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# Optimal Control

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# Optimal Control

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# SENSITIVITY SYNTHESIS OF OPTIMAL CONTROL UNDER CHANGES OF SYSTEM ORDER

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## 1. Introduction

An ideal but unrealistic assumption that a real physical system coincides perfectly with its mathematical model has usually been made in the analysis and the synthesis of control systems. In practice, however, there often arises the discrepancy between the dynamical characteristics of the real physical system and its mathematical model, partly because of the unskilled identification or parameter estimation technique, or partly because of the inherent fluctuation effect contained in the physical system.

It is, thus, necessary to analyse quantitative aspects of the discrepancy itself or its effect on the control performance, and further it is desirable to develop a new synthesis method to reduce such discrepancies.

The concept of the dynamic system sensitivity[1] plays an important role in the aspect mentioned above. Chang[2] classified the problems associated with the sensitivity analysis into the following three major categories, from the viewpoint of parameter variations:

(1) $\alpha$-variations --- parameter variations which do not alter the order of the system or its initial conditions.

(2) $\beta$-variations --- variations of the initial conditions or variations of system characteristic due to external disturbances.

(3) $\lambda$-variations --- parameter variations which lead to changes in the system order.

The $\alpha$-variation problem has been discussed by a number of researchers. The $\lambda$-variation problem was, however, rarely studied from the control engineer's viewpoint.[2,3] The $\lambda$-variation problem should attract more interest, partly because the problem is closely connected with the reduction, or simplification, of the system model, or partly because there often exist such parameter variations in actual practice.

In this paper, we extend the concept of sensitivity in the synthesis of optimal control to the minimum energy problem subjected to changes of system order. For the foundation of the present study, we firstly introduce "the $\lambda$-combined system", which consists of both the model of a given

physical system and its sensitivity model with respect to "the λ-parameter", and then we examine the controllability of the λ-combined system. Secondly, for the controllable λ-combined system, a new synthesis method of minimum energy control with zero sensitive terminal constraints against changes of the system order is developed and its simple examples are given to demonstrate the advantage of the present method over the conventional one.

## 2. Basic concept of λ-sensitivity analysis

Let a physical system to be controlled be given by the following (n+1)-th order differential equations

$$\begin{cases} \dot{\tilde{x}}_i = f_i(\tilde{x}_1, \cdots, \tilde{x}_n, \tilde{x}_{n+1}, u, t), \quad (i=1,\cdots,n), & (2.1)_1 \\ \lambda \dot{\tilde{x}}_{n+1} = g(\tilde{x}_1, \cdots, \tilde{x}_n, \tilde{x}_{n+1}, u, t), & (2.1)_2 \end{cases}$$

with the initial condition

$$\tilde{x}_i(0) = \tilde{x}_i^0 \quad , \quad (i=1,\cdots,n+1) \quad , \qquad (2.2)$$

for $0 \leq t \leq T$, where $\tilde{x}_i$ is a state variable of the physical system, and T is a finite positive constant. The parameter $\lambda$ is assumed to be a sufficiently small positive constant. For the security of the following discussion, we assume that the functions $f_i$ (i=1,...,n) and g are continuously differentiable up to the second order with respect to their arguments. The symbol u represents a control function.

Letting $\lambda=0$ in Eq. $(2.1)_2$, we have the degenerate system

$$\begin{cases} \dot{x}_i = f_i(x_1, \cdots, x_n, x_{n+1}, u, t), \quad (i=1,\cdots,n), & (2.3)_1 \\ x_{n+1} = h(x_1, x_2, \cdots, u, t), & (2.3)_2 \end{cases}$$

where for simplicity it is assumed that $g(x_1,\ldots,x_{n+1},u,t)=0$ can be solved with respect to $x_{n+1}$ as in Eq. $(2.3)_2$.

