

POSITIVE STABLE REALIZATIONS OF FRACTIONAL CONTINUOUS-TIME LINEAR SYSTEMS

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Conditions for the existence of positive stable realizations with system Metzler matrices for fractional continuous-time linear systems are established. A procedure based on the Gilbert method for computation of positive stable realizations of proper transfer matrices is proposed. It is shown that linear minimum-phase systems with real negative poles and zeros always have positive stable realizations.

Keywords: fractional, positive, stable, realization, system Metzler matrix, procedure, linear continuous-time.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of the state of the art in positive system theory is given in the monographs of Farina and Rinaldi (2000) as well as Kaczorek (2002). A variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc. An overview of the positive realization problem is given by Benvenuti and Farina (2004), Farina and Rinaldi (2000), or Kaczorek (2002; 2009b). The realization problem for positive continuous-time and discrete-time linear systems has been considered by Kaczorek (2004; 2006a; 2006b; 2006c) along with the positive minimal realization problem for singular discrete-time systems with delays (2005) as well as the realization problem for fractional linear systems (2008b; 2011) and for positive 2D hybrid systems (2008c). Mathematical fundamentals of the fractional calculus and some applications are given by Kilbas *et al.* (2006) and Podlubny (1999), while some selected problems in fractional systems theory are presented by Kaczorek (2008a; 2009a; 2011). In this paper sufficient conditions will be established for the existence of positive stable realizations with the Metzler system matrix of fractional continuous-time linear systems and a procedure for computation of the realizations of proper transfer matrices will be proposed.

The paper is organized as follows. In Section 2 so-

me definitions and theorems concerning fractional positive continuous-time linear systems are recalled and the problem formulation is given. A problem solution is presented in Sections 3 and 4. In Section 3, a procedure based on the Gilbert method is presented and in Section 4, the problem is solved for minimum-phase systems with negative real poles and zeros. Concluding remarks and open problems are presented in Section 5. The following notation is used: \mathbb{R} means the set of real numbers, $\mathbb{R}^{n \times m}$ stands for the set of $n \times m$ real matrices, $\mathbb{R}_+^{n \times m}$ is the set of $n \times m$ matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$, $\mathbb{R}^{n \times m}[s]$ signifies the set of $n \times m$ polynomial matrices in s with real coefficients, M_n denotes the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n is the $n \times n$ identity matrix.

2. Preliminaries and problem formulation

Consider the continuous-time linear system

$${}_0D_t^\alpha x(t) = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \quad (1a)$$

$$y(t) = Cx(t) + Du(t), \quad (1b)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$,

$${}_0D_t^\alpha x(t) = \frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} d\tau, \quad (2)$$
$$\dot{x}(\tau) = \frac{dx(\tau)}{d\tau}$$

is the Caputo definition of the α -th order derivative, $\alpha \in (0, 1)$, while

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \quad (3)$$

is the Euler gamma function.

Definition 1. (Kaczorek, 2011) The fractional system (1) is called (internally) *positive* if $x(t) \in \mathbb{R}_+^n, y(t) \in \mathbb{R}_+^m, t \geq 0$ for any initial conditions $x(0) = x_0 \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m, t \geq 0$.

Theorem 1. (Kaczorek, 2011) *The fractional system (1) is positive if and only if*

$$A \in M_n, \quad B \in \mathbb{R}_+^{n \times m}, \quad C \in \mathbb{R}_+^{p \times n}, \quad D \in \mathbb{R}_+^{p \times m}. \quad (4)$$

The transfer matrix of the fractional system (1) is given by

$$\begin{aligned} T(s^\alpha) &= C[I_n s^\alpha - A]^{-1} B + D \\ &= C[I_n \lambda - A]^{-1} B + D, \quad s^\alpha = \lambda. \end{aligned} \quad (5)$$

The transfer matrix is called *proper* if

$$\lim_{\lambda \rightarrow \infty} T(\lambda) = K \in \mathbb{R}^{p \times m}, \quad (6)$$

and it is called *strictly proper* if $K = 0$.

Definition 2. The matrices (4) are called a *positive realization* of the transfer matrix $T(\lambda) \in \mathbb{R}^{p \times m}(\lambda)$ if they satisfy Eqn. (5). The realization is called *minimal* if the dimension of A is minimal among all realizations of $T(\lambda)$. The realization is called (asymptotically) *stable* if and only if all eigenvalues λ_i of the matrix A satisfy the conditions $\arg \lambda_i > \alpha\pi/2, i = 1, \dots, n$.

