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FRICTIONAL CONTACT PROBLEMS FOR NONLINEAR ELASTIC MATERIALS

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We are interested in the static problem of modelling the frictional contact between an elastic body and a rigid foundation. We assume that the elastic constitutive law is nonlinear, that the contact is bilateral and that the friction is described by Tresca's law. Two equivalent weak formulations of the problem are established and the existence of a unique solution is proved in each case. A regularized problem is also studied and a strong convergence result is proved.

1. Introduction

Only recently progress has been made towards the modelling and analysis of contact processes between deformable bodies. This is due to the considerable difficulties that the process of frictional contact presents in the modelling and analysis due to the complicated surface phenomena involved. Contact problems with or without friction were already studied for instance in (Burguera and Viaño, 1995; Drabla *et al.*, 1998; Duvaut and Lions, 1972; Haslinger and Hlaváček, 1980; 1982; Hlaváček and Nečas, 1981; 1983; Kikuchi and Oden, 1988; Licht, 1985; Shillor and Sofonea, 1998), see also the references therein, in the case of elastic or viscoelastic materials. The case of elasto-visco-plastic materials was considered for instance in (Amassad and Sofonea, 1998; Drabla *et al.*, 1997; Rochdi, 1997; Rochdi and Sofonea, 1997; Sofonea, 1997) and the works cited therein.

In this work, we consider the process of frictional contact between an elastic body which is acted upon by volume forces and surface tractions, and a rigid foundation. We assume that the forces and tractions change slowly in time so that the accelerations in the system are negligible. Neglecting sufficiently the inertial terms in the equations of motion leads to a static approximation of the process. The material's constitutive law is assumed to be nonlinear elastic. The same constitutive law was recently used in (Drabla *et al.*, 1998) for the study of a frictionless contact problem with Signorini's contact conditions. The contact is modelled here with a bilateral condition and the friction with the associated Tresca law. These contact and friction conditions were considered for instance in (Duvaut and Lions, 1972; Licht, 1985) in the case of linear elastic or viscoelastic bodies and in (Amassad and Sofonea, 1998) in the case of elasto-visco-plastic bodies.

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This paper is organized as follows. Section 2 contains the notations and some preliminary material. Section 3 deals with the description of the model for the process and the mathematical statement of the problem. In Section 4, we list the assumptions on the data and set the problem in two variational forms. These are two elliptic variational inequalities: Problems P_1 and P_2. The unknown in the first problem is the displacement field and in the second one it is the stress field. The existence of a unique solution to each problem (Theorem 1, Theorem 2) as well as an equivalence result between the problems P_1 and P_2 (Theorem 3) are established in Section 5. In the last section, we introduce for each nonnegative parameter μ a regularized problem P^μ of Problem P_1 and we prove a strong convergence result of its solution to the solution of Problem P_1 when $\mu \to 0$ (Theorem 4).

The purpose of this work is to extend some known results in linear elasticity to the nonlinear case and to point out the second variational formulation which is important in engineering since it is related to the stress field. Moreover, it deals with a regularization of the problem considered, which is of interest from the numerical point of view.

2. Notation and Preliminaries

In this short section, we present the notation we will use and some preliminary material. For further details we refer the reader to (Duvaut and Lions, 1972; Ionescu and Sofonea, 1993; Kikuchi and Oden, 1988; Panagiotopoulos, 1985). $\mathbb{S}_N$ represents the set of second-order symmetric tensors in $\mathbb{R}^N$. We denote by '$\cdot$' and $|\cdot|$ the inner product and the Euclidean norm on $\mathbb{S}_N$ and $\mathbb{R}^N$. We also use the following notations:

$$H = \left\{ v = (v_i) \mid v_i \in L^2(\Omega) \right\} = L^2(\Omega)^N$$

$$H_1 = \left\{ v = (v_i) \mid v_i \in H^1(\Omega) \right\} = H^1(\Omega)^N$$

$$\mathcal{H} = \left\{ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega) \right\} = L^2(\Omega)_s^{N\times N}$$

$$\mathcal{H}_1 = \left\{ \tau \in \mathcal{H} \mid \operatorname{Div} \tau \in H \right\}$$

where $i, j = 1, \ldots, N$. H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the inner products given by

$$\langle u, v \rangle_H = \int_\Omega u_i v_i \, \mathrm{d}x$$

$$\langle \sigma, \tau \rangle_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij} \, \mathrm{d}x$$

$$\langle u, v \rangle_{H_1} = \langle u, v \rangle_H + \big\langle \varepsilon(u), \varepsilon(v) \big\rangle_{\mathcal{H}}$$

$$\langle \sigma, \tau \rangle_{\mathcal{H}_1} = \langle \sigma, \tau \rangle_{\mathcal{H}} + \langle \operatorname{Div} \sigma, \operatorname{Div} \tau \rangle_H$$

respectively, where $\varepsilon : H_1 \longrightarrow \mathcal{H}$ and $\text{Div} : \mathcal{H}_1 \longrightarrow H$ are the *deformation* and the *divergence* operators, respectively, defined by

$$\varepsilon(v) = \big(\varepsilon_{ij}(v)\big), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad \text{Div}\,\sigma = (\sigma_{ij,j})$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively.

