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# IMPROVEMENT OF THE GRAPHICAL METHOD FOR PLOTTING THE SHEAR AND MOMENT DIAGRAMS FOR MEMBERS SUBJECTED TO LINEARLY VARYING LOADS

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This study presents an improvement of the graphical method for plotting the shear and moment diagrams for the structural members under linearly varying loads (triangular and trapezoidal loads). Based on the parabolic nature of the shear function, when the loading varies linearly, and on the relations among load, shear, and moment, a mathematical equation is developed to locate the zero-shear point, while a geometric technique is presented to calculate the parabolic shear area. Five comprehensive examples of beams loaded with linearly varying loads are selected to illustrate the steps of the solution for the proposed techniques. The results demonstrated the applicability of the presented method, and gave exact diagrams compared with the basic graphical method. It is concluded that the proposed improved method is generally more convenient, less time-consuming, and has less computational efforts because it does not require sectioning, solving equilibrium equations, and quadratic formulas compared with the basic graphical method.

**Key words:** shear diagrams, moment diagram, graphical method, improvement, varying loads.

## 1. Introduction

When a structural member is subjected to external loads, internal forces are developed in order to maintain the equilibrium [1, 2]. The internal shear forces and the internal bending moments are considered to be the foremost imperative internal forces in the analysis and design of the structural members because the computations of stresses and deformations depend mainly on these two essential internal forces [3]. Since the internal shear forces and bending moments are varying from point to point along the member, it is helpful to visualize this variation by constructing plots called the shear and moment diagrams [1, 2, 4, 5].

The shear and moment diagrams are graphs with an abscissa representing the locations of the sections along the member, and an ordinate representing the values of the internal shears and internal moments at the corresponding sections, respectively [2]. The shear diagram ($V$-diag. or $S$-diag.) and the moment diagram (M-diag.) provide detailed information about the change in shear and moment throughout the member as well as the maximum values and their locations [1].

There are many approaches of constructing (or plotting) the shear and moment diagrams for the loaded members. However, two methods are the most common methods that are discussed in the literature; the method of sections and the graphical method.

The method of sections is the basic method for constructing the shear and moment diagrams. This method was presented by Hibbeler [1], Onouye and Kane [2], Goodno and Gere [4], Kassimali [5], Pytel and Kiusalaas [6], Muvdi and Elhour [7], Beer *et al.* [8], Hibbeler [9], Limbrunner and D'Allaird [10], Mott and Untener [11], and Ranzi and Gilbert [12]. In this method, the functions of the shear and the moment are developed for each segment between the discontinuity points throughout the member. The discontinuity points represent the points of the sudden change in loading and the support reactions. Each segment is sectioned at a distance $x$, usually from the starting point on the left of the member, then a free body diagram is constructed

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for each segment to establish the equations of the shear and the moment in terms of the distance *x* by applying the equilibrium equations. Finally, the equations of the shear and moment for each segment are plotted to achieve the shear and moment diagrams.

Another common method is the graphical method; which is generally preferred by engineers in plotting the shear and moment diagrams because it is faster and simpler compared with the method of sections. This method was presented by Hibbeler [1], Onouye and Kane [2], Kassimali [5], Pytel and Kiusalaas [6], Beer *et al.* [8], and Hibbeler [9]. The graphical method is based on the relations among the loading, shear force and bending moment, in which the values and the properties of the shear and moment diagrams are determined according to the relations below:

- The slope of the shear diagram equals the intensity of the load diagram and the change in the shear equals the area under the load diagram.
- The slope of the moment diagram equals the intensity of the shear diagram and the change in the moment equals the area under the shear diagram.

The values of the shear due to the concentrated force and the moment due to couple moments are established according to a sign convention, while in the case of the distributed loads these values are calculated from the areas under the load diagram and the shear diagram, respectively. The degree of the diagram lines is also identified based on the above relations.

The method of integration is one of the approaches for constructing the shear and moment diagrams. This method is reported by Goodno and Gere [4], Kassimali [5], Beer *et al.* [8], and Mott and Untener [11]. In this method, the functions of the shear and the moment are developed by integrating the load and the shear functions, respectively, and then plotting the diagrams.

Boedo [3] and Beer *et al.* [8] presented a technique of using the polynomial-based singularity function, in mathematics, to construct the shear and moment diagrams. In this technique, the load diagram is represented by a singularity function, then the integration is used to develop the functions of the shear and moment, and finally plotting these functions as diagrams. This approach is characterized by some complexity due to using long polynomial functions and mathematical processes to achieve the diagrams.

On the other hand, as a result of the challenges that researchers confront in sketching the shear and moment diagrams, in their courses, Lumsdaine and Ratchukool [13] developed multimedia tools, whereas Philpot *et al.* [14] investigated the influence of using the computer-aid tools for improving the skills of constructing the shear and moment diagrams. Moreover, Le *et al.* [15] presented, to the students, a simple graphical procedure for plotting the shear and moment diagrams for the members subjected to simple loading cases; concentrated loads and uniformly distributed loads.

The structural members are always exposed to various types of loadings while performing their functions in the structures, such as the concentrated loads, uniformly distributed loads, and linearly varying distributed loads. The linearly varying distributed loads (triangular and trapezoidal loads) which can be applied to the structural members due to supported two-way roofs and the hydrostatic pressures, are considered to be one of the common loading cases in civil engineering structures. But the task of plotting the shear and moment diagrams for the members under these loadings is characterized by some difficulty and is time-consuming compared with plotting these diagrams for concentrated and uniformly distributed loads.

Due to the parabolic nature of the shear diagrams, dealing with linearly varying loads (triangular and trapezoidal loads) still involves several slow stages. Even if the graphical method is employed, the locating of the zero-shear point, which is a necessary value in plotting the shear and moment diagrams, requires sectioning the member and applying equilibrium equations, which is a time-consuming process with the potential for errors. However, Onouye and Kane [2] adopted a quicker approach in utilizing the graphical method for the beams under triangular loadings by computing the area under the loading diagram in terms of the location for the zero-shear point, then finding the location based on equating the load intensity with the change in shear forces. In addition, for calculating the area under the parabolic curve, they used the formula of the area under the parabola (the area equals one third of the base times the height) to obtain the values of the moment diagram. Although the technique presented by Onouye and Kane [2] is helpful, dealing with members under trapezoidal loads was not considered in their technique.

The purpose of this study is to present an improvement for the graphical method of plotting the shear and moment diagrams for the structural members under linearly varying distributed loads (triangular and trapezoidal loads). The proposed method intends to contribute to a quick and helpful procedure based on the parabolic shape and the relations among the external loading, internal shear, and internal moment, rather than employing the monotonous sectioning technique. The study is motivated by the need of simpler and more convenient techniques for locating the zero-shear point and computing the area under the shear parabolic diagram for the structural members under triangular and trapezoidal loads, as the available methods are utilizing the tedious sectioning technique to achieve this goal.

## 2. Methodology

This study aims to provide a quick geometric approach for drawing the shear and moment diagrams of the structural members under linearly varying loads (triangular and trapezoidal loads). The proposed approach is based on the general graphical method but with two improvements relevant to the zero-shear point and the area under the shear diagram. Hence, this work includes two specific points; formulation of a technique for locating the zero-shear point on the shear diagram, and presenting a geometric technique for calculating the area under the parabolic shear diagram.

It is necessary to establish the sign convention for the loading, shear, and moment in order to define the positive and negative values for these quantities. The sign convention adopted in this study is the same as that often used in practice, and it was adopted by Hibbeler [1], Goodno and Gere [4], Kassimali [5], Muvdi and Elhouar [7], Beer *et al.* [8], and Hibbeler [9]. In this sign convention, the loading that acts upward, the shear force that causes a clockwise rotation of the member segment, and the moment that causes sagging of the member segment are all defined as positive values. The opposite directions are considered as negative values.

### 2.1. Formulation of a technique for locating the zero-shear point

In the construction of the shear and moment diagrams, determining the location of the zero-shear section is necessary because it is utilized in plotting the moment diagram and identifying the position of the peak value of the moment. This subsection presents the formulation of a geometric technique to locate the zero-shear point for the members under linearly varying loads, that is adopted in this study. The formulation of the geometric technique is based on the equation of the parabola and the relation between the load and the shear functions.

The equation of a parabola, in the *x-y* plane, with a vertex at the origin can be defined as:

$$y = ax^2 , \tag{2.1}$$

in which, $a$ represents the constant of the parabola [16].