The problem under consideration can be stated as follows: Given a rational matrix $T(\lambda) \in \mathbb{R}^{p \times m}(\lambda)$, find a positive stable realization with the system Metzler matrix A of $T(\lambda)$, i.e.,

$$A \in M_{nS}, \quad B \in \mathbb{R}_+^{n \times m}, \quad C \in \mathbb{R}_+^{p \times n}, \quad D \in \mathbb{R}_+^{p \times m}, \quad (7)$$

where M_{nS} is the set of $n \times n$ (asymptotically) stable Metzler matrices.

3. Problem solution

Using the Gilbert method (Kaczorek, 1992), a procedure for finding positive stable realizations with system Metzler matrices will be presented for transfer matrices with real negative poles. Consider a stable positive continuous-time linear system (1) with a given proper transfer matrix of the form

$$T(\lambda) = \begin{bmatrix} T_{11}(\lambda) & \dots & T_{1,m}(\lambda) \\ \vdots & \dots & \vdots \\ T_{p,1}(\lambda) & \dots & T_{p,m}(\lambda) \end{bmatrix} \in \mathbb{R}^{p \times m}(\lambda),$$

$$T_{i,j}(\lambda) = \frac{n_{i,j}(\lambda)}{d_{i,j}(\lambda)}, \quad i = 1, \dots, p, \quad j = 1, \dots, m, \quad (8)$$

where $\mathbb{R}^{p \times m}(\lambda)$ is the set of proper rational real matrices in $s^\alpha = \lambda$. The matrix D can be found with the use of the formula

$$D = \lim_{\lambda \rightarrow \infty} T(\lambda), \quad (9)$$

and the strictly proper transfer matrix

$$T_{sp}(\lambda) = T(\lambda) - D, \quad (10)$$

which can be written in the form

$$T_{sp}(\lambda) = \frac{N(\lambda)}{d(\lambda)} \in \mathbb{R}^{p \times m}(\lambda), \quad (11)$$

where $N(\lambda) \in \mathbb{R}^{p \times m}[\lambda]$ and

$$d(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0. \quad (12)$$

It is assumed that the equation $d(\lambda) = 0$ has only distinct real negative roots $\lambda_1, \lambda_2, \dots, \lambda_n$ ($\lambda_i \neq \lambda_j$ for $i \neq j$), i.e., $d(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$. In this case, the transfer matrix (11) can be written in the form

$$T_{sp}(\lambda) = \sum_{i=1}^n \frac{T_i}{\lambda - \lambda_i}, \quad (13)$$

where

$$\begin{aligned} T_i &= \lim_{\lambda \rightarrow \lambda_i} (\lambda - \lambda_i) T_{sp}(\lambda) \\ &= \frac{N(\lambda_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (\lambda_i - \lambda_j)}, \quad i = 1, \dots, n. \end{aligned} \quad (14)$$

Let

$$\text{rank } T_i = r_i \leq \min(p, m). \quad (15)$$

It is easy to show (Kaczorek, 1992) that

$$T_i = C_i B_i, \quad \text{rank } C_i = \text{rank } B_i = r_i, \quad (16a)$$

$i = 1, \dots, n$, where

$$\begin{aligned} C_i &= [C_{i,1} \quad C_{i,2} \quad \dots \quad C_{i,r_i}] \in \mathbb{R}^{p \times r_i}, \\ B_i &= \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,r_i} \end{bmatrix} \in \mathbb{R}^{r_i \times m}. \end{aligned} \quad (16b)$$

We shall show that the matrices are the desired positive stable realization with the system Metzler matrix

$$\begin{aligned} A &= \text{blockdiag} [I_{r_1} \lambda_1 \quad \dots \quad I_{r_n} \lambda_n], \\ B &= \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}, \quad C = [C_1 \quad \dots \quad C_n]. \end{aligned} \quad (17)$$

Using (13), (16) and (17), we obtain

$$\begin{aligned}
 T(\lambda) &= C[I\lambda - A]^{-1}B \\
 &= [C_1 \ \dots \ C_n] \text{blockdiag}[I_{r_1}(\lambda - \lambda_1)^{-1} \\
 &\dots I_{r_n}(\lambda - \lambda_n)^{-1}] \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} \\
 &= \sum_{i=1}^n \frac{C_i B_i}{\lambda - \lambda_i} = \sum_{i=1}^n \frac{T_i}{\lambda - \lambda_i}.
 \end{aligned} \tag{18}$$

From (17), it follows that

(i) if $\lambda_1, \lambda_2, \dots, \lambda_n$ are real negative, then the matrix A is stable and is a Metzler matrix;