Let $H_\Gamma = H^{1/2}(\Gamma)^N$ and let $\gamma : H_1 \longrightarrow H_\Gamma$ be the trace map. Let also ν be the outward unit normal to Γ. For every element $v \in H_1$ we use, when no confusion is likely, the notation v for the trace γv of v on Γ. We denote by v_ν and v_τ the *normal* and the *tangential* components of v on Γ given by $v_\nu = v \cdot \nu$ and $v_\tau = v - v_\nu \nu$, respectively. Let H'_Γ be the dual of H_Γ and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between H'_Γ and H_Γ. For every $\sigma \in \mathcal{H}_1$ let $\sigma\nu$ be the element of H'_Γ given by

$$\langle \sigma\nu, \gamma v\rangle = \big\langle \sigma, \varepsilon(v)\big\rangle_{\mathcal{H}} + \langle \text{Div}\,\sigma, v\rangle_H \qquad \forall\, v \in H_1 \tag{1}$$

We also denote by σ_ν and σ_τ the *normal* and *tangential* traces of σ (see e.g. Kikuchi and Oden, 1988; Panagiotopoulos, 1985). We recall that if σ is a regular function (say C^1), then

$$\langle \sigma\nu, \gamma v\rangle = \int_\Gamma \sigma\nu \cdot v \,\mathrm{d}a \qquad \forall\, v \in H_1 \tag{2}$$

where $\mathrm{d}a$ is the surface measure element, $\sigma_\nu = (\sigma\nu)\cdot\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$.

3. Problem Modelling

We model the static process when a nonlinear elastic body is being acted upon by forces and surface tractions and as a result it contacts a rigid foundation. The elastic body occupies a domain Ω of $\mathbb{R}^N$ ($N = 1,2,3$) with surface Γ. A volume force of density f_0 is applied on Ω. We assume that Γ is Lipschitz and is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $\text{meas}\,\Gamma_1 > 0$. We assume that the body is clamped on Γ_1 and thus the displacement field vanishes there and that surface tractions f_2 act on Γ_2. The solid is always maintained in frictional contact with a rigid foundation on Γ_3, which means that the body and the foundation have a compliant shape on Γ_3.

We denote by u the displacement vector, σ the stress field and $\varepsilon = \varepsilon(u)$ the small strain tensor. The elastic constitutive law that we consider is $\sigma = \mathcal{F}(\varepsilon(u))$, in which $\mathcal{F}$ is a given nonlinear constitutive function. The condition of bilateral contact between the body and the foundation along Γ_3 is given by $u_\nu = 0$, where u_ν represents the normal displacement. The associated friction law is the static Tresca law:

$$|\sigma_\tau| \le g \quad \text{on} \quad \Gamma_3$$
$$|\sigma_\tau| < g \Longrightarrow u_\tau = 0$$
$$|\sigma_\tau| = g \Longrightarrow \sigma_\tau = -\lambda u_\tau, \quad \lambda \ge 0$$

Here, σ_τ represents the tangential force on the contact boundary Γ_3, g denotes the friction yield limit and u_τ represents the tangential displacement. This friction law, which was already considered by Duvaut and Lions (1972), Licht (1985) and Panagiotopoulos (1985), states that the tangential shear cannot exceed the maximal frictional resistance g. Then, if the inequality holds the surface adheres completely to the foundation and is in the so-called *stick* state, and when the equality holds there is relative sliding, the so-called *slip* state. Therefore, at each time instant the contact surface Γ_3 is divided into two zones: the stick zone and the slip zone.

The mechanical problem of frictional contact between a nonlinear elastic body and a rigid foundation may be formulated classically as follows:

Problem P: Find a displacement field $u\colon \Omega \longrightarrow \mathbb{R}^N$ and a stress field $\sigma\colon \Omega \longrightarrow \mathbb{S}_N$ such that

$$\sigma = \mathcal{F}\big(\varepsilon(u)\big) \quad \text{in} \quad \Omega \tag{3}$$

$$\operatorname{Div} \sigma + f_0 = 0 \quad \text{in} \quad \Omega \tag{4}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \tag{5}$$

$$\sigma\nu = f_2 \quad \text{on} \quad \Gamma_2 \tag{6}$$

$$\begin{cases} u_\nu = 0 \quad \text{on} \quad \Gamma_3 \\ |\sigma_\tau| \le g \quad \text{on} \quad \Gamma_3 \\ |\sigma_\tau| < g \Longrightarrow u_\tau = 0 \\ |\sigma_\tau| = g \Longrightarrow \sigma_\tau = -\lambda u_\tau, \quad \lambda \ge 0 \end{cases} \tag{7}$$

To study Problem P, we need the following additional notation. Let V denote the closed subspace of H_1 given by

$$V = \Big\{ v \in H_1 \,|\, v = 0 \text{ on } \Gamma_1, \;\; v_\nu = 0 \text{ on } \Gamma_3 \Big\}$$

Now, Korn's inequality holds, since $\operatorname{meas}\Gamma_1 > 0$. Thus (Duvaut and Lions, 1972; Hlaváček and Nečas, 1981)

$$\big|\varepsilon(v)\big|_{\mathcal{H}} \ge C|v|_{H_1} \qquad \forall\, v \in V \tag{8}$$

Here and below C denotes a positive generic constant which may depend on Ω, Γ_1, Γ_2, Γ_3 and $\mathcal{F}$, but does not depend on the input data f_0, f_2, g, and whose value may vary from place to place.