The general equation of the parabola ( $y = ax^2 + bx + c$ ) represents a shifted equation from the parabola $y = ax^2$ [17] as shown in Fig.1. This means that the equation of each shifted parabola becomes $y = ax^2$ when the location of its vertex is transferred to the origin.

Since the change in a shear diagram between two points represents the integration of the loading diagram throughout these points [5, 9], then the first-degree of the loading diagram, which occurs in linearly varying loads, leads to a second-degree (parabolic) shear diagram. Therefore, the function of the internal shear for a member under linearly varying loads can be expressed in the form of the parabola $y = ax^2$ as follows:

$$V = ax^2 , \tag{2.2}$$

in which, $V$ is the value of the shear with respect to the vertex of the parabola, $x$ is the location of the section on the member with respect to the vertex of the parabola.

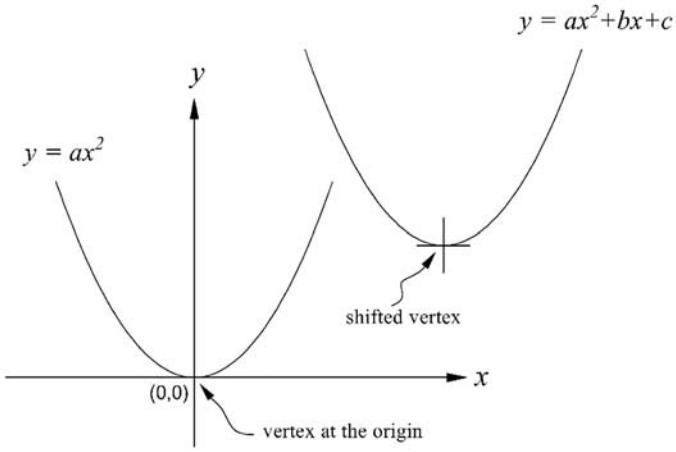


Fig.1. The parabola $y = ax^2$ and its shifted parabola.

The vertex of the parabola in Eq.(2.2) has a zero-slope tangent at the corresponding zero-load point because, as aforementioned in the previous section, the slope of the shear diagram equals the intensity of the load diagram. Hence, the location of the vertex for the parabola in Eq.(2.2) can be identified based on the location of the zero-load point. For the case of triangular loading, the location of the zero-load point is known as the point of the zero-load is available, but in the case of trapezoidal loading, the location of the zero-load point is located at an imaginary point (the point where the load diagram intersects the member). Figures 2 and 3 show the difference in the locations of the parabola vertices under triangular and trapezoidal loadings respectively, noting that the local coordinate axes for the parabola are $x$ and $V$, while the global coordinate axes for the shear diagram are $X$ and $S$.

To find the constant $a$ of the parabola in Eq.(2.2), the first step is computing the slope of the shear diagram by the differentiation of the shear $V$ in Eq.(2.2) with respect to the distance $x$ to obtain:

$$\frac{dV}{dx} = 2ax\,, \tag{2.3}$$

Since the slope of the shear diagram is equal to the load intensity (as aforementioned), then Eq.(2.3) can be expressed in terms of the load intensity on the member as follows:

For the triangular loading with a maximum load intensity $w$, as shown in Fig.2, the vertex of the shear parabola is located below the zero-load point. If the local coordinates of the parabola ( $x$ and $V$ ) are established such that the vertex of the parabola is located at the origin as shown in Fig.2, then the intensity of the load when $x = L$ is equal to $w$. Thus, Eq.(2.3) becomes:

$$\mathrm{w} = 2aL\,, \tag{2.4}$$

which leads to the value of the constant $a$ to be:

$$a = \frac{w}{2L}\,. \tag{2.5}$$

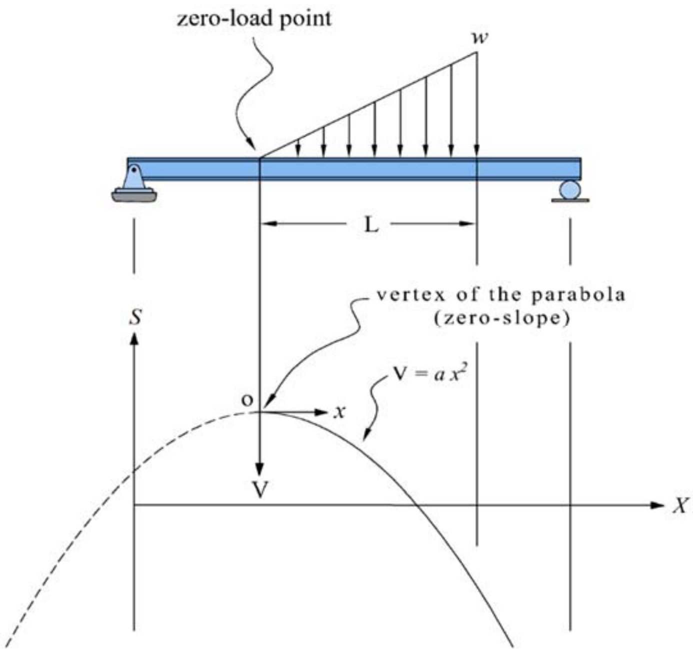


Fig.2. Location of the vertex of the parabola under triangular loading.

In the case of the trapezoidal loading, the constant $a$ can also be determined by following the same steps for the triangular loading. Here, if $s$ represents the distance from the vertex of the parabola to the point of the maximum intensity of the trapezoidal load $w_2$ as shown in Fig.3., then the slope of the shear diagram in Eq.(2.3) is equal to $w_2$ at $x = s$ . Thus, Eq.(2.3) becomes:

$$w_2 = 2as , \tag{2.6}$$

then the constant of the parabola $a$ can be determined as:

$$a = \frac{w_2}{2s} . \tag{2.7}$$

It can be clearly noticed that the value of $(w/L)$ in Eq.(2.5) and the value of $(w_2 / s)$ in Eq.(2.7) are both equal to the slope of the loading diagram $m$ for the triangular and trapezoidal loadings, respectively. Therefore, the constant $a$ of the parabola in Eq.(2.5) and Eq.(2.7) can be expressed in terms of the loading slope $m$ as follows:

$$a = \frac{m}{2} . \tag{2.8}$$

By substituting the value of the parabola constant $a$ from Eq.(2.8) into Eq.(2.2), the equation of the parabola for the shear diagram becomes:

$$V = \frac{m}{2} x^2 \tag{2.9}$$

The above equation represents an equation of a parabola, with a vertex at the origin. It relates the value of the shear $V$ to the location of the section $x$ by a constant and represents the half of the loading diagram slope $(m/2)$, provided that both $V$ and $x$ are measured with respect to the vertex of the parabola. This equation is considered the key equation for the proposed approach in this study because the shear value can be calculated easily from this equation as the slope of the linearly varying loading $m$ can be calculated easily, noting that the value of the shear $V$ in this equation represents the ordinate of the shear diagram with respect to the parabola vertex, not the actual shear value $(S)$ on the member, since the actual shear value on the member is measured according to the main global centroidal axis of the member ( $X$ *-axis*) as shown in Figs 2 and 3. The parabola in Eq.(2.9) represents the curve of the shear diagram in the region below the triangular or trapezoidal loads.

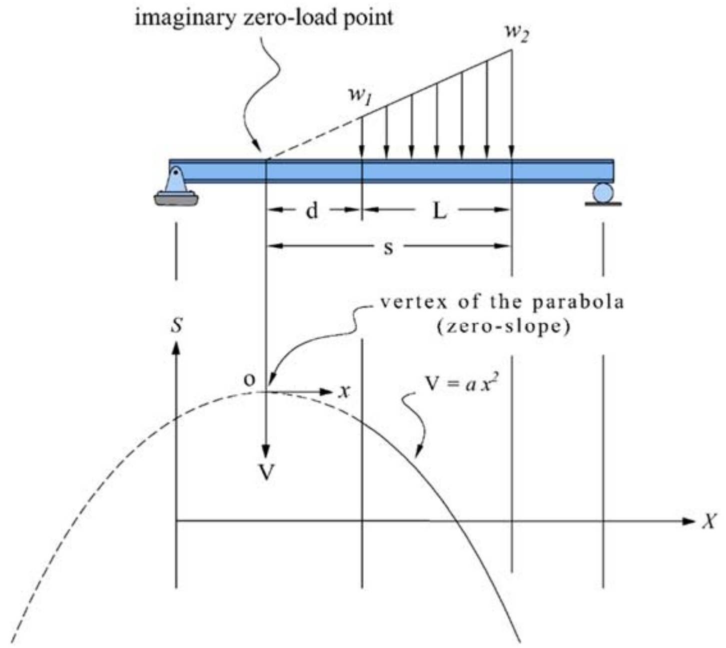


Fig.3. Location of the vertex of the parabola under trapezoidal loading.