(ii) if

$$T_i \in \mathbb{R}_+^{p \times m} \quad \text{for } i = 1, \dots, n, \tag{19}$$

then we can choose

$$C_i \in \mathbb{R}_+^{p \times r_i}, \quad B_i \in \mathbb{R}_+^{r_i \times m} \tag{20}$$

for $i = 1, \dots, n$ and

$$B \in \mathbb{R}_+^{\bar{n} \times m}, \quad C \in \mathbb{R}_+^{p \times \bar{n}}, \quad \bar{n} = \sum_{i=1}^n r_i.$$

If $T(\infty) \in \mathbb{R}_+^{p \times m}$, then from (9) we have $D \in \mathbb{R}_+^{p \times m}$. Therefore, the following theorem has been proved.

Theorem 2. *There exists a positive stable realization (17), (9) of the proper transfer matrix (8) if the following conditions are satisfied:*

(i) *The poles of $T(\lambda)$ are distinct, real and negative, i.e., $\lambda_i \neq \lambda_j$ for $i \neq j$, $\lambda_i < 0$, $i = 1, \dots, n$.*

(ii) $T_i \in \mathbb{R}_+^{p \times m}$ for $i = 1, \dots, n$.

(iii) $T(\infty) \in \mathbb{R}_+^{p \times m}$.

If the conditions of Theorem 1 are satisfied, the following procedure can be used to find the desired positive stable realization with the system Metzler matrix.

Procedure 1.

Step 1. Using (9), find the matrix D and the strictly proper transfer matrix (10) and write it in the form (11).

Step 2. Find the real zeros $\lambda_1, \lambda_2, \dots, \lambda_n$ of the polynomial (12).

Step 3. Using (14), find the matrices T_1, \dots, T_n and their decomposition (16).

Step 4. Using (17), find the matrices A, B and C .

Example 1. Using Procedure 1, find a positive stable realization with the system Metzler matrix of the transfer matrix

$$T(\lambda) = \begin{bmatrix} \frac{\lambda + 3}{\lambda + 1} & \frac{2\lambda + 5}{\lambda + 2} \\ \frac{1}{\lambda + 2} & \frac{\lambda + 4}{\lambda + 3} \end{bmatrix}. \tag{21}$$

Employing Procedure 1, we deliver the following.

Using (9) for (20), we obtain

$$D = \lim_{\lambda \rightarrow \infty} T(\lambda) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \tag{22}$$

and the strictly proper transfer matrix

$$T_{sp}(\lambda) = T(\lambda) - D = \begin{bmatrix} \frac{2}{\lambda + 1} & \frac{1}{\lambda + 2} \\ \frac{1}{\lambda + 2} & \frac{1}{\lambda + 3} \end{bmatrix},$$

which can be written in the form

$$\begin{aligned}
 T_{sp}(s) &= \frac{1}{(\lambda + 1)(\lambda + 2)(\lambda + 3)} \\
 &\times \begin{bmatrix} 2(\lambda + 2)(\lambda + 3) & (\lambda + 1)(\lambda + 3) \\ (\lambda + 1)(\lambda + 3) & (\lambda + 2)(\lambda + 1) \end{bmatrix} \\
 &= \frac{N(\lambda)}{d(\lambda)}.
 \end{aligned} \tag{23}$$

In this case, $d(\lambda) = (\lambda + 1)(\lambda + 2)(\lambda + 3)$, $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$ and the condition (i) of Theorem 1 is met. Using (14) and (16), we obtain

$$\begin{aligned}
 T_1 &= \frac{1}{(\lambda + 2)(\lambda + 3)} \\
 &\times \left[\begin{array}{cc} 2(\lambda + 2)(\lambda + 3) & (\lambda + 1)(\lambda + 3) \\ (\lambda + 1)(\lambda + 3) & (\lambda + 2)(\lambda + 1) \end{array} \right] \Big|_{\lambda=-1} \\
 &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \\
 r_1 &= \text{rank } T_1 = 1, \quad T_1 = C_1 B_1, \\
 B_1 &= [1 \ 0], \quad C_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix},
 \end{aligned} \tag{24a}$$

$$T_2 = \frac{1}{(\lambda + 1)(\lambda + 3)} \times \left[\begin{array}{cc} 2(\lambda + 2)(\lambda + 3) & (\lambda + 1)(\lambda + 3) \\ (\lambda + 1)(\lambda + 3) & (\lambda + 2)(\lambda + 1) \end{array} \right] \Big|_{\lambda=-2}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$r_2 = \text{rank } T_2 = 2, \quad T_2 = C_2 B_2,$$

$$B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$C_2 = [C_{21} \quad C_{22}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