We consider the inner product $\langle \cdot\,, \cdot \rangle_V$ on V, given by

$$\langle v, w \rangle_V = \big\langle \varepsilon(v), \varepsilon(w) \big\rangle_{\mathcal{H}} \tag{9}$$

It follows from (8) that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V. Therefore $(V, |\cdot|_V)$ is a Hilbert space.

4. Variational Formulations

In this section, we give two variational formulations to Problem P. For that purpose, we assume that the *elasticity operator*

$$\mathcal{F}\colon \Omega \times \mathbb{S}_N \longrightarrow \mathbb{S}_N$$

satisfies the following set of conditions:

$$\left\{\begin{array}{l}\text{(a) there exists } L>0 \text{ such that} \\ \quad \big|\mathcal{F}(\cdot\,,\varepsilon_1)-\mathcal{F}(\cdot\,,\varepsilon_2)\big| \le L|\varepsilon_1-\varepsilon_2| \quad \forall\ \varepsilon_1,\varepsilon_2\in\mathbb{S}_N, \text{ a.e. in } \Omega \\ \\ \text{(b) there exists } M>0 \text{ such that} \\ \quad \big(\mathcal{F}(\cdot\,,\varepsilon_1)-\mathcal{F}(\cdot\,,\varepsilon_2)\big)\cdot(\varepsilon_1-\varepsilon_2)\ge M|\varepsilon_1-\varepsilon_2|^2 \quad \forall\varepsilon_1,\varepsilon_2\in\mathbb{S}_N, \text{ a.e. in } \Omega \\ \\ \text{(c) } x\longmapsto\mathcal{F}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \quad \forall\varepsilon\in\mathbb{S}_N \\ \\ \text{(d) } x\longmapsto\mathcal{F}(x,0)\in\mathcal{H}.\end{array}\right. \tag{10}$$

The forces and the tractions satisfy

$$f_0\in H, \quad f_2\in L^2(\Gamma_2)^N \tag{11}$$

Moreover, the *friction yield limit* satisfies

$$g\ge 0 \tag{12}$$

Remark 1. Using (10) it is straightforward to show that for all $\tau\in\mathcal{H}$, the function $x\longmapsto\mathcal{F}(x,\tau(x))$ belongs to $\mathcal{H}$. Consequently, it is possible to consider $\mathcal{F}$ as an operator from $\mathcal{H}$ into $\mathcal{H}$. Moreover, $\mathcal{F}:\mathcal{H}\longrightarrow\mathcal{H}$ is a strongly monotone and Lipschitz operator (Sofonea, 1993, p.53). Therefore $\mathcal{F}$ is invertible and its inverse operator $\mathcal{F}^{-1}\colon\mathcal{H}\longrightarrow\mathcal{H}$ is also strongly monotone and Lipschitz.

Next, using (11) and the Riesz representation theorem, we may define the element $f\in V$ by

$$\langle f,v\rangle_V=\langle f_0,v\rangle_H+\langle f_2,\gamma v\rangle_{L^2(\Gamma_2)^N} \qquad \forall v\in V \tag{13}$$

Let also $j\colon V\longrightarrow\mathbb{R}_+$ be the functional

$$j(v)=g\int_{\Gamma_3}|v_\tau|\,da, \quad v\in V \tag{14}$$

Finally, we define the set of "*statically admissible stress fields*" $\Sigma_{\rm ad}$ by

$$\Sigma_{\rm ad}=\Big\{z\in\mathcal{H}\,\big|\,\big(z,\varepsilon(v)\big)_{\mathcal{H}}+j(v)\ge\langle f,v\rangle_V \quad \forall v\in V\Big\} \tag{15}$$

Lemma 1. *If (u,σ) is a regular solution to Problem P, then*

$$u\in V, \quad \big\langle\mathcal{F}(\varepsilon(u)),\ \varepsilon(v)-\varepsilon(u)\big\rangle_{\mathcal{H}}+j(v)-j(u)\ge\langle f,v-u\rangle_V \qquad \forall v\in V \tag{16}$$

$$\sigma\in\Sigma_{\rm ad}, \quad \big\langle\mathcal{F}^{-1}(\sigma),\ \tau-\sigma\big\rangle_{\mathcal{H}}\ \ge\ 0 \qquad \forall\tau\in\Sigma_{\rm ad} \tag{17}$$

Proof. First, from (5) and (13) we deduce that $u \in V$. Let $v \in V$. Using (4), (1) and (2), we have

$$\big\langle \sigma,\ \varepsilon(v) - \varepsilon(u) \big\rangle_{\mathcal{H}} = \langle f_0, v - u \rangle_H + \int_\Gamma \sigma\nu \cdot (v - u)\, da$$

and, from (5), (6) and (11), we obtain

$$\big\langle \sigma,\ \varepsilon(v) - \varepsilon(u) \big\rangle_{\mathcal{H}} = \langle f, v - u \rangle_V + \int_{\Gamma_3} \sigma\nu \cdot (v - u)\, da$$

Using now (3) and (7), the previous equality leads to

$$\big\langle \mathcal{F}\big(\varepsilon(u)\big),\ \varepsilon(v) - \varepsilon(u) \big\rangle_{\mathcal{H}} = \langle f, v - u \rangle_V + \int_{\Gamma_3} \sigma_\tau (v_\tau - u_\tau)\, da \tag{18}$$

The inequality in (16) follows from (18) and (14) since (7) implies that

$$\sigma_\tau (v_\tau - u_\tau) \ \geq\ g\big(|u_\tau| - |v_\tau|\big) \quad \text{a.e. on } \Gamma_3$$

Putting now $v = 2u$ and $v = 0$ in (16) and taking (14) and (3) into account, we obtain

$$\big\langle \sigma, \varepsilon(u) \big\rangle_{\mathcal{H}} + j(u) = \langle f, u \rangle_V \tag{19}$$

Hence, by using (16), (3), (19) and (15) it follows that $\sigma \in \Sigma_{\text{ad}}$. The inequality in (17) is now a consequence of (19) and (15) since $u \in V$ and $\mathcal{F}$ is invertible. ∎

Lemma 1 leads to the following weak formulations for Problem P.