Let us suppose here that we regard, consciously or unconsciously, Eq. (2.3) as a simplified model system for Eq. (2.1), because of the assumption that the parameter $\lambda$ is sufficiently small if it exists. Then, how is the discrepancy between the solution of Eq. (2.1) and that of Eq. (2.3) ? The difference between the two state $\tilde{x}_i(t,\lambda)$ and $x_i(t)$ may formally be expressed by

$$\Delta x_i(t,\lambda) = \tilde{x}_i(t,\lambda) - x_i(t) = w_i(t)\cdot\lambda + o(\lambda^2), \quad (i=1,\cdots,n+1), \qquad (2.4)$$

where

$$w_i(t) = \frac{\partial \tilde{x}_i(t,\lambda)}{\partial \lambda}\Big|_{\lambda \to +0}, \quad (i = 1, 2, \cdots, n+1), \tag{2. 5}$$

is the λ-sensitivity function of the state $x_i$ with respect to the λ-parameter. Under suitable conditions[4,5] on the form of the physical system described by Eq. (2. 1), $w_i(t)$ becomes the solution of the differential equations

$$\begin{cases} \dot{w}_i = \sum_{j=1}^{n+1} \frac{\partial f_i}{\partial x_j} w_j, \quad (i = 1, \cdots, n), & (2. 6)_1 \\ \sum_{j=1}^{n+1} \frac{\partial g}{\partial x_j} w_j - \dot{x}_{n+1} = 0, & (2. 6)_2 \end{cases}$$

with the initial condition

$$w_i(0) = \int_{\tilde{x}^\circ_{n+1}}^{h(\tilde{x}^\circ_1, \cdots, \tilde{x}^\circ_n, u(0), 0)} \frac{f_i(\tilde{x}^\circ_1, \cdots, \tilde{x}^\circ_n, x_{n+1}, u(0), 0) - f_i(\tilde{x}^\circ_1, \cdots, \tilde{x}^\circ_n, h^\circ, u(0), 0)}{g(\tilde{x}^\circ_1, \cdots, \tilde{x}^\circ_n, x_{n+1}, u(0), 0)} dx_{n+1}, \tag{2. 7}$$
$$(i = 1, \cdots, n).$$

where it should be noted that in Eq. (2. 6), both $\partial f_i/\partial x_j$ and $\partial g/\partial x_j$ must be evaluated along with the solution $x_i$ (i=1,...,n+1) of Eq. (2. 3).

In order to make clear the meaning of Eq. (2. 6), we shall proceed to manipulate Eq. (2. 6) in the following manner. From Eq. $(2. 6)_2$, the function $w_{n+1}(t)$ can be solved to be

$$w_{n+1} = \frac{1}{\left(\frac{\partial g}{\partial x_{n+1}}\right)} \left\{ \dot{x}_{n+1} - \sum_{j=1}^{n} \frac{\partial g}{\partial x_j} w_j \right\}, \tag{2. 8}$$

where we can assume $\partial g/\partial x_{n+1}=0$ without any loss of generality. Substitution of Eq. (2. 8) into Eq. $(2. 6)_1$ gives

$$\dot{w}_i = \sum_{j=1}^{n} \left\{ \frac{\partial f_i}{\partial x_j} - F_i \frac{\partial g}{\partial x_j} \right\} w_j + F_i \dot{x}_{n+1}, \quad (i = 1, \cdots, n), \tag{2. 9}$$

where, for simplicity, we denote

$$F_i(x_1, \cdots, x_{n+1}, u, t) = (\partial f_i/\partial x_{n+1})/(\partial g/\partial x_{n+1}), \quad (i = 1, \cdots, n). \tag{2.10}$$

By using Eq. (2. 3), Eq. (2. 9) becomes

$$\dot{w}_i = \sum_{j=1}^{n} \left\{ \frac{\partial f_i}{\partial x_j} - F_i \frac{\partial g}{\partial x_j} \right\} w_j + F_i \left[ \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} f_j + \frac{\partial h}{\partial u} \dot{u} + \frac{\partial h}{\partial t} \right],$$
$$(i = 1, \cdots, n). \tag{2.11}_1$$

$$w_{n+1} = \frac{1}{\left(\frac{\partial g}{\partial x_{n+1}}\right)} \left\{ \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} f_j + \frac{\partial h}{\partial u} \dot{u} + \frac{\partial h}{\partial t} - \sum_{j=1}^{n} \frac{\partial g}{\partial x_j} w_j \right\} \qquad (2.11)_2$$

By solving Eq. (2.11) with the initial condition (2. 7), which we call the λ-sensitivity equation, we can obtain the λ-sensitivity function $w_i(t)$.

