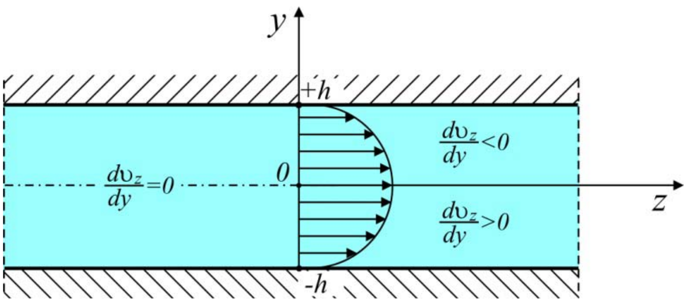

Fig.1. Channel between two parallel plates.

The flow field is given by the following relations

$$\upsilon_x = 0, \quad \upsilon_y = 0, \quad \upsilon_z = \upsilon_z(y), \quad p = p(z). \tag{2.1}$$

The equations of motion given by Eqs (1.6) ÷ (1.8) reduce to

$$\frac{dp}{dz} = \frac{d\Lambda_{yz}}{dy} \tag{2.2}$$

but the constitutive Eq.(1.5) takes the form

$$\mu_0 \frac{d\upsilon_z}{dy} = \Lambda_{yz}\left(1 + k_i \left|\Lambda_{yz}\right|^{n_i}\right). \tag{2.3}$$

The boundary conditions are stated as follows

$$\upsilon_z(\pm h) = 0, \qquad \left.\frac{\partial \upsilon_z}{\partial y}\right|_{y=0} = 0 \ . \tag{2.4}$$

A single integration of Eq.(2.2) gives

$$\Lambda_{yz} = C_1 + y\frac{dp}{dz}. \tag{2.5}$$

Upon putting this result into Eq.(2.3), one obtains the following expression

$$\mu_0 \frac{d\upsilon_z}{dy} = \left(C_1 + y\frac{dp}{dz}\right)\left[1 + k_i\left|C_1 + y\frac{dp}{dz}\right|^{n_i}\right]. \tag{2.6}$$

Introducing here the second boundary condition (2.5) one obtains the following equation

$$0 = C_1\left(k_i + \left|C_1\right|^{n_i}\right),$$

whose unique real root is $C_1 = 0$, then Eq.(2.5) takes the form

$$\mu_0 \frac{d\upsilon_z}{dy} = -y\left(-\frac{dp}{dz}\right) + k_i\left|-y\left(-\frac{dp}{dz}\right)\right|^{n_i+1}. \tag{2.7}$$

Integration of this equation yields

$$\upsilon_z = C_2 - \frac{y^2}{2\mu_0}\left(-\frac{dp}{dz}\right) - \frac{k_i}{\mu_0}\frac{\left|-y\right|^{n_i+2}}{n_i+2}\left(-\frac{dp}{dz}\right)^{n_i+1}.$$

Upon determination of the constant of integration from the first boundary condition (2.4), we obtain finally

$$\upsilon_z = \frac{h^2 - y^2}{2\mu_0}\left(-\frac{dp}{dz}\right) - \frac{k_i}{\mu_0}\frac{h^{n_i+2} - y^{n_i+2}}{n_i+2}\left(-\frac{dp}{dz}\right)^{n_i+1}. \tag{2.8}$$

The flow rate Q per unit width of the channel is defined as

$$Q = 2\int_0^h \upsilon_z \, dy$$

Upon using relation (2.8), we will obtain

$$Q = \frac{2h^3}{3\mu_0}\left(-\frac{dp}{dz}\right)\left[1 + \frac{3k_i h^{n_i}}{(n_i+3)}\left(-\frac{dp}{dz}\right)^{n_i}\right]; \tag{2.9}$$

Note that for the Newtonian flow we have $k_i = 0$ and

$$Q = Q_{Newt} = \frac{2h^3}{3\mu_0}\left(-\frac{dp}{dz}\right). \tag{2.10}$$

3. Poiseuille flow through a circular pipe

Let us consider the steady laminar flow of a DeHaven fluid in a circular pipe of radius R (Fig.2).