Problem P_1: Find a displacement field $u\colon \Omega \longrightarrow \mathbb{R}^N$ such that

$$u \in V, \quad \big\langle \mathcal{F}\big(\varepsilon(u)\big),\ \varepsilon(v) - \varepsilon(u) \big\rangle_{\mathcal{H}} + j(v) - j(u) \geq \langle f, v - u \rangle_V \qquad \forall v \in V \tag{20}$$

Problem P_2: Find a stress field $\sigma\colon \Omega \longrightarrow \mathbb{S}_N$ such that

$$\sigma \in \Sigma_{\text{ad}}, \quad \big\langle \mathcal{F}^{-1}(\sigma),\ \tau - \sigma \big\rangle_{\mathcal{H}} \geq 0 \qquad \forall \tau \in \Sigma_{\text{ad}} \tag{21}$$

Remark 2. Let us remark that Problems P_1 and P_2 are formally equivalent to Problem P. Indeed, if u represents a regular solution to the variational problem P_1 and σ is defined by $\sigma = \mathcal{F}(\varepsilon(u))$, using arguments in (Duvaut and Lions, 1972) it follows that $\{u, \sigma\}$ is a solution to Problem P. In a similar way, if σ represents a regular solution to the variational problem P_2 and $u \in V$ is given by $\sigma = \mathcal{F}(\varepsilon(u))$ then, using the same arguments, it follows that $\{u, \sigma\}$ is a solution to Problem P. For this reason, we may consider Problems P_1 and P_2 as *variational formulations* to Problem P.

Under the assumptions (10)–(12), in the next section we give the existence and uniqueness results for the variational problems P_1 and P_2 followed by an equivalence result between these two problems.

5. Existence and Uniqueness Results

Theorem 1. *Let (10)–(12) hold. Then there exists a unique solution to Problem P_1.*

Proof. Using the Riesz representation theorem, we may consider the operator A: $V \longrightarrow V$ defined by

$$\langle Av, w\rangle_V = \big\langle \mathcal{F}\big(\varepsilon(v)\big), \varepsilon(w)\big\rangle_{\mathcal{H}} \qquad \forall v,\ w \in V$$

Theorem 1 is now a consequence of the theory of elliptic variational inequalities (Brezis, 1968; Kikuchi and Oden, 1980), since (10) and (8) imply that the operator A is strongly monotone and Lipschitz and, since the functional j defined by (14) is proper, convex and lower semicontinuous. ■

Theorem 2. *Let (10)–(12) hold. Then there exists a unique solution to Problem P_2.*

Proof. Using (9) and the fact that the functional j is nonnegative, we deduce that $\varepsilon(f) \in \Sigma_{\mathrm{ad}}$. Thus, Σ_{ad} given by (15) is a nonempty convex subset of $\mathcal{H}$. Moreover, from Remark 1 we obtain that $\mathcal{F}^{-1}$ is a strongly monotone and Lipschitz operator. Hence, using arguments of the theory of elliptic variational inequalities, it follows that Problem P_2 has a unique solution $\sigma \in \Sigma_{\mathrm{ad}}$. Let us prove now that $\sigma \in \mathcal{H}_1$. Indeed, since $\sigma \in \Sigma_{\mathrm{ad}}$, it results from (15) that $\langle \sigma, \varepsilon(v)\rangle_{\mathcal{H}} + j(v) \geq \langle f, v\rangle_V$ for all $v \in V$. Putting in this inequality $v = \pm\varphi$ where $\varphi \in \mathcal{D}(\Omega)^N$ and using (13), we obtain that $\langle \sigma, \varepsilon(\varphi)\rangle_{\mathcal{D}'(\Omega)^N \times \mathcal{D}(\Omega)^N} = \langle f_0, \varphi\rangle_H$ for all $\varphi \in \mathcal{D}(\Omega)^N$. Thus using (1) yields $\mathrm{Div}\,\sigma + f_0 = 0$ a.e. in Ω. Finally, the regularity of $\sigma \in \mathcal{H}_1$ is a consequence of the last equality and (11). ■

The following result deals with the study of the link between the variational problems P_1, P_2 and the constitutive law (3).

Theorem 3. *Let (10)–(12) hold and let (u, σ) be such that $u \in V$ and $\sigma \in \mathcal{H}_1$. Consider the following properties:*

(i) u is the solution to Problem P_1 given in Theorem 1,

(ii) σ is the solution to Problem P_2 given in Theorem 2, and

(iii) u and σ are connected with the elastic constitutive law $\sigma = \mathcal{F}(\varepsilon(u))$.

Then two among these properties imply the third one.