The λ-sensitivity equation given by Eq. (2.11) clearly shows the linearity with respect to the λ-sensitivity function $w_i$ (i=1,...,n), which is similar to the so-called α-sensitivity equation. The initial value of the λ-sensitivity function is given by Eq. (2. 7), which is, in general, not zero, whereas that of the α-sensitivity function is zero. Moreover, the fact that the λ-sensitivity equation (2.11) contains $\dot{u}$, the derivative of the control $u(t)$ with respect to time $t$, should be emphasized as a distinctive feature of the λ-sensitivity analysis. If the order of the highest derivative multiplied by $\lambda$ in Eq. (2. 1) is higher than n by r, $u^{(r)}(t)$ comes out in the λ-sensitivity equation.

## 3. λ-sensitivity synthesis of optimal control

We shall introduce the fundamental concept of the λ-sensitivity synthesis of optimal control into the minimum energy problem with terminal constraints, as a typical example.

The minimum energy problem is to find out the control that achieves the following two aims:

[1] to transfer the initial state of the system to the desired state at a preassigned time, and in so doing,

[2] to minimize the consumption of the control energy.

Such an optimal control problem is usually solved for a model system of the real physical system by applying the calculus of variations, Dynamic Programming or Maximum Principle etc. However, the discrepancy often arises between the real physical system and its model system, because the complete mathematical description of the real physical system is almost impossible, or because the model system cannot always represent the real physical system whose dynamical characteristics vary with time.

In such a case, even if the optimal control $\hat{u}(t)$ which performs [1] and [2] on the model system could be obtained, the control $\hat{u}(t)$ cannot achieve the aim [1] on the real physical system. Our most important requirement in this minimum energy problem is the aim [1]. Therefore, it is not reasonable to call the control $\hat{u}(t)$ the optimal control for the real physical system.

In order to make up for the defect involved in the conventional synthesis of optimal control, it has been already proposed in the case of the α-variation problem that we should introduce the sensitivity aspect into the optimization procedure.[6]

Consider the model equation (2. 3) and its λ-sensitivity equation (2.11) simultaneously

$$\begin{cases} \dot{x}_i = f_i(x_1, \cdots, x_n, h(x_1, \cdots, x_n, u, t), u, t), & (i = 1, \cdots, n), \quad (3.1)_1 \\ \dot{w}_i = \sum_{j=1}^{n} \left\{ \frac{\partial f_i}{\partial x_j} - F_i \frac{\partial g}{\partial x_j} \right\} w_j + F_i \left\{ \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} f_j + \frac{\partial h}{\partial u} \dot{u} + \frac{\partial h}{\partial t} \right\}, & (3.1)_2 \end{cases}$$

with the initial conditions

$$\begin{cases} x_i(0) = \tilde{x}_i^{\circ}, & (3.2)_1 \\ w_i(0) = \int_{\tilde{x}_{n+1}^{\circ}}^{h^{\circ}} \frac{f_i(\tilde{x}_1^{\circ}, \cdots, \tilde{x}_n^{\circ}, x_{n+1}, u(0), 0) - f_i(\tilde{x}_1^{\circ}, \cdots, \tilde{x}_n^{\circ}, h^{\circ}, u(0), 0)}{g(\tilde{x}_1^{\circ}, \cdots, \tilde{x}_n^{\circ}, x_{n+1}, u(0), 0)} dx_{n+1}, \\ \qquad (i = 1, \cdots, n) \quad (3.2)_2 \end{cases}$$

which we simply call the λ-combined system, then it may be recognized that there is a possibility to control at our disposal not only the model state $x_i$ but also its sensitivity function $w_i$ by choosing a proper control function u(t). That is, we can take the λ-variation effect into consideration at the initial stage of the synthesis. This is the basis of the λ-sensitivity synthesis of optimal control.