Finally, to plot this parabola, it is required to determine the coordinates of its vertex. The vertical coordinate of the vertex can be determined from the change in the shear $(\Delta S)$ by using simple adding and/or subtracting for the shear values on the shear diagram, while the horizontal coordinate of the vertex is determined in different ways for the triangular and trapezoidal loadings.

For the triangular loading, the vertex of the parabola is located horizontally at the same distance of the zero-load point as shown in Fig.2, while for the trapezoidal loading, if the vertex of the parabola is assumed to be located at a distance $d$ from the point of the minimum load intensity of the trapezoidal loading, as shown in Fig.3, then this distance can be calculated from the slope $m$ of the load diagram as:

$$d = \frac{w_1}{m} \quad . \tag{2.10}$$

It can be noticed from the preceding formulation that the zero-shear point can be determined by solving Eq.(2.9) for the distance $x$. The parameters that are needed to solve Eq.(2.9) are: the slope of loading diagram $m$, the location of the parabola vertex, and the value of the shear $V$ with respect to the parabola vertex.

**2.2. Calculating the area under the shear diagram**

Since the function of the shear diagram of a member subjected to a triangular or trapezoidal load is a parabolic equation, then it is convenient to find the area under the shear diagram geometrically, rather than using the method of sections.

If the parabola opens upward with a vertex located at the origin, as shown in Fig.4a, the area under the parabola can be calculated by the following equation [1, 9]:

$$A_1 = \frac{1}{3} b\, h \;, \qquad (2.11)$$

in which, $A_1$ is the area under the parabola which opens upward with a vertex at the origin, $b$ is the horizontal distance from the origin to a given point (the base length), $h$ is the vertical distance from the origin to a given point (the height).

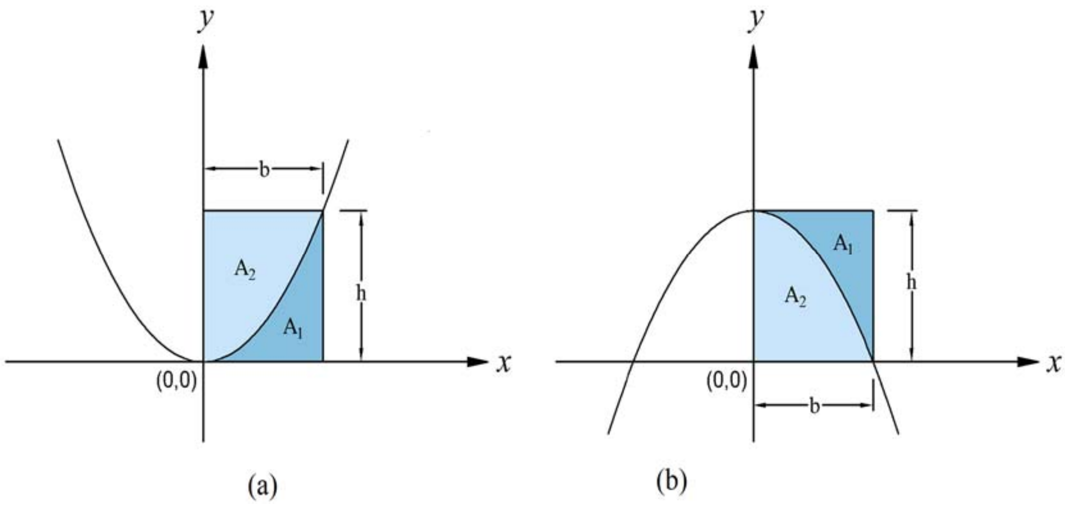


Fig.4. The area under the parabola.

The area of the spandrel portion $A_2$, which is located above the area $A_1$ as shown in Fig.4a, can be calculated by subtracting the area $A_1$ from the rectangular area $(b \times h)$ to get:

$$A_2 = \frac{2}{3} bh \;. \qquad (2.12)$$

When the parabola opens downward as shown in Fig.4b, the areas $A_1$ and $A_2$ can be calculated by using the same equations, Eq.(2.11) and Eq.(2.12), respectively.

Equations (2.11) and (2.12) can be easily proven by applying the concept of the area under curves using the definite integration. But one should pay attention that the areas calculated from these equations were developed based on the location of the vertex of the parabola (the zero-slope point). Therefore, these equations are employed to calculate the area of any portion under or above the parabola in the shear diagram, provided that the base $b$ and the height $h$ of the parabola are measured from the vertex (at zero-slope point). If the parabolic shear diagram has not zero-slope tangent, it is necessary to locate the point of the zero-slope, which represents the location of the vertex of the parabola, in order to apply Eq.(2.11) and Eq.(2.12) correctly.

It is necessary to mention that Onouye and Kane [2] have used the above principle in calculating the area under the shear diagram for the case of the triangular load only (calculating the area under trapezoidal load was not presented).

In the current study, this geometric approach of calculating the parabolic areas is presented for the members under both the triangular loading and the trapezoidal loading.

### 2.3. Procedure for the improved graphical method

After upgrading the method with the techniques presented in this paper, the suggested name of the graphical method is chosen to be the improved graphical method. The proposed procedure of plotting the shear and moment diagrams has the same steps that were utilized within the basic graphical method presented by Hibbeler [1], Kassimali [5], Pytel and Kiusalaas [6], Beer *et al.* [8] except two differences in determining the location of the zero-shear point and the area under the parabolic shear curve. The main steps for determining the location of the zero-shear point and the parabolic area by using the improved graphical method can be summarized as follows:

***Step 1:***
Computing the slope of the loading diagram $m$.
***Step 2:***
Developing the equation of the shear parabola (the equation of the shear $V$ as a function of the distance $x$ with respect to the parabola vertex) by applying Eq.(2.9).
***Step 3:***
Identifying the vertex of the parabola, which it located horizontally corresponding to the zero-load point (for triangular loading) or to the imaginary zero-load point (for trapezoidal loading), and located vertically according to the shear value at a given horizontal distance. In the case of the trapezoidal loading, Eq.(2.10) is used to find the horizontal location, while the parabola equation that was developed in *Step 2* is utilized to find the vertical location of the parabola vertex.
***Step 4:***
Locating the zero-shear point by substituting the value of the shear $V$, at which the parabola intersects the $X$ *-axis,* into the parabolic equation that was developed in *Step 2,* and solving the equation for the distance $x$. The zero-shear point is evaluated geometrically based on the distance $x$.
***Step 5:***
Calculating the area under the parabolic shear diagram by using Eq.(2.11) and/or Eq.(2.12) based on the location of the parabola vertex. The area is utilized to evaluate the maximum and/or minimum points on the moment diagram.

Furthermore, to simplify the calculations of the proposed method, the positive direction of the vertical axis of the parabola ($V$ *-axis*) is assumed in the direction of the focus of the parabola (the positive direction of the vertical axis is assumed to be downward if the parabola opens downward and vice versa), while the positive direction of the horizontal axis of the parabola (x-axis) is assumed to be directed towards the zero-shear point.

## 3. Applications

In order to examine the applicability of the proposed improved graphical method, and to study the differences in solutions obtained by the basic graphical method and improved method, five examples of beams subjected to various cases of linearly varying loads are selected to construct their shear and moment diagrams. In the first two examples, two solutions are presented for locating the zero-shear point and computing the maximum moment values under the regions of linearly varying load; The basic graphical method is used in the first solution since it was used in most of the literature, while the proposed improved graphical method is used as the other solution to illustrate its procedure and to compare the two solutions. The remaining three examples are solved by using only the proposed improved graphical method to illustrate how to apply its procedure. The procedure mentioned in the previous section is followed in the solutions of the proposed method. Other details of how to construct the shear and moment diagrams are not mentioned in these examples because these details are widely available in the literature that deal with the graphical method.

These five examples introduce comprehensive cases of beams that are found in practice. Regarding the frame members, since the procedure of constructing the shear and moment diagrams for the frame members is completely identical to the procedure for the beams, only the examples of the beams are presented in this study to avoid long computations in the frame cases.