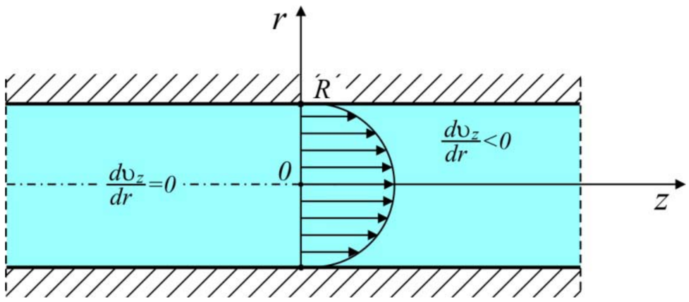

Fig.2. Geometry of a circular pipe.

The flow field has the form of

$$\upsilon_r = 0, \qquad \upsilon_\vartheta = 0, \qquad \upsilon_z = \upsilon_z(r), \qquad p = p(z). \tag{3.1}$$

The equations of motion given by Eqs (1.6) ÷ (1.8) reduce to

$$\frac{dp}{dz} = \frac{1}{r}\frac{d(r\Lambda_{rz})}{dr} \tag{3.2}$$

whereas the constitutive Eq.(1.5) takes the form

$$\mu_0 \frac{d\upsilon_z}{dr} = \Lambda_{rz}\left(1 + k_i \left|\Lambda_{rz}\right|^{n_i}\right). \tag{3.3}$$

The boundary conditions are stated as follows

$$\upsilon_z = 0, \quad \text{for} \quad r = R \quad \text{and} \quad \frac{\partial \upsilon_z}{\partial r} = 0 \quad \text{for} \quad r = 0. \tag{3.4}$$

A single integration of Eq.(3.2) gives

$$\Lambda_{rz} = \frac{C_1}{r} + \frac{r}{2}\frac{dp}{dz}$$

and after introducing this result into Eq.(3.3), we have

$$\mu_0 \frac{d\upsilon_z}{dr} = \left(\frac{C_1}{r} + \frac{r}{2}\frac{dp}{dz}\right)\left[1 + k_i \left|\frac{C_1}{r} + \frac{r}{2}\frac{dp}{dz}\right|^{n_i}\right]. \tag{3.5}$$

Taking into account the second boundary condition (3.4), one obtains that

$$C_1 = 0$$

and Eq.(3.5) takes the form

$$\frac{d\upsilon_z}{dr} = -\frac{r}{2\mu_0}\left(-\frac{dp}{dz}\right) + \frac{k_i}{\mu_0}\left(\frac{r}{2}\right)^{n_i+1}\left(-\frac{dp}{dz}\right)^{n_i+1}, \tag{3.6}$$

whose integral is equal to

$$\upsilon_z = C_2 - \frac{r^2}{4\mu_0}\left(-\frac{dp}{dz}\right) - \frac{k_i}{\mu_0}\frac{r^{n_i+2}}{2^{n_i+1}(n_i+2)}\left(-\frac{dp}{dz}\right)^{n_i+1}.$$

Upon determination of the constant of integration from the first boundary condition (3.4), we will obtain finally

$$\upsilon_z = \frac{R^2 - r^2}{4\mu_0}\left(-\frac{dp}{dz}\right) - \frac{k_i}{\mu_0}\frac{R^{n_i+2} - r^{n_i+2}}{2^{n_i+1}(n_i+2)}\left(-\frac{dp}{dz}\right)^{n_i+1}. \tag{3.7}$$

The flow rate Q is defined us

$$Q = 2\pi\int_0^R \upsilon_z r dr.$$

Upon using relation (3.7), we will find

$$Q = \frac{\pi R^4}{8\mu_0}\left(-\frac{dp}{dz}\right)\left[1 + \frac{8k_i R^{n_i}}{2^{n_i+1}(n_i+4)}\left(-\frac{dp}{dz}\right)^{n_i}\right]; \tag{3.8}$$