Proof. We start by proving that (i) and (iii) imply (ii). Putting $v = 2u \in V$ and $v = 0 \in V$ in (20) and using (14) and (iii), we deduce that

$$\big\langle \sigma, \varepsilon(u)\big\rangle_{\mathcal{H}} + j(u) = \langle f, u\rangle_V \tag{22}$$

Therfore, (20), (22) and (iii) imply that

$$\sigma \in \Sigma_{\mathrm{ad}} \tag{23}$$

Let now $\tau \in \Sigma_{\text{ad}}$. From (iii) it follows that

$$\left\langle \mathcal{F}^{-1}(\sigma),\ \tau - \sigma \right\rangle_{\mathcal{H}} = \Big(\left\langle \tau, \varepsilon(u) \right\rangle_{\mathcal{H}} + j(u) \Big) - \Big(\left\langle \sigma, \varepsilon(u) \right\rangle_{\mathcal{H}} + j(u) \Big) \tag{24}$$

Property (ii) is now a consequence of (15) and (22)–(24) since $u \in V$.

Let us prove now that (i) and (ii) imply (iii). For this, let $\widetilde{\sigma} \in \mathcal{H}$ be the function $\widetilde{\sigma} = \mathcal{F}(\varepsilon(u))$. Using now the previous step of the proof, it follows that $\widetilde{\sigma}$ is a solution to Problem P_2. The uniqueness of the solution σ to this problem yields Property (iii).

Finally, we will establish that (ii) and (iii) imply (i). For that purpose, we introduce the spaces W and $\mathcal{W}$ defined by

$$W = \Big\{ v \in H_1 \,\big|\, v = 0 \ \text{ on } \ \Gamma_1 \Big\} \supset V$$

$$\mathcal{W} = \Big\{ z \in \mathcal{H} \,\big|\, \text{Div}\ z = 0 \ \text{ in } \ \Omega, \ \ z\nu = 0 \ \text{ on } \ \Gamma_2 \cup \Gamma_3 \Big\}$$

Using (1), it is straightforward to show that the orthogonal complement of $\mathcal{W}$ in $\mathcal{H}$ is the subspace $\varepsilon(W)$, i.e.

$$\mathcal{W}^{\perp} = \varepsilon(W) \quad \text{in } \ \mathcal{H} \tag{25}$$

Thus it follows from (15) and (25) that $\sigma \pm z \in \Sigma_{\text{ad}}$ for all $z \in \mathcal{W}$. Consequently, taking $\tau = \sigma \pm z$ in (21), it may be concluded that $\langle z, \mathcal{F}^{-1}(\sigma) \rangle_{\mathcal{H}} = 0$ for all $z \in \mathcal{W}$. This implies, by using (25), that there exists $\widetilde{u} \in W$ such that

$$\mathcal{F}^{-1}(\sigma) = \varepsilon(\widetilde{u}) \tag{26}$$

Let us prove that $\widetilde{u} \in V$. For this, let us suppose that $\widetilde{u} \notin V$. Hence, since V is a closed subspace of W, there exists $\widetilde{\tau} \in \mathcal{H}$ such that

$$\left\langle \widetilde{\tau}, \varepsilon(v) \right\rangle_{\mathcal{H}} = 0 \qquad \forall\, v \in V \tag{27}$$

and

$$\left\langle \widetilde{\tau}, \varepsilon(\widetilde{u}) \right\rangle_{\mathcal{H}} < 0 \tag{28}$$

Since the functional j is nonnegative, it follows from (9) that $\lambda \widetilde{\tau} + \varepsilon(f) \in \Sigma_{\text{ad}}$ for all $\lambda \geq 0$. Therefore, if we set $\tau = \lambda \widetilde{\tau} + \varepsilon(f)$ in (21) and use (26), we obtain

$$\left\langle \sigma - \varepsilon(f),\ \varepsilon(\widetilde{u}) \right\rangle_{\mathcal{H}} \leq \lambda \left\langle \widetilde{\tau}, \varepsilon(\widetilde{u}) \right\rangle_{\mathcal{H}} \qquad \forall\, \lambda \geq 0$$

Passing to the limit as $\lambda \to +\infty$, it follows from (28) that $\langle \sigma - \varepsilon(f),\ \varepsilon(\widetilde{u}) \rangle_{\mathcal{H}} \leq -\infty$ which is absurd. Consequently, $\widetilde{u} \in V$. Assertion (iii) and (26) yield $\sigma = \mathcal{F}(\varepsilon(u)) = \mathcal{F}(\varepsilon(\widetilde{u}))$. Hence, using (10) and (8), we obtain

$$0 = \left\langle \mathcal{F}\big(\varepsilon(u)\big) - \mathcal{F}\big(\varepsilon(\widetilde{u})\big),\ \varepsilon(u) - \varepsilon(\widetilde{u}) \right\rangle_{\mathcal{H}} \geq C \big|\varepsilon(u) - \varepsilon(\widetilde{u})\big|_{\mathcal{H}}^2 \geq C \big|u - \widetilde{u}\big|_{H_1}^2$$

Thus we deduce that $u = \widetilde{u} \in V$.