**Example 1:** This example is presented by Hibbeler [1]. It is a simply supported beam subjected to a triangular loading along its span as shown in Fig.5. The reactions are calculated as shown in Fig.6a, and the shear and moment diagrams are shown in Figs 6c and 6d, respectively.

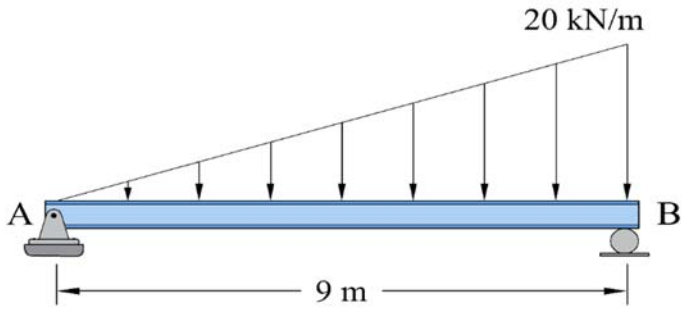


Fig.5: The beam of example 1.

***Solution I (by using the basic graphical method):***

The steps of solution were as follows:

***Step 1:***
Sectioning the beam at a distance $x$ where the internal shear force $S$ equals zero as shown in Fig.6b (the distance $x$ represents the location of the zero-shear point measured from the point $A$).

***Step 2:***
Finding the intensity of the loading at a distance $x$ based on the similarity of triangles, $w = (20/9)x$.

***Step 3:***
Applying the equilibrium equation for the vertical forces in the free body diagram, Fig.6b, to find the shear $S$ as a function of the distance $x$, $S = 30 - (10/9)x^2$.

***Step 4:***
Substituting $S = 0$ into the equation in *step 3* to obtain $x = 5.2\ m$.

***Step 5:***
Evaluating the maximum moment, which occurs at the zero-shear point (at $x = 5.2m$), by using the section technique. The equilibrium equation for the moments in the free body diagram shown in Fig.6b was applied to get the maximum moment of $104\,kN.m$.

***Solution II (by using the improved graphical method):***

Referring to Fig.6, the steps for locating the zero-shear point and computing the value of the maximum moment by using the proposed technique are as follows:

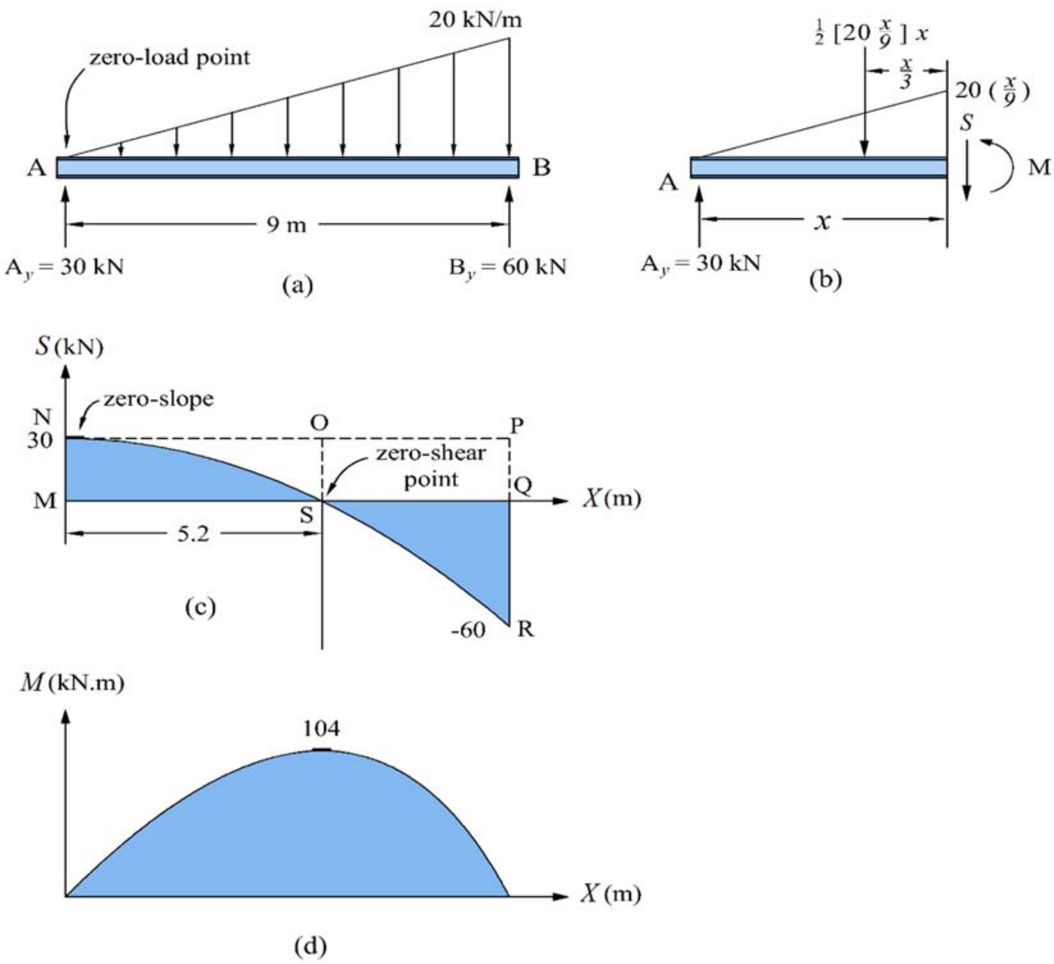


Fig.6. The diagrams of example 1.

***Step 1:***
Computing the slope of the loading diagram *m* by dividing the maximum intensity of the load by the loading length to get $m = 20/9$.

***Step 2:***
Developing the shear parabola by applying Eq.(2.9) to obtain $V = (10/9)x^2$.

***Step 3:***
Identifying the vertex of the parabola to be horizontal at point *A* (below the zero-load point) and vertical at the value of $V = 30kN$ (the value of the reaction at *A*).

***Step 4:***
The parabolic function in *step 2* intersects the *X*-axis when $V = 30kN$ as shown in Fig.6c, thus substituting $V = 30kN$ in the parabolic function $V = (10/9)x^2$ yields the distance $x = 5.2\ m$, which represents the location of the zero-shear point.

***Step 5:***
To find the maximum moment, the shaded area $A_{MNS}$ in Fig.6c is calculated directly according to Eq.(2.12) since the slope at the vertex $N$ is zero (the case of $A_2$ in Fig.4b). By substituting $b = 5.2m$ and $h = 30kN$ in Eq.(2.12), we get a maximum moment of $104\,kN.m$.

This example represents the case of a beam subjected to a common and simple triangular loading. It can be seen that the method proposed in this study (Solution II) gave the same results as the method presented by Hibbeler [1] (Solution I). For comparison, the method proposed in this study seems to be more convenient compared with the method of Solution I because it does not require taking a section and applying equilibrium equations, but it depends on the parabolic equation, $V=(10/9)x^2$, which it developed simply from knowing the slope of the loading diagram. Regarding computing the parabolic areas, the proposed geometric method is characterized by a systematic process depending on simple equations dealing with a simple geometry. It is necessary to note that the distances $b$ and $h$ of the parabola in Fig.6c are measured according to the vertex (point $N$), which has a zero-slope tangent. Even if the number of the steps for the two solutions appears equal, actually Solution I took more computation efforts and time since both the sectioning and the equilibrium equations were utilized twice for determining the zero-shear point and the maximum moment.

**Example 2:** This example is presented by Ranzi and Gilbert [12]. It is a simply supported beam subjected to a trapezoidal loading along its span as shown in Fig.7. The reactions are calculated as shown in Fig.8a, and the shear and moment diagrams are shown in Figs 8c and 8d, respectively.

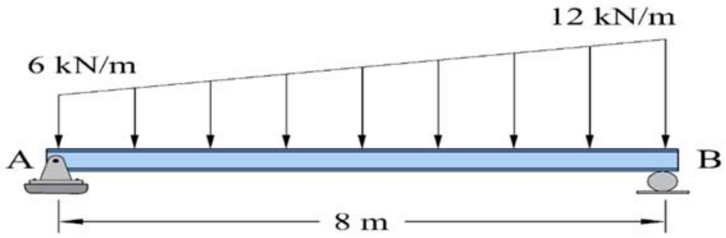


Fig.7. The beam of example 2.