(24b)

$$T_3 = \frac{1}{(\lambda + 1)(\lambda + 2)} \times \left[\begin{array}{cc} 2(\lambda + 2)(\lambda + 3) & (\lambda + 1)(\lambda + 3) \\ (\lambda + 1)(\lambda + 3) & (\lambda + 2)(\lambda + 1) \end{array} \right] \Big|_{\lambda=-3}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$r_3 = \text{rank } T_3 = 1, \quad T_3 = C_3 B_3,$$

$$B_3 = [0 \quad 1], \quad C_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(24c)

From (23), it follows that the conditions (ii) of Theorem 1 are satisfied. Using (17) and (23), we obtain

$$A = \begin{bmatrix} I_{r_1} \lambda_1 & 0 & 0 \\ 0 & I_{r_2} \lambda_2 & 0 \\ 0 & 0 & I_{r_1} \lambda_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C = [C_1 \quad C_2 \quad C_3] = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The desired positive stable realization of (20) is given by (25) and (22). This approach can be extended to transfer matrices with multiple real negative poles (Shaker and Dixon, 1977). ♦

4. Minimum-phase systems with real poles and zeros

Consider the stable strictly proper irreducible transfer function

$$T_{sp}(\lambda) = \frac{\bar{b}_{n-1} \lambda^{n-1} + \dots + \bar{b}_1 \lambda + \bar{b}_0}{\lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0}$$

$$= \frac{\bar{b}_{n-1}(\lambda - z_1) \dots (\lambda - z_{n-1})}{(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)} \quad (26)$$

where $\lambda_1, \dots, \lambda_n$ are the real negative poles and z_1, \dots, z_{n-1} are real negative zeros of the transfer function.

Theorem 3. *There exists a positive stable realization of (26) if*

$$\lambda_k < z_k < \lambda_{k+1} \quad \text{for } k = 1, \dots, n - 1. \quad (27)$$

Proof. From (14), we have

$$T_i = \frac{(\lambda_i - z_1)(\lambda_i - z_2) \dots (\lambda_i - z_{n-1})}{(\lambda_i - \lambda_1) \dots (\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1}) \dots (\lambda_i - \lambda_n)} > 0 \quad (28)$$

for $i = 1, \dots, n$ if the condition (ii) is satisfied. By Theorem 1, the matrices

$$A = \text{diag}[\lambda_1 \quad \dots \quad \lambda_n],$$

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad C = [c_1 \quad \dots \quad c_n], \quad (29)$$

$$T_i = b_i c_i, \quad i = 1, \dots, n$$

are a positive stable realization of the transfer function (26). ■

Example 2. Let us find a positive realization of the strictly proper transfer function

$$T_{sp}(\lambda) = \frac{\lambda + 2}{\lambda^2 + 4\lambda + 3}. \quad (30)$$

In this case, $\lambda_1 = -1, \lambda_2 = -3, z_1 = -2$ and the condition (27) is satisfied. Using (14), we obtain

$$T_1 = \frac{\lambda + 2}{\lambda + 3} \Big|_{\lambda=-1} = \frac{1}{2}, \quad T_2 = \frac{\lambda + 2}{\lambda + 1} \Big|_{\lambda=-3} = \frac{1}{2}$$

and

$$T_1 = b_1 c_1 = \frac{1}{2}, \quad b_1 = 1, \quad c_1 = \frac{1}{2},$$

$$T_2 = b_2 c_2 = \frac{1}{2}, \quad b_2 = 1, \quad c_2 = \frac{1}{2}.$$

The desired positive realization has the form

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix},$$

$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad (31)$$

$$C = [c_1 \quad c_2] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Now let us consider the strictly proper transfer matrix (11) rewritten in the form

$$T_{sp}(\lambda) = \frac{1}{(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)} \times \begin{bmatrix} (\lambda - z_{11}^1) \cdots (\lambda - z_{11}^{n_{11}}) & \cdots \\ \vdots & \cdots \\ (\lambda - z_{p,1}^1) \cdots (\lambda - z_{p,1}^{n_{p,1}}) & \cdots \\ \vdots & \cdots \\ (\lambda - z_{1,m}^1) \cdots (\lambda - z_{1,m}^{n_{1,m}}) \\ \vdots \\ (\lambda - z_{p,m}^1) \cdots (\lambda - z_{p,m}^{n_{p,m}}) \end{bmatrix}. \quad (32)$$

where $\lambda_1, \dots, \lambda_n$ are real negative poles and $z_{ij}^k, i = 1, \dots, p, j = 1, \dots, m, k = 1, \dots, n_{ij}$ are real negative zeros.