Let us establish now the inequality in (20). Since the functional j is subdifferentiable, there exists $\overline{\tau} \in \mathcal{H}$ such that

$$\left\langle \overline{\tau},\ \varepsilon(v) - \varepsilon(u) \right\rangle_{\mathcal{H}} + j(v) - j(u) \geq \langle f, v - u \rangle_V \qquad \forall\, v \in V \tag{29}$$

Taking $v = 2u \in V$ and $v = 0 \in V$ in this inequality, we have

$$\langle \overline{\tau}, \varepsilon(u) \rangle_{\mathcal{H}} + j(u) = (f, u)_V \tag{30}$$

and, from (29), (30) and (15), we deduce that $\overline{\tau} \in \Sigma_{\mathbf{ad}}$. Taking now $\tau = \overline{\tau}$ in (21) and, using Assertion (iii) and (30), it follows that

$$\langle \mathcal{F}(\varepsilon(u)), \varepsilon(u) \rangle_{\mathcal{H}} + j(u) \leq (f, u)_V \tag{31}$$

Moreover, since $\sigma = \mathcal{F}(\varepsilon(u)) \in \Sigma_{\mathbf{ad}}$, we have

$$\langle \mathcal{F}(\varepsilon(u)), \varepsilon(u) \rangle_{\mathcal{H}} + j(u) \geq (f, u)_V \tag{32}$$

and

$$\langle \mathcal{F}(\varepsilon(u)), \varepsilon(v) \rangle_{\mathcal{H}} + j(v) \geq (f, v)_V \qquad \forall v \in V \tag{33}$$

The inequality in (20) is finally a consequence of (31)–(33). This concludes the proof of Theorem 3. ■

Remark 3. A mechanical interpretation of the result obtained in Theorem 3 is the following:

1. If the displacement field u is the solution to Problem P_1, then the stress field σ connected to u by the elastic constitutive law $\sigma = \mathcal{F}(\varepsilon(u))$ is the solution to Problem P_2.
2. If the stress field σ is the solution to Problem P_2, then the displacement field u connected to σ by the elastic constitutive law $\sigma = \mathcal{F}(\varepsilon(u))$ is the solution to Problem P_1.
3. If the displacement field u is the solution to Problem P_1 and the stress field σ is the solution to Problem P_2, then u and σ are connected by the elastic constitutive law $\sigma = \mathcal{F}(\varepsilon(u))$.

6. A Regularized Problem

Due to the nondifferentiability of the functional j given by (14), we introduce a regularized problem P^μ of Problem P_1, depending on a nonnegative parameter μ. We prove the existence of a unique solution u_μ to this problem and we obtain a convergence result of u_μ to the solution of Problem P_1 as $\mu \to 0$.

Indeed, for every parameter $0 \leq \mu < 1$, let $j_\mu : V \longrightarrow \mathbb{R}_+$ be the functional defined by

$$j_\mu(v) = \frac{g}{1+\mu} \int_{\Gamma_3} |v_\tau|^{1+\mu} \, da \qquad \forall v \in V \tag{34}$$

Replacing the functional j by j_μ in Problem P_1, we obtain the following regularized problem:

Problem P^μ: Find a displacement field $u_\mu \in H_1$ such that

$$u_\mu \in V, \quad \langle \mathcal{F}(\varepsilon(u_\mu)), \, \varepsilon(v) - \varepsilon(u_\mu) \rangle_{\mathcal{H}} + j_\mu(v) - j_\mu(u_\mu) \geq (f, v - u_\mu)_V \qquad \forall v \in V \tag{35}$$

Since the functional j_μ is proper, convex and lower semicontinuous, using the same arguments as those used in the proof of Theorem 1, we have

Theorem 4. *Let (10)–(12) hold. Then there exists a unique solution to Problem P^μ.*

Our main interest in this section lies in the behaviour of the solution u_μ of Problem P^μ as $\mu \to 0$. This is the subject of the following result:

Theorem 5. *Let (10)–(12) hold. Then the solution u_μ of Problem P^μ converges in V to the solution u of Problem P_1 as $\mu \to 0$, i.e.*

$$u_\mu \longrightarrow u \quad \textit{in } V \quad \textit{as } \mu \to 0 \tag{36}$$

Proof. If $v = 0$ in (35), then

$$\langle \mathcal{F}(\varepsilon(u_\mu)),\, \varepsilon(u_\mu)\rangle_{\mathcal{H}} + j_\mu(u_\mu) \;\le\; \langle f, u_\mu\rangle_V \quad \text{for all} \quad 0 \le \mu < 1$$

and, using (10), (9) and the nonnegativity of the functional j_μ, we deduce that the sequence $(u_\mu)_\mu$ is bounded in V. Thus there exist a subsequence denoted again by $(u_\mu)_\mu$ and an element $\tilde{u} \in V$ such that

$$u_\mu \rightharpoonup \tilde{u} \quad \text{weakly in } V \quad \text{as } \mu \to 0 \tag{37}$$

In order to pass to the limit in (35) as $\mu \to 0$, we remark that using (37), (34) and (14) we have

$$\lim_{\mu\to 0} \langle f, v - u_\mu\rangle_V = \langle f, v - \tilde{u}\rangle_V \qquad \forall v \in V \tag{38}$$

and

$$\lim_{\mu\to 0} j_\mu(v) = j(v) \qquad \forall v \in V \tag{39}$$

We will prove now that

$$\liminf_{\mu\to 0} j_\mu(u_\mu) \;\ge\; j(\tilde{u}) \tag{40}$$

Due to the differentiability and the convexity of the functional j_μ given by (34), it follows that

$$j_\mu(u_\mu) - j_\mu(\tilde{u}) \;\ge\; g\int_{\Gamma_3} |\tilde{u}_\tau|^\mu (u_\mu - \tilde{u})\, da \tag{41}$$