***Solution I (by using the basic graphical method):***

The steps of solution were as follows:

***Step 1:***
Sectioning the beam at a distance $x$ where the internal shear force $S$ equals zero as shown in Fig.8b (the distance $x$ represents the location of the zero-shear point measured from the point $A$).

***Step 2:***
Finding the intensity of the loading at a distance $x$ based on the similarity of triangles, $w=0.75x$.

***Step 3:***
Applying the equilibrium equation for the vertical forces in the free body diagram, Fig.8b, to find the shear $S$ as a function of the distance $x$, $S=(-3/8)x^2-6x+32$.

***Step 4:***
Substituting $S=0$ into the equation in *step 3* and solving the quadratic formula to obtain $x=4.22m$.

***Step 5:***
Evaluating the maximum moment, which occurs at the zero-shear point $(\text{at } x=4.22m)$, by using the section technique. The equilibrium equation for the moments in the free body diagram shown in Fig.8b was applied to get the maximum moment of $72.2\,kN.m$.

***Solution II (by using the improved graphical method):***

Referring to Fig.8, the steps for locating the zero-shear point and computing the value of the maximum moment by using the proposed technique are as follows:

***Step 1:***
Computing the slope of the loading diagram *m* by dividing the difference between the maximum and the minimum load intensities of the trapezoidal load by the loading length to get the slope $m=0.75$

***Step 2:***
Applying Eq.(2.9) to obtain the shear parabolic function $V = 0.375 x^2$.

***Step 3:***
From the minimum load intensity ($w_1=6\ kN/m$) and the loading slope ($m=0.75$), the location of the parabola vertex *d* is obtained by applying Eq.(2.10) to get $d=8\ m$ (to the left of point *A*) as shown in Fig.8c. The ordinate of the parabola vertex is computed by adding the vertical distances *QR* and *RK* in Fig.8c. The distance *QR* represents the reaction at the support *A* (*32 kN*), while the distance *RK* is obtained by substituting the distance $d=8\ m$ into the parabolic function from *step 2* to get $V=24\ kN$. Thus, the ordinate of the vertex becomes ( $32 + 24 = 56\ kN$ ).

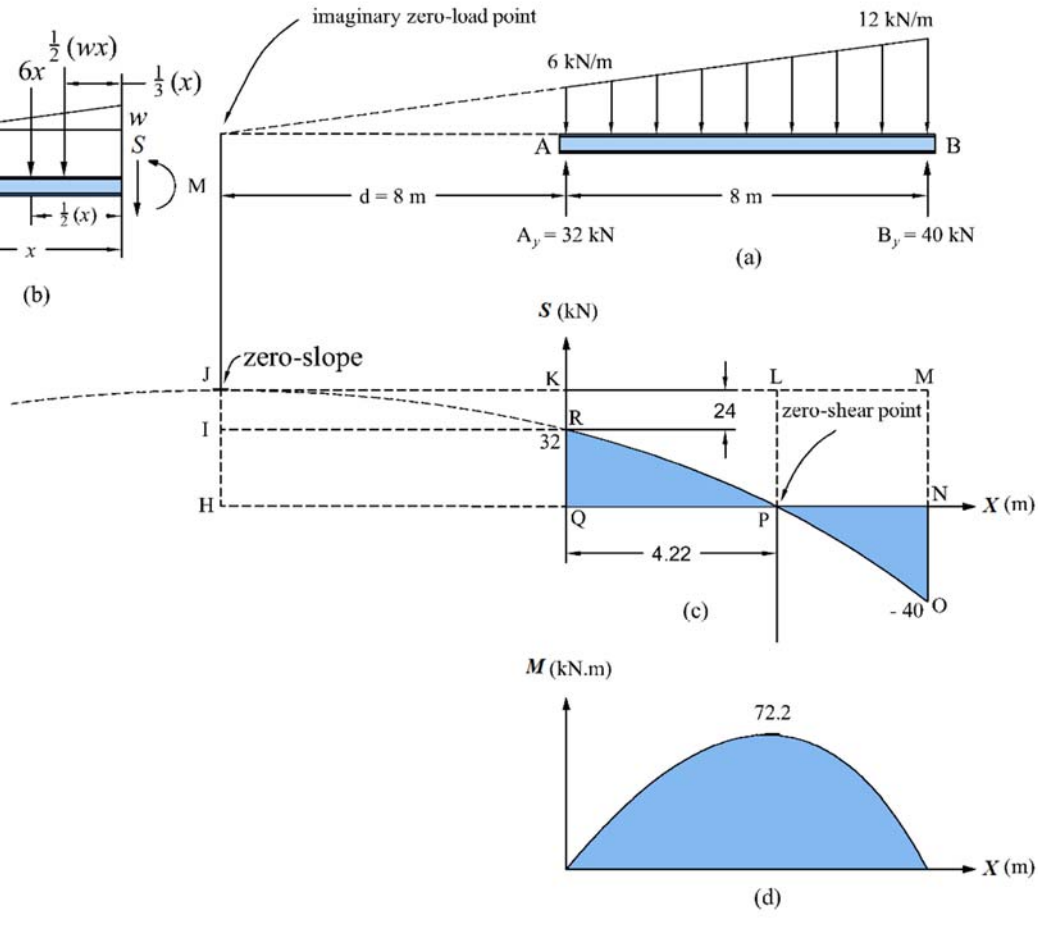


Fig.8. The diagrams of example 2.

***Step 4:***
The parabolic function in *step 2* intersects the *X*-axis when $V=56\ kN$ as shown in Fig.8c, thus, substituting $V=56\ kN$ in the parabolic function $V = 0.375 x^2$ yields the distance $x=12.22\ m$ (the location of the zero-shear point with respect to the vertex). Finally, the location of the zero-shear point is calculated by subtracting the distance *8 m* from the total distance *12.22 m* to get the distance of *4.22 m*.

***Step 5:***
Based on the location of the parabola vertex $J$ (the zero-slope point), the shaded area $A_{QRP}$ in Fig.8c is calculated geometrically from the parabolic areas $A_{HJP}$ and $A_{IJR}$, and the rectangular area $A_{HIRQ}$ as follows:

$$A_{QRP} = A_{HJP} - A_{IJR} - A_{HIRQ} . \tag{3.1}$$

The parabolic areas $A_{HJP}$ and $A_{IJR}$ are calculated according to Eq.(2.12), and the area $A_{HIRQ}$ is an area of a rectangle. By substituting the numerical values in Eq.(3.1), we get:

$$A_{QRP} = \frac{2}{3}\times(8+4.22)\times(24+32)-\frac{2}{3}\times 8\times 24-8\times 32 . \tag{3.2}$$

Equation (3.2) yields the value $A_{QRP}=72.2$, which represents the maximum moment of *72.2 kN.m*.

It can be noticed from example 2 that the common method for locating the zero-shear point (Solution I) requires using the quadratic formula to solve the equation developed from applying the equilibrium equations for the sectioned beam, which is in the form of ( $ax^2+bx+c=0$ ). The trapezoidal loading always makes utilizing the common method more complex and time-consuming in both sectioning and solving the quadratic equations. On the other hand, the application of the proposed improved method does not need sectioning the beam and/or solving quadratic equations, which give it an advantage compared with the common method.

**Example 3:** This example is presented by Kassimali [5]. It is an overhanging beam subjected to a triangular loading along its length as shown in Fig.9. The reactions are shown in Fig.10a, and the shear and moment diagrams are shown in Figs10b and 10c, respectively.

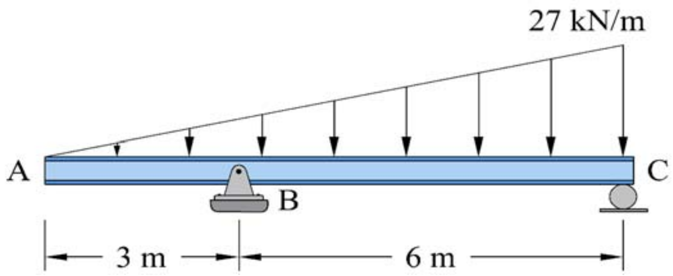


Fig.9. The beam of example 3.