Since the variation of the terminal constraints due to the existence of the small parameter λ is approximated to be $\Delta x_i(\lambda, T) = w_i(T) \cdot \lambda$, it is, then, desirable to choose the control in such a way that it should satisfy $w_i(T) = 0$ (i=1,...,n). From this point of view, the minimum energy problem with terminal constraints is, then, restated as follows:

For the λ-combined system described by Eq. (3. 1), choose the optimal control u*(t) that

[1] satisfies the new terminal constraint

$$w_i(T) = 0, \quad (i = 1, \cdots, n), \quad (3.3)_1$$

together with

$$x_i(T) = x_i^T, \quad (i = 1, \cdots, n), \quad (3.3)_2$$

where $x_i^T$ is the desired final state, and

[2] minimizes the energy consumption

$$J(u) = \int_0^T u^2(t)\,dt \,. \tag{3. 4}$$

Here, a question arises whether this rewritten optimization problem may have a solution or not. It depends upon the controllability of the λ-combined system. Therefore, before trying to solve this problem, we must examine the controllability of the λ-combined system. Since the λ-combined system contains $\dot{u}(t)$ besides $u(t)$, it is, however, rather difficult to apply the conventional controllability theory and the conventional optimization theory to this problem.

Now, when the coefficient of $\dot{u}$ does not depend on $x_{n+1}$ and $u$, or more precisely, when $F_i \cdot (\partial h/\partial u)$ is independent of $x_{n+1}$ and $u$, we can develop our discussion by introducing a new function $y_i(t)$ defined by

$$y_i = w_i - F_i \frac{\partial h}{\partial u} u \,, \quad (i = 1, \cdots, n) \,. \tag{3. 5}$$

Rewriting Eq. $(3.\ 1)_2$ in terms of $y_i$, we have

$$\begin{aligned} \dot{y}_i = {} & \sum_{j=1}^{n} \left\{ \frac{\partial f_i}{\partial x_j} - F_i \frac{\partial g}{\partial x_j} \right\} y_j + F_i \left\{ \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} f_j + \frac{\partial h}{\partial t} \right\} \\ & + \left[ \sum_{j=1}^{n} \left\{ \frac{\partial f_i}{\partial x_j} - F_i \frac{\partial g}{\partial x_j} \right\} F_i \frac{\partial h}{\partial u} + \sum_{j=1}^{n} \left\{ \frac{\partial F_i}{\partial x_j} f_j + \frac{\partial F_i}{\partial t} \right\} \frac{\partial h}{\partial u} \right. \\ & \left. + \left\{ \sum_{j=1}^{n} \frac{\partial^2 h}{\partial u \partial x_j} f_j + \frac{\partial^2 h}{\partial u \partial t} \right\} F_i \right] u \,, \quad (i = 1, \cdots, n) \,, \end{aligned} \tag{3. 6}$$

with the initial condition

$$y_i(0) = w_i(0) - F_i \frac{\partial h}{\partial u}\Big|_{\substack{t=0 \\ x_i = x_i^0}} \cdot u(0) \,, \quad (i = 1, \cdots, n) \,. \tag{3. 7}$$

Both the simultaneous differential equations $(3.\ 1)_1$ and (3. 6) with the initial conditions $(3.\ 2)_1$ and (3. 7) and the relation (3. 5) may be called the modified λ-combined system. We call the function $y_i(t)$ defined by Eq. (3. 5) the modified λ-sensitivity function. It should be noted, here, that $\dot{u}$ is eliminated from the modified λ-combined system.

By regarding $x_i$ and $y_i$ as the state variables and $x_i$ and $w_i$ as the output variables, we can examine the controllability of the λ-combined system, and moreover we can obtain the new optimal control by use of the conventional controllability and optimization theory.

## 4. Example: Linear time-invariant systems

As a physical system, consider the system described by (n+1)-th order linear time-invariant differential equations

$$\begin{cases} \dot{\tilde{x}}_i = \tilde{x}_{i+1} , & (i = 1, \cdots, n) , \quad (4.1)_1 \\ \lambda \dot{\tilde{x}}_{n+1} = \sum_{j=1}^{n} a_j \tilde{x}_j + bu - \tilde{x}_{n+1} , & (b \neq 0) . \quad (4.1)_2 \end{cases}$$

with the initial condition

$$\tilde{x}_i(0) = \tilde{x}_i^0 , \quad (i = 1, \cdots, n+1) , \qquad (4.2)$$

where $a_1, \ldots, a_n$ and $b$ are constant parameters. Let $\lambda$ be a sufficiently small positive constant parameter.