Consequently, taking $v = \tilde{u} =$ in (39) and using (41), we deduce that in order to establish (40) it suffices to prove that

$$g\int_{\Gamma_3} |\tilde{u}_\tau|^\mu (u_\mu - \tilde{u})\, da \longrightarrow 0 \quad \text{as } \mu \to 0 \tag{42}$$

Indeed, since the trace map is linear and continuous from H_1 into $L^2(\Gamma)^N$, one can easily deduce from (37) that

$$u_\mu \rightharpoonup \tilde{u} \quad \text{weakly in } L^2(\Gamma_3)^N \quad \text{as } \mu \to 0 \tag{43}$$

Moreover, from the Lebesgue theorem we obtain

$$|\tilde{u}_\tau|^\mu \longrightarrow 1 \quad \text{in} \quad L^2(\Gamma_3)^N \quad \text{as} \quad \mu \to 0 \tag{44}$$

Therefore, using (43) and (44), we establish (42) and consequently (40). In order to pass to the limit in (35) as $\mu \to 0$, we need to prove that

$$\liminf_{\mu\to 0} \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(v) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} \leq \langle \mathcal{F}(\varepsilon(\tilde{u})),\ \varepsilon(v) - \varepsilon(\tilde{u})\rangle_{\mathcal{H}} \qquad \forall v \in V \tag{45}$$

For this, taking $v = \tilde{u}$ in (35) on the one hand and using the monotonicity of the operator $\mathcal{F}$ (see (10)) on the other hand, we obtain

$$\langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} \geq j_\mu(u_\mu) - j_\mu(\tilde{u}) + \langle f,\ \tilde{u} - u_\mu\rangle_V$$

and

$$\langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} \leq \langle \mathcal{F}(\varepsilon(\tilde{u})),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}}$$

Passing to the limit in these inequalities as $\mu \to 0$, from (37)–(40) we see that

$$\liminf_{\mu\to 0} \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} \geq 0$$

and

$$\limsup_{\mu\to 0} \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} \leq 0$$

Therefore

$$\lim_{\mu\to 0} \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} = 0 \tag{46}$$

Let $v \in V$ and $\theta \in (0,1)$. The monotonicity assumption in (10) applied with u_μ and $w \in V$ given by

$$w = (1-\theta)\tilde{u} + \theta v \tag{47}$$

implies that

$$\begin{aligned}\langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} &+ \theta\langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(v) - \varepsilon(\tilde{u})\rangle_{\mathcal{H}} \\ &\leq \langle \mathcal{F}(\varepsilon(w)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} + \theta\langle \mathcal{F}(\varepsilon(w)),\ \varepsilon(v) - \varepsilon(\tilde{u})\rangle_{\mathcal{H}}\end{aligned} \tag{48}$$

Using now (46), (37) in (48), we obtain

$$\liminf_{\mu\to 0} \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(v) - \varepsilon(\tilde{u})\rangle_{\mathcal{H}} \leq \langle \mathcal{F}(\varepsilon(w)),\ \varepsilon(v) - \varepsilon(\tilde{u})\rangle_{\mathcal{H}} \tag{49}$$

Moreover, since

$$\begin{aligned}\langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(v) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} &= \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} \\ &\quad + \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(v) - \varepsilon(\tilde{u})\rangle_{\mathcal{H}}\end{aligned}$$

from (46) and (49) it follows that

$$\liminf_{\mu\to 0} \langle \mathcal{F}(\varepsilon(u_\mu)),\, \varepsilon(v)-\varepsilon(u_\mu)\rangle_{\mathcal{H}} \le \langle \mathcal{F}(\varepsilon(w)),\, \varepsilon(v)-\varepsilon(\tilde{u})\rangle_{\mathcal{H}} \tag{50}$$

The inequality (45) may be deduced by introducing (47) in (50) and passing to the limit as $\theta \to 0$.

Using now (38)–(40) and (45), we may pass to the limit in (35) as $\mu \to 0$ and obtain that $\tilde{u}$ is a solution to the variational problem (20). Therefore, from the uniqueness of the solution to this problem (see Theorem 1) we deduce that $\tilde{u} = u$. Thus u is the unique weak limit of any subsequence of $(u_\mu)_\mu$. Consequently, the whole sequence $(u_\mu)_\mu$ is weakly convergent in V to u, i.e.

$$u_\mu \rightharpoonup u \quad \text{weakly in } V \quad \text{as } \mu \to 0 \tag{51}$$

In order to obtain (36), let us remark that from (10) and (8) it follows that

$$C|u_\mu - u|_V^2 \le \langle \mathcal{F}(\varepsilon(u)),\, \varepsilon(u)-\varepsilon(u_\mu)\rangle_{\mathcal{H}} - \langle \mathcal{F}(\varepsilon(u_\mu)),\, \varepsilon(u)-\varepsilon(u_\mu)\rangle_{\mathcal{H}} \tag{52}$$

where $C > 0$ is a positive constant independent of μ. The strong convergence (36) is finally a consequence of (51) and (46) since $\tilde{u} = u$. ■

Remark 4. Let u and u_μ be the solutions to the problems P and P^μ given in Theorems 1 and 4, respectively. We define the associated stress fields by

$$\sigma = \mathcal{F}(\varepsilon(u)) \tag{53}$$

and

$$\sigma_\mu = \mathcal{F}(\varepsilon(u_\mu)) \tag{54}$$

Then we have

$$\sigma_\mu \longrightarrow \sigma \quad \text{in } \mathcal{H}_1 \quad \text{as } \mu \to 0 \tag{55}$$