***Solution (by using the improved graphical method):***

Referring to Fig.10, the steps for locating the zero-shear point and computing the value of the maximum moment by using the proposed technique are as follows:

***Step 1:***
Computing the slope of the loading diagram $m$ by dividing the maximum intensity of the load by the loading length to get the slope $m=3$.
***Step 2:***
Applying Eq.(2.9) to obtain the shear parabolic function $V = 1.5x^2$ .
***Step 3:***
Identifying the vertex of the parabola to be horizontal under the zero-load point (point $K$ in Fig.10b) and vertical at the value of $V=60.75$ *kN*, which is calculated by adding the vertical distance *13.5* (by substituting $x=3$ *m* into the parabola from *step 2*) with the vertical distance *47.25* (the value of the shear at point $B$).

***Step 4:***
The parabolic function in *step 2* intersects the *X*-axis when *V=60.75 kN* as shown in Fig.10b, thus substituting *V=60.75 kN* in the parabolic function $V = 1.5x^2$ yields the distance *x=6.364 m*. Finally, the location of the zero-shear point is calculated by subtracting the distance *3m*, shown in Fig.10a, from the total distance *6.364 m*, shown in Fig.10b, to get the location of the zero-shear point as *3.364 m* (with respect to point *B*).

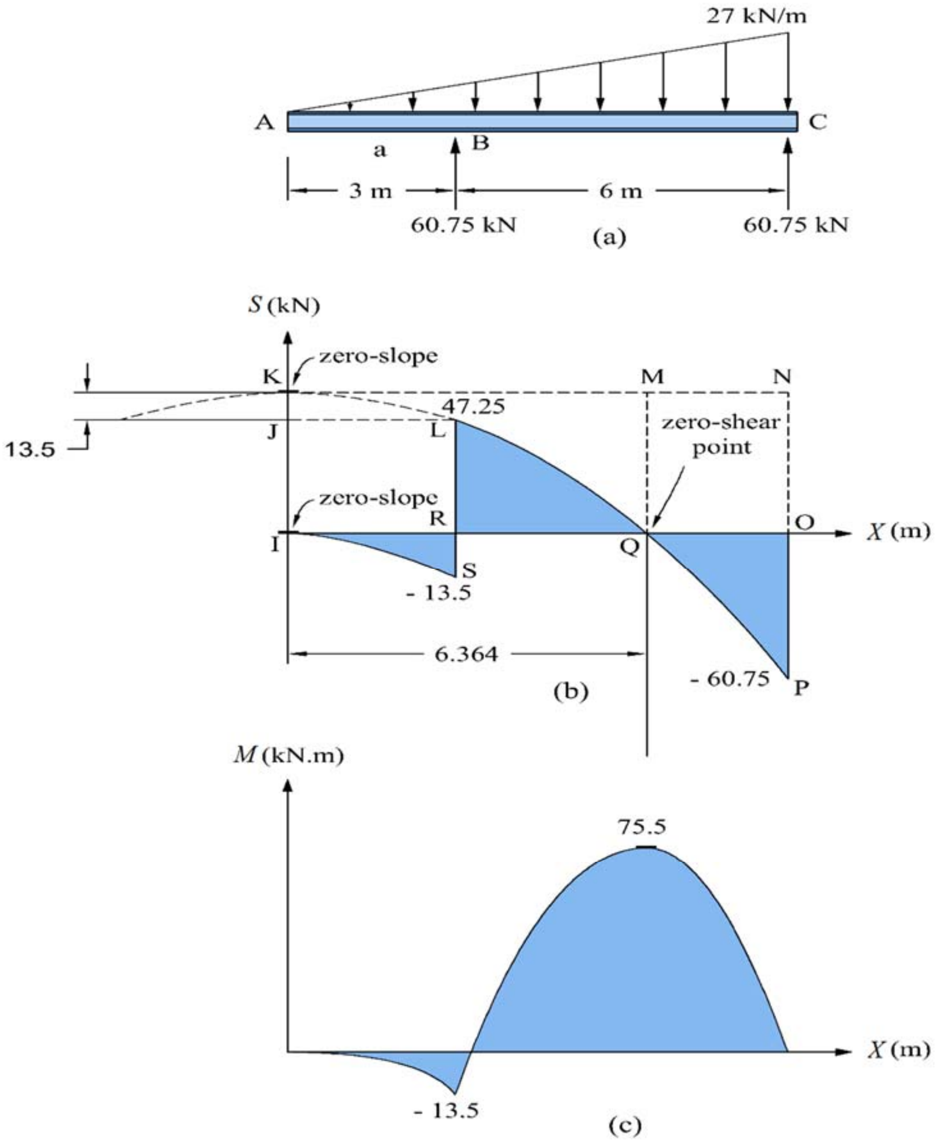


Fig.10. The diagrams of example 3.

***Step 5:***
Based on the location of the parabola vertex *K* (the zero-slope point), the shaded area $A_{RLQ}$ in Fig.10.b is calculated as follows:

$$A_{RLQ} = A_{IKQ} - A_{JKL} - A_{IJLR}\,. \tag{3.3}$$

The parabolic areas $A_{IKQ}$ and $A_{JKL}$ are calculated according to Eq.(2.12), and the area $A_{IJLR}$ is an area of a rectangle. By substituting the numerical values in Eq.(3.3), we get:

$$A_{RLQ} = \frac{2}{3}\times 6.364\times\left(47.25+13.5\right)-\frac{2}{3}\times 3\times 13.5-3\times 47.25\ . \tag{3.4}$$

Equation (3.4) yields the value $A_{RLQ} = 88.992$, then the maximum moment is obtained from calculating the area $A_{IRS}$ to be *13.5*, according to Eq.(2.11):

$$A_{IRS} = \frac{1}{3}\times 3\times 13.5 \tag{3.5}$$

and finally subtracting the value *13.5*, Eq.(3.5), from the value *88.992 kN.m*, Eq.(3.4), to get the maximum moment of *75.5 kN.m*.

In this example, the parabola of the shear diagram has a jump at the support *B* due to the existence of the concentrated vertical reaction at the roller support. The jump caused two parabolic areas to the left of the zero-shear point *Q*, having two identical curves due to an identical loading slope *m* which leads to the same coefficients for the two parabolas. Furthermore, the parabolic areas *IRS* and *RLQ* are calculated to evaluate the negative moment over the roller support *B* (*-13.5*) and the maximum moment (*75.5 kN.m*), noting that these parabolic areas calculations are based on the vertices *I* and *K*, respectively.

**Example 4:** This example is created by the authors of this study to give a comprehensive case of a loaded beam. In this example, an overhanging beam is subjected to various types of loadings as shown in Fig.11. The reactions are shown in Fig.12a, and the shear and moment diagrams are shown in Figs 12b and 12c, respectively.

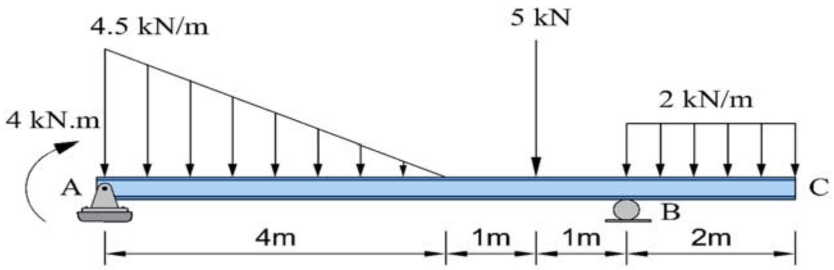


Fig.11. The beam of example 4.

***Solution (by using the improved graphical method):***

Referring to Fig.12, the steps for locating the zero-shear point and finding the value of the maximum moment by using the proposed technique are as follows:

***Step 1:***

Computing the slope of the loading diagram *m* by dividing the maximum intensity of the load by the loading length to get the slope $m=4.5/4$.

***Step 2:***

Applying Eq.(2.9) to obtain the shear parabolic function $V = \left(9/16\right)x^2$.

***Step 3:***

Identifying the vertex of the parabola to be horizontal at point *Q* (below the zero-load point) and vertically at the value of $V=2.5\ kN$ (the value of the shear at *Q*).

***Step 4:***
The parabolic function in *step 2* intersects the *X*-axis when *V=2.5 kN* as shown in Fig.12b, thus substituting *V=2.5 kN* in the parabolic function $V = (9/16)x^2$ yields the distance *x=2.108 m*. Finally, subtracting the distance *2.108 m* from the total distance *4 m*, in Fig.12a, gives the location of the zero-shear point as *1.892 m* with respect to point *A*.