Note that for the Newtonian flow, we have $k_i = 0$ and

$$Q = Q_{Newt} = \frac{\pi R^4}{8\mu_0}\left(-\frac{dp}{dz}\right). \tag{3.9}$$

4. Rotating Couette flow between two coaxial cylinders

Let us consider the flow of a DeHaven fluid in the clearance between two coaxial cylinders shown in Fig.3.

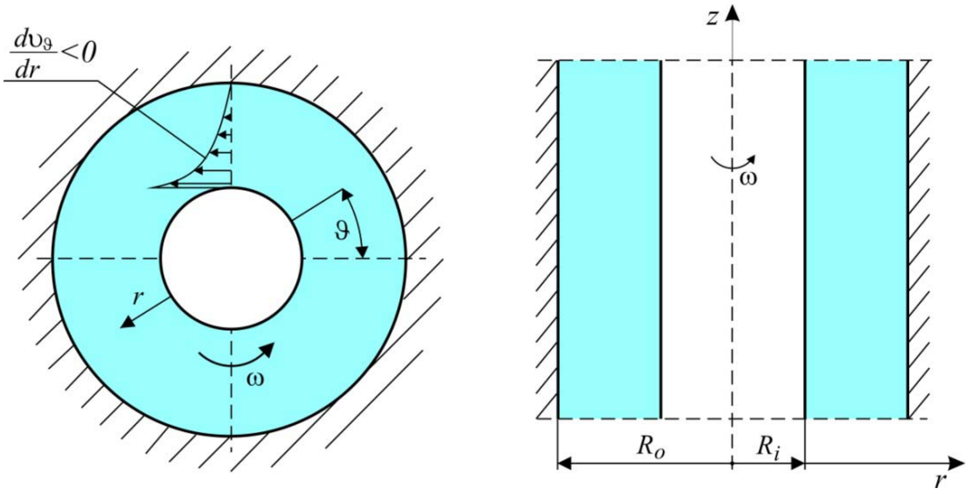

Fig.3. Geometry of the rotational flow between cylindrical surfaces.

The flow field is given by the relationships

$$\upsilon_r = 0, \qquad \upsilon_\vartheta = \upsilon_\vartheta(r), \qquad \upsilon_z = 0, \qquad p = p(r). \tag{4.1}$$

The equations of motion (1.6) ÷ (1.8) now take the form

$$\rho\frac{\upsilon_\vartheta^2}{r} = \frac{dp}{dr}, \tag{4.2}$$

$$\frac{d}{dr}\left(r^2\Lambda_{r\vartheta}\right) = 0 \tag{4.3}$$

but the constitutive Eq.(1.5) takes the form

$$\mu_0 r\frac{d}{dr}\left(\frac{\upsilon_\vartheta}{r}\right) = \Lambda_{r\vartheta}\left(1 + k_i\left|\Lambda_{r\vartheta}\right|^{n_i}\right). \tag{4.4}$$

The boundary conditions are as follows

$$\upsilon_\vartheta = R_i\omega, \quad \text{for} \quad r = R_i, \qquad \upsilon_\vartheta = 0, \quad \text{for} \quad r = R_o. \tag{4.5}$$

Upon integration of Eq.(4.3), we will obtain

$$\Lambda_{r\vartheta} = \frac{C_1}{r^2}$$

and after introducing this result into Eq.(4.4), we will have

$$\mu_0 r \frac{d}{dr}\left(\frac{\upsilon_\vartheta}{r}\right) = -\frac{(-C_1)}{r^2} - \frac{k_i(-C_1)^{n_i+1}}{r^{2n_i+2}} \tag{4.6}$$

because $C_1 < 0$. The next integration gives

$$\upsilon_\vartheta(r) = C_2 r + \frac{(-C_1)}{2\mu_0 r^2} + \frac{k_i}{\mu_0}\frac{(-C_1)^{n_i+1}}{(2n_i+2)r^{2n_i+1}}. \tag{4.7}$$