By letting $\lambda = 0$, as a model of Eq. (4.1), we have

$$\begin{cases} \dot{x}_i = x_{i+1} , & (i = 1, \cdots, n-1) , \quad (4.3)_1 \\ \dot{x}_n = \sum_{j=1}^{n} a_j x_j + bu , & \quad (4.3)_2 \end{cases}$$

with the initial condition

$$x_i(0) = \tilde{x}_i^0 , \quad (i = 1, \cdots, n) . \qquad (4.4)$$

The $\lambda$-sensitivity equation of Eq. (4.1) turns out to be

$$\begin{cases} \dot{w}_i = w_{i+1} , & (i = 1, \cdots, n-1) , \quad (4.5)_1 \\ \dot{w}_n = \sum_{j=1}^{n} a_j w_j - \sum_{j=1}^{n} (a_{j-1} + a_j a_n) x_j - a_n b u - b \dot{u} , & \quad (4.5)_2 \end{cases}$$

with the initial condition

$$\begin{cases} w_i(0) = 0 , & (i = 1, \cdots, n-1) , \quad (4.6)_1 \\ w_n(0) = \tilde{x}_{n+1}^0 - \sum_{j=1}^{n} a_j \tilde{x}_j^0 - b u(0) . & \quad (4.6)_2 \end{cases}$$

In order to eliminate $\dot{u}$ in Eq. (4.5), we introduce the modified $\lambda$-sensitivity function defined by

$$\begin{cases} y_i = w_i , & (i = 1, \cdots, n-1) , \quad (4.7)_1 \\ y_n = w_n + bu . & \quad (4.7)_2 \end{cases}$$

Eq. (4.5) hence becomes

$$\dot{y}_i = y_{i+1} , \quad (i = 1, \cdots, n-2) \qquad (4.8)_1$$

$$\begin{cases} \dot{y}_{n-1} = y_n - b u , & (4.8)_2 \\ \dot{y}_n = \sum_{j=1}^{n} a_j y_j - \sum_{j=1}^{n} (a_{j-1} + a_j a_n) x_j - 2 a_n b u , & (4.8)_3 \end{cases}$$

and the initial conditions are

$$\begin{cases} y_i(0) = 0 , \quad (i = 1, \cdots, n-1) , & (4.9)_1 \\ y_n(0) = \tilde{x}^{\circ}_{n+1} - \sum_{j=1}^{n} a_j \tilde{x}^{\circ}_j . & (4.9)_2 \end{cases}$$

The 2n-dimensional modified λ-combined system is, thus, written in the vector-matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u , \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x^{\circ} \\ y^{\circ} \end{bmatrix} , \qquad (4.10)_1$$

$$\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} E_n & 0 \\ 0 & E_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ d \end{bmatrix} u , \qquad (4.10)_2$$

where $x$, $y$, $w$, $b_1$, $b_2$ and $d$ denote n-dimensional vectors described by

$$x(t) = (x_1, x_2, \ldots, x_n)^T; \qquad y(t) = (y_1, y_2, \ldots, y_n)^T;$$

$$w(t) = (w_1, w_2, \ldots, w_n)^T; \qquad b_1 = (0, 0, \ldots, 0, b)^T;$$

$$b_2 = (0, \ldots, 0, -b, -2a_n b)^T; \qquad d = (0, 0, \ldots, 0, -b)^T;$$

A and C denote $n \times n$ constant matrices represented by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & & & & \cdots \\ \cdots & & & & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ -a_1 a_n & -(a_1 + a_2 a_n) & \cdots & & -(a_{n-1} + a_n^2) \end{bmatrix}$$

and $E_n$ denotes an $n \times n$ unit matrix.

At first, let us examine the controllability of the λ-combined system (4.10), that is, the controllability of the (output) vectors $x$ and $w$ by a scalar control u. For this purpose, we can apply the output controllability theorem introduced by Kreindler and Sarachik.[7] The theorem says that if and only if the rank of the $2n \times (2n+1)$ matrix G defined by

$$G = \left[ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \vdots \begin{bmatrix} A & 0 \\ C & A \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \vdots \cdots \vdots \begin{bmatrix} A & 0 \\ C & A \end{bmatrix}^{2n-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \vdots \begin{matrix} 0 \\ d \end{matrix} \right] , \tag{4.11}$$

is $2n$, then the $\lambda$-combined system (4.10) is completely controllable for the outputs $x$ and $w$.