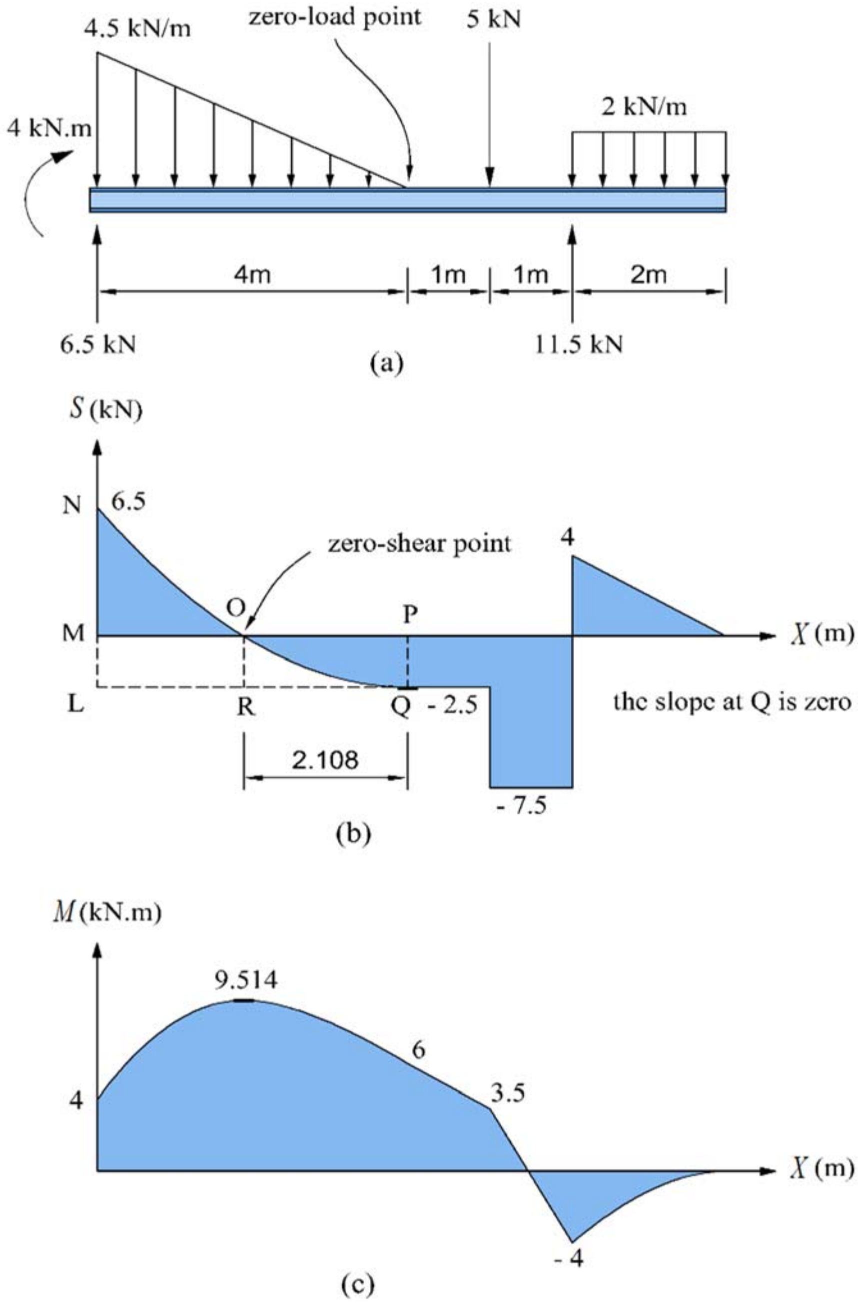


Fig.12. The diagrams of example 4.

***Step 5:***
Based on the location of parabola vertex *Q* (the zero-slope point), The shaded area $A_{MNO}$ in Fig.12b is calculated geometrically as follows:

$$A_{MNO} = A_{LNQ} - A_{ROQ} - A_{LMOR} . \tag{3.6}$$

The parabolic areas $A_{LNQ}$ and $A_{ROQ}$ are calculated according to Eq.(2.11), and the area $A_{LMOR}$ is an area of a rectangle. By substituting the numerical values in Eq.(3.6), we get:

$$A_{MNO} = \frac{1}{3} \times 4 \times (6.5 + 2.5) - \frac{1}{3} \times 2.108 \times 2.5 - (4 - 2.108) \times 2.5 . \tag{3.7}$$

Equation (3.7) gives the value of $A_{MNO} = 5.514$, which is added to the value of the couple moment (*4 kN.m*) at *A* to get the maximum moment of *9.514 kN.m*.

The moment decreases by the value of $\Delta M$ which represents the shaded area $A_{OPQ}$. This area is calculated according to Eq.(2.12) to get the value of *3.514*, which is subtracted from the value of the maximum moment (*9.514 kN.m*) to get a moment of *6 kN.m* as shown in Fig.12c.

This example evidenced that the improved method is applicable when the beam is subjected to various types of loadings. In addition, the example presented the case of decreasing linearly varying load which leads to an open-up parabola rather than open-down parabolas that were presented in previous examples. It can be seen that the direction of the parabola opening, whether up or down, does not affect the calculations of the proposed method because the sign convention of this method states that the positive direction of the ordinate is in the direction of the parabola focus (or opening) as mentioned in the previous section.

**Example 5**: This example is presented by Kassimali [5]. It is an overhanging beam subjected to a trapezoidal loading along its length as shown in Fig.13. The reactions are shown in Fig.14a, and the shear and moment diagrams are shown in Figs 14b and 14c, respectively.

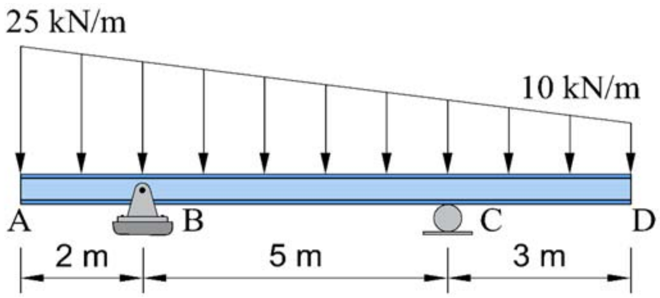


Fig.13. The beam of example 5.

**Solution (by using the improved graphical method):**

Referring to Fig.14, the steps for locating the zero-shear point and finding the value of the maximum moment by using the proposed technique are as follows:

***Step 1:***

Computing the slope of the loading diagram *m* by dividing the difference between the maximum and the minimum load intensities of the trapezoidal load by the loading length to get the slope *m=1.5*.

***Step 2:***

Applying Eq.(2.9) to obtain the shear parabolic function $V = 0.75\,x^2$.

***Step 3:***

From the minimum load intensity ($w_1 = 10\ kN/m$) and the loading slope (*m=1.5*), the location of the parabola vertex *d* is obtained by applying Eq.(2.10) to get *d=6.667 m* (to the right of point *D*) as shown in Fig.14a. To locate the zero-shear point in Fig.14b, it is required to find the ordinate of the vertex for the middle parabola. The ordinate of the vertex is evaluated as follows.