Indeed, it follows from (53), (20) applied with $v = \pm\varphi \in \mathcal{D}(\Omega)^N$ and (1) that

$$\operatorname{Div} \sigma + f_0 = 0 \quad \text{a.e. in } \Omega \tag{56}$$

A similar argument used for (54) and (35) implies that

$$\operatorname{Div} \sigma_\mu + f_0 = 0 \quad \text{a.e. in } \Omega \tag{57}$$

Therefore, by (53)–(54) and (56)–(57) we deduce that

$$|\sigma_\mu - \sigma|_{\mathcal{H}_1} = |\sigma_\mu - \sigma|_{\mathcal{H}} = \left|\mathcal{F}(\varepsilon(u_\mu)) - \mathcal{F}(\varepsilon(u))\right|_{\mathcal{H}} \tag{58}$$

The strong convergence (55) is finally a consequence of (58), (10) and (36).

Remark 5. Let us consider the following contact and friction conditions:

$$u_\nu = 0 \quad \text{on } \Gamma_3, \quad |\sigma_\tau| = -g|u_\tau|^{\mu-1}u_\tau \quad \text{on } \Gamma_3 \tag{59}$$

Using arguments similar to those used in the proof of Lemma 1, one can prove that the solution u_μ to Problem P_μ and the associated stress field σ_μ given by (54) represent a weak solution (in the sense of Lemma 1) to the frictional contact problem (3)–(6), (59).

Remark 6. The strong convergence (36), (55) may be interpreted as follows: the weak solution $\{u, \sigma\}$ to problem (3)–(7) modelling the frictional contact between an elastic body and a rigid foundation may be approximated by the weak solution $\{u_\mu, \sigma_\mu\}$ to problem (3)–(6), (59) which models the frictional contact between the elastic body and the rigid foundation using a more regular friction law. The regularization used here may be of a strong interest in the numerical study of such a type of contact problems.

References

Amassad A. and Sofonea M. (1998): *Analysis of a quasistatic viscoplastic problem involving Tresca friction law.* — Discr. Cont. Dynam. Syst., Vol.4, No.1, pp.55–72.

Brezis H. (1968): *Equations et inéquations non linéaires dans les espaces vectoriels en dualité.* — Annales Inst. Fourier, Vol.18, pp.115–175.

Burguera M. and Viaño J. M. (1995): *Numerical solving of frictionless contact problems in perfectly plastic bodies.* — Comp. Meth. Appl. Mech. Eng., Vol.121, pp.303–322.

Drabla S., Rochdi M. and Sofonea M. (1997): *On a frictionless contact problem for elastic-viscoplastic materials with internal state variables.* — Math. Comp. Model., Vol.26, No.12, pp.31–47.

Drabla S., Sofonea M. and Teniou B. (1998): *Analysis of a frictionless contact problem for elastic bodies.* — Annales Polonici Mathematici, Vol.LXIX, No.1, pp.75–88.

Duvaut G. and Lions J.L. (1972): *Les Inéquations en Mécanique et en Physique.* — Paris: Dunod.

Haslinger J. and Hlaváček I. (1980): *Contact between elastic bodies. I. Continuous problem.* — Applik. Math., Vol.25, pp.324–347.

Haslinger J. and Hlaváček I. (1982): *Contact between elastic perfectly plastic bodies.* — Applik. Math., Vol.27, pp.27–45.

Hlaváček I. and Nečas J. (1981): *Mathematical Theory of Elastic and Elastoplastic Bodies: an Introduction.* — Amsterdam: Elsevier.

Hlaváček I. and Nečas J. (1983): *Solution of Signorini's contact problem in the deformation theory of plasticity by secant modules method.* — Applik. Math., Vol.28, pp.199–214.

Ionescu I.R. and Sofonea M. (1993): *Functional and Numerical Methods in Viscoplasticity.* — Oxford: Oxford University Press.

Kikuchi N. and Oden T.J. (1980): *Theory of variational inequalities with application to problems of flow through porous media.* — Int. J. Eng. Sci., Vol.18, No.10, pp.1173–1284.

Kikuchi N. and Oden T.J. (1988): *Contact Problems in Elasticity.* — Philadelphia: SIAM.

Licht C. (1985): *Un problème d'élasticité avec frottement visqueux non linéaire.* — J. Méc. Th. Appl., Vol.4, No.1, pp.15–26.

Panagiotopoulos P.D. (1985): *Inequality Problems in Mechanics and Applications.* — Basel: Birkhäuser.

Rochdi M. (1997): *Analyse Variatio nnelle de Quelques Problèmes aux Limites en Viscoplasticité.* — Ph.D. Thesis, University of Perpignan, France.

Rochdi M. and Sofonea M. (1997): *On frictionless contact between two elastic-viscoplastic bodies.* — QJMAM, Vol.50, No.3, pp.481–496.

Shillor M. and Sofonea M. (1998): *A quasistatic viscoelastic contact problem with friction.* — preprint.

Sofonea M. (1993): *Problèmes Mathématiques en Elasticité et Viscoplasticité.* — Cours de DEA de Mathématiques Appliquées, Laboratoire de Mathématiques Appliquées, Blaise Pascal University, Clermont-Ferrand, France.

Sofonea M. (1997): *On a contact problem for elastic-viscoplastic bodies.* — Nonlin. Anal. TMA, Vol.29, No.9, pp.1037–1050.

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