Theorem 4. *There exists a positive stable realization (17) of (32) if*

$$\lambda_k \leq z_{i,j}^k \leq \lambda_{k+1} \quad (33)$$

for $i = 1, \dots, p, j = 1, \dots, m,$ and $k = 1, \dots, n_{ij}.$

The proof is similar to that of Theorem 3. If the condition (33) is satisfied, then a positive stable realization (17) of (32) can be found with the use of Procedure 1.

Example 3. Using Procedure 1, we wish to find a positive realization of the strictly proper transfer matrix

$$T_{sp}(\lambda) = \frac{1}{(\lambda + 1)(\lambda + 3)(\lambda + 5)} \times \begin{bmatrix} (\lambda + 2)(\lambda + 4) & (\lambda + 1)(\lambda + 4) \\ (\lambda + 2)(\lambda + 5) & (\lambda + 2)(\lambda + 4) \end{bmatrix}. \quad (34)$$

In this case, we have $\lambda_1 = -1, \lambda_2 = -3, \lambda_3 = -5, z_{11}^1 = -2, z_{11}^2 = -4, z_{12}^1 = -1, z_{12}^2 = -4, z_{21}^1 = -2, z_{21}^2 = -5, z_{22}^1 = -2, z_{22}^2 = -4,$ and the conditions (33) are satisfied. Therefore, by Theorem 4, there exists a positive stable realization of the transfer matrix (34). Using (14) and (34), we obtain

$$T_1 = \frac{1}{(\lambda + 3)(\lambda + 5)} \times \begin{bmatrix} (\lambda + 2)(\lambda + 4) & (\lambda + 1)(\lambda + 4) \\ (\lambda + 2)(\lambda + 5) & (\lambda + 2)(\lambda + 4) \end{bmatrix} \Big|_{\lambda=-1} = \begin{bmatrix} \frac{3}{8} & 0 \\ \frac{1}{2} & \frac{3}{8} \end{bmatrix},$$

$$\text{rank } T_1 = 2, \quad T_1 = C_1 B_1, \quad C_1 = \begin{bmatrix} \frac{3}{8} & 0 \\ \frac{1}{2} & \frac{3}{8} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (35a)$$

$$T_2 = \frac{1}{(\lambda + 1)(\lambda + 5)} \times \begin{bmatrix} (\lambda + 2)(\lambda + 4) & (\lambda + 1)(\lambda + 4) \\ (\lambda + 2)(\lambda + 5) & (\lambda + 2)(\lambda + 4) \end{bmatrix} \Big|_{\lambda=-3} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix},$$

$$\text{rank } T_2 = 2, \quad T_2 = C_2 B_2, \quad C_2 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (35b)$$

$$T_3 = \frac{1}{(\lambda + 1)(\lambda + 3)} \times \begin{bmatrix} (\lambda + 2)(\lambda + 4) & (\lambda + 1)(\lambda + 4) \\ (\lambda + 2)(\lambda + 5) & (\lambda + 2)(\lambda + 4) \end{bmatrix} \Big|_{\lambda=-5} = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} \\ 0 & \frac{3}{8} \end{bmatrix},$$

$$\text{rank } T_3 = 2, \quad T_3 = C_3 B_3, \quad C_3 = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} \\ 0 & \frac{3}{8} \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (35c)$$

The desired positive stable realization has the form

$$A = \begin{bmatrix} I_{r_1} \lambda_1 & 0 & 0 \\ 0 & I_{r_2} \lambda_2 & 0 \\ 0 & 0 & I_{r_3} \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C = [C_1 \ C_2 \ C_3] = \frac{1}{8} \begin{bmatrix} 3 & 0 & 2 & 4 & 3 & 4 \\ 4 & 3 & 4 & 2 & 0 & 3 \end{bmatrix}. \quad (36)$$

5. Concluding remarks

Conditions for the existence of positive stable realizations with system Metzler matrices of fractional continuous-time linear systems have been established (Theorem 2). A procedure based on the Gilbert method for computation of positive stable realizations of proper transfer matrices has been proposed. It has been shown that minimum-phase systems with real negative poles and zeros always have the positive stable realizations (Theorems 3 and 4). The deliberations have been illustrated by numerical examples.

The following are open problems for fractional continuous-time linear systems:

1. Find necessary and sufficient conditions for the existence of positive stable realizations with system Metzler matrices of proper transfer matrices.
2. Give a method for finding positive stable realizations with system Metzler matrices which is not based on the Gilbert method of proper transfer matrices.

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