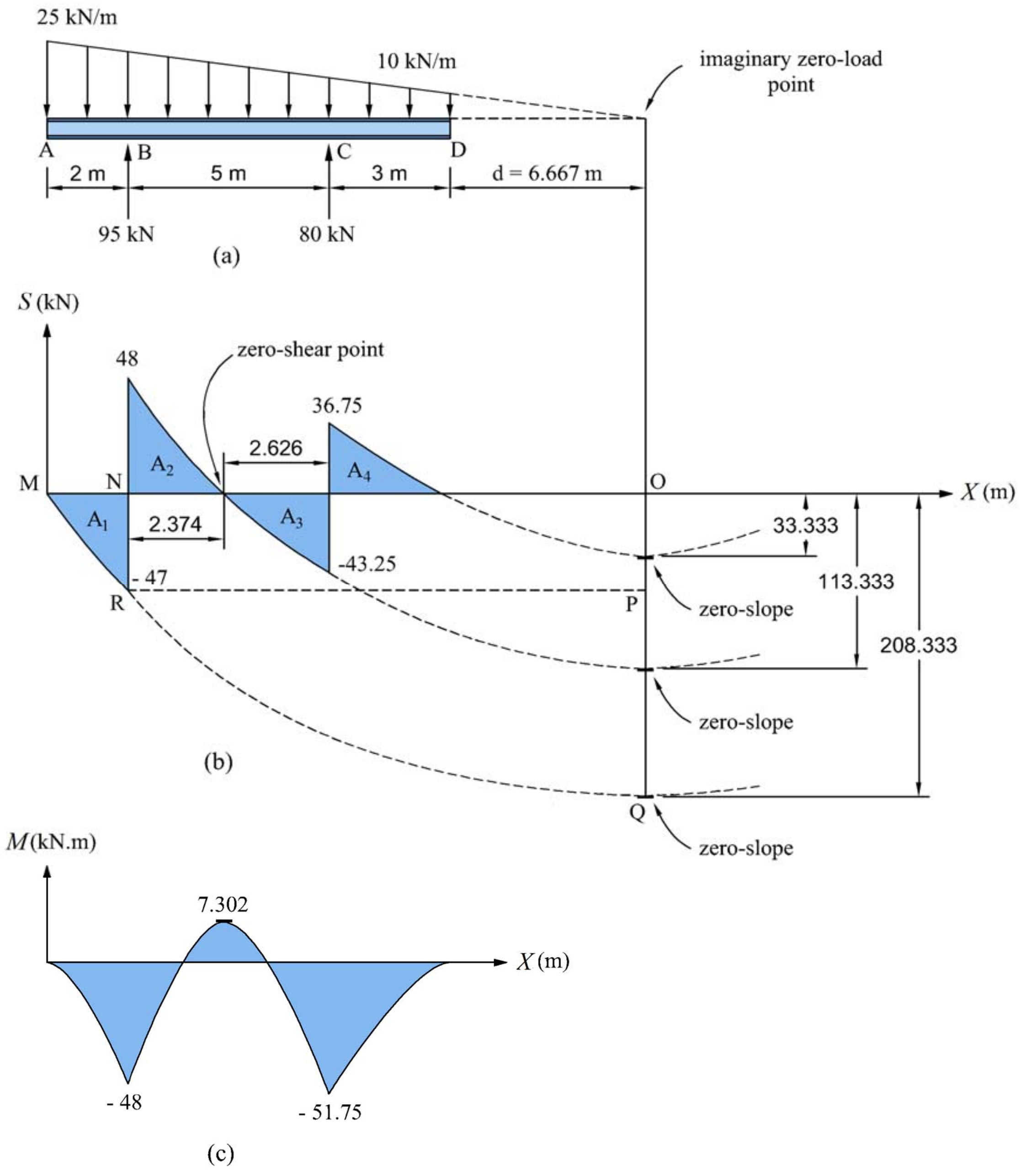


Fig.14. The diagrams of example 5.

The height of the middle parabola (vertical distance) is determined by substituting the value of its base (horizontal distance) into the equation of the parabola $V = 0.75\,x^2$. It can be seen from Fig.14b that the distance from point *N* to point *O* is equal to ( $5+3+6.667 = 14.667\,m$ ), which represents the base of the parabola when the shear value equals *48 kN*. Thus, by substituting the value of (*x=14.667 m*) into the parabola equation $V = 0.75\,x^2$, we get *V=161.333 kN*. Finally, subtracting *48 kN* from *161.333 kN* gives the ordinate of *113.333 kN* (downward with respect to the *X*-axis). Similarly, the location of the vertex of the upper parabola is obtained to be *33.333 kN* (downward with respect to *X*-axis), and the location of the vertex of the lower parabolas is obtained to be *208.333 kN* (downward with respect to the *X*-axis), as shown in Fig.14b.

***Step 4:***
The parabola equation in *step 2* represents the equation for the upper, middle, and lower parabolas as the loading slope $m$ is constant. Only the middle parabola intersects the $X$-axis at the interior point on the beam, as the upper and the lower parabolas pass through the free ends of the beam as shown in Fig.14b. The middle parabola intersects the $X$-axis when $V=113.333\ kN$, thus, substituting this value in the parabolic function in *step 2* yields the distance $x=12.293\ m$. Finally, the location of the zero-shear point with respect to point $C$ is calculated by subtracting the distance ( $3+6.667=9.667\,m$ ) from the total distance *12.293 m* to get the location of the zero-shear point as *2.626 m* to the left of point $C$.
***Step 5:***
There are four parabolic areas; $A_1, A_2, A_3$ and $A_4$ as shown in Fig.14b. Each of these areas is calculated based on the location of the parabola vertex (the zero-slope point).

The area $A_1$, which represents the shaded area of the parabolic portion *MNR* in Fig.14b, is calculated geometrically as follows:

$$A_1 = A_{MOQ} - A_{RPQ} - A_{RNOP}\,. \tag{3.8}$$

The parabolic areas $A_{MOQ}$ and $A_{RPQ}$ are calculated according to Eq.(2.12), and the area $A_{RNOP}$ is an area of a rectangle. By substituting the numerical values in Eq.(3.8), we get:

$$A_1 = \frac{2}{3}\times\left(10+6.667\right)\times 208.333 - \frac{2}{3}\times\left(8+6.667\right)\times\left(208.333-47\right) - \left(8+6.667\right)\times 47. \tag{3.9}$$

Eq.(3.9) gives the value of $A_1 = 48\ kN.m$, which represents the moment of *-48 kN.m* in Fig.14c.
By using the same manner that is employed for $A_1$, the other areas $\left(A_2, A_3 \text{ and } A_4\right)$ are calculated to be: $\left(A_2 = 55.302\,kN.m, A_3 = 59.052\,kN.m \text{ and } A_4 = 51.75\,kN.m\right)$.

Based on the obtained values of the parabolic areas, the moment diagram is constructed normally as shown in Fig.14c.

This example shows a complex case of a loaded beam as the beam has interior support reactions which cause jumps in the shear diagrams, and also due to the applied trapezoidal load which is considered a more troublesome loading case compared with the triangular load. The jumps in shear diagrams lead to numerous parabolic curves for a given one linearly varying loading case. These parabolic curves have the same coefficient of the parabola equation, as the slope of the loading diagram is constant, but with different vertical locations of their vertices. As a result, four parabolic areas $A_1, A_2, A_3$ and $A_4$ are developed due to the three parabolas curves. Hence, relatively long calculations in constructing the shear and moment diagram are carried out compared with other simpler cases of loaded beams. It should be noted that if the common sectioning technique is utilized for this example, then it is required to cut the beam three times, applying the equilibrium equations for each section, and solving three quadratic formulas, in order to obtain the values of the moment diagram shown in Fig.14c. This leads to a time-consuming process with the potential for errors, compared with the geometrical method proposed in this study.

## 4. Conclusions:

The conclusions of this study can be summarized as follows:

1. The graphical method of plotting the shear and moment diagrams for the structural members subjected to linearly varying distributed loads (triangular and trapezoidal loads) can be improved by using the geometric approach presented in this study. The proposed improved method gives completely the same results and diagrams for all cases of the structural members compared with the basic graphical method.

2. The improved graphical method does not require sectioning and applying the equilibrium equations for the structural members to construct the shear and moment diagrams. Therefore, utilizing this method cancels the possibility of making mistakes which can occur in sectioning and applying the equilibrium equations.
3. Generally, the improved graphical method is more convenient, less time-consuming, and requires less computational efforts because it does not need sectioning and solving equations. Furthermore, in the case of trapezoidal loads, solving quadratic formulas becomes not needed as is the case in the basic graphical method.
4. If the analyst and/or the designer of the structural members prefers using the basic graphical method, since it is a commonly used method, then he or she can use the improved graphical method as a good option to verify or check the shear and moment diagrams that are constructed by using the basic method.

## Nomenclature

$a$ – coefficient of the parabola

$A_1$ – area under the parabola opens upward with vertex at the origin

$A_2$ – area above the parabola opens upward with vertex at the origin

$b$ – horizontal distance from the origin to a given point on the parabola

$d$ – length of imaginary loading

$h$ – vertical distance from the origin to a given point on the parabola

$L$ – length of actual loading

$m$ – slope of the loading diagram

$M$ – value of the internal moment in a member

$s$ – length of the actual plus the imaginary loadings

$S$ – value of the internal shear in the member

$V$ – value of the shear with respect to the parabola vertex

$w$ – intensity of the triangular loading

$w_1$ – minimum intensity of the trapezoidal loading

$w_2$ – maximum intensity of the trapezoidal loading

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