On the basis of the second boundary condition (4.5) we will obtain

$$C_2 = -\frac{(-C_1)}{2\mu_0 R_o^2} + \frac{k_i}{\mu_0}\frac{(-C_1)^{n_i+1}}{(2n_i+2)R_o^{2n_i+2}} \tag{4.8}$$

and finally

$$\omega = \frac{(-C_1)}{2\mu_0}\left(\frac{1}{R_i^2} - \frac{1}{R_o^2}\right) + \frac{k_i}{\mu_0}\frac{(-C_1)^{n_i+1}}{(2n_i+2)}\left(\frac{1}{R_i^{2n_i+2}} - \frac{1}{R_o^{2n_i+2}}\right). \tag{4.9}$$

The angular velocity $\omega(r) = \dfrac{\upsilon_\vartheta}{r}$ at any position r is expressed as

$$\omega(r) = C_2 + \frac{(-C_1)}{2\mu_0 r^2} + \frac{k_i}{\mu_0}\frac{(-C_1)^{n_i+1}}{(2n_i+2)r^{2n_i+2}}. \tag{4.10}$$

The unit torque acting on the cylindrical surface of radius r is equal to

$$T = 2\pi r^2 \Lambda_{r\vartheta} = 2\pi C_1. \tag{4.11}$$

Note that T_s is also the torque which has to be applied to the inner cylinder to maintain its motion. Denoting by T_r the anti-torque which must be applied to the outer cylinder to maintain its rest we have

$$T = \begin{cases} T_r = -2\pi C_1, \\ T_s = 2\pi C_1. \end{cases} \tag{4.12}$$

Introducing the notation

$$\beta = \frac{R_i}{R_o}, \qquad \tilde{r} = \frac{r}{R_o} \tag{4.13}$$

one can write:

$$\omega-\omega(r)=\frac{1}{2\mu_0}\left(\frac{T}{2\pi R_o^2}\right)\left(\frac{1}{\beta^2}-\frac{1}{\tilde{r}^2}\right)+ \\ +\frac{k_i}{\mu_0\left(2n_i+2\right)}\left(\frac{T}{2\pi R_o^2}\right)^{n_i+1}\left(\frac{1}{\beta^{2n_i+2}}-\frac{1}{\tilde{r}^{2n_i+2}}\right) \tag{4.14}$$

the formula which can be used for determining the material constants in the DeHaven model or similar models from measurements of torque and angular velocity in a coaxial annular viscosimeter [12] if it is equipped with a suitable software [17].

5. Conclusions

The simple flows of pseudoplastic fluids based on DeHaven model may find many applications in different branches of technology and industry.

It can cite e.g. the theory of lubrication. Basing on the method of solution the flow in a plane channel one may obtain the solution of the flow in bearing clearances; the solution for the flow in a circular pipe may be used to find the flow in bearing clearances with porous walls [11].

Nomenclature

$\boldsymbol{A}_1$ – the first Rivlin-Ericksen kinematic tensor
e – Naperian logarithm base
k, k_i – pseudo-plasticity coefficients
n – exponential rheological parameter
p – pressure
Q – flow rate
$\boldsymbol{T}$ – shear stress tensor
T – torque
t – time
$\mathbf{v}$ – velocity vector
υ_k – components of velocity vector
$\boldsymbol{1}$ – unit tensor
$\dot{\gamma}$ – shear strain rate
Λ – extra stress tensor
$|\Lambda|$ – magnitude of extra stress tensor
μ – shear viscosity
μ_0, μ_∞ – limiting values of shear viscosity
ρ – fluid density
τ – shear stress
ω – angular velocity

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Received: June 23, 2017

Revised: July 5, 2017