A somewhat tedious calculation of the determinant of $G'$, an $2n \times 2n$ submatrix of $G$, which is derived by taking away the $(2n+1)$-th column $[0,d]^T$ from $G$, leads to

$$|G'| = - a_1^{n+1} b^{2n} . \tag{4.12}$$

We can say, thus, as a sufficient condition that the $\lambda$-combined system, that is, both the state $x$ and the $\lambda$-sensitivity $w$ are controllable by $u$ if $a_1 \neq 0$.

To demonstrate the advantage of the $\lambda$-sensitivity synthesis method over the conventional one, let us consider a simple illustrative example.

Suppose that the model system is given by

$$\dot{x}_1 = a x_1 + b u , \quad x_1(0) = \tilde{x}_1^0 , \tag{4.13}$$

which is the degenerate system of

$$\begin{cases} \dot{\tilde{x}}_1 = \tilde{x}_2 , \quad \tilde{x}_1(0) = \tilde{x}_1^0 , & (4.14)_1 \\ \lambda \dot{\tilde{x}}_2 = a \tilde{x}_1 + b u - \tilde{x}_2 , \quad \tilde{x}_2(0) = \tilde{x}_2^0 . & (4.14)_2 \end{cases}$$

then we have the $\lambda$-sensitivity equation

$$\dot{w}_1 = a w_1 - a^2 w_1 - ab u - b \dot{u} , \tag{4.15}$$

with the initial condition

$$w_1(0) = \tilde{x}_2^0 - a \tilde{x}_1^0 - b u(0) . \tag{4.16}$$

The conventional optimal control $\hat{u}(t)$ which transfers the model system from the initial state $\tilde{x}_1^0$ to the origin in time $T$ and in the minimum energy is given by

$$\hat{u}(t) = - \frac{a \cdot e^{a(T-t)}}{b \cdot \sinh(aT)} \cdot \tilde{x}_1^0 , \tag{4.17}$$

and the corresponding (nominal) optimal trajectory $\hat{x}_1(t)$ is given by

$$\hat{x}_1(t) = \frac{\sinh\{a(T-t)\}}{\sinh(aT)} \cdot \tilde{x}_1^0 . \tag{4.18}$$

which is shown in Fig. 1 by the broken line.

On the other hand, the λ-sensitivity optimal control $u^*(t)$ which transfers the λ-combined system from its initial state $(\tilde{x}_1^o, w_1(0))$ to the origin (0,0) in time T and in the minimum energy is expressed by

$$u^*(t) = -b(\pi_1 - 2a\pi_2 + a^2\pi_2 t)\cdot e^{-at} \quad , \tag{4.19}$$

and the corresponding (sensitivity) optimal trajectory is given by

$$x_1^*(t) = [\tilde{x}_1^o + b^2(\tfrac{3}{4}\pi_2 - \tfrac{1}{2a}\pi_1)]e^{at} + b^2[\tfrac{1}{2a}\pi_1 - \tfrac{3}{4}\pi_2 + \tfrac{at}{2}\pi_2]e^{-at} \quad , \tag{4.20}$$

which is illustrated in Fig. 2 by the broken line. In Eqs. (4.19) and (4.20), $\pi_1$ and $\pi_2$ are

$$\pi_1 = \frac{(12\tilde{x}_2^o + 8a\tilde{x}_1^o)e^{2aT} - 8a^3T^2\tilde{x}_1^o + 8aT\tilde{x}_2^o - 12\tilde{x}_2^o + 24a\tilde{x}_1^o}{b^2(e^{2aT} - 3e^{-2aT} - 8aT - 4a^2T^2 + 2)} \quad , \tag{4.21}_1$$

$$\pi_2 = \frac{(8\tilde{x}_2^o + 4a\tilde{x}_1^o)e^{2aT} + 8a^2T\tilde{x}_1^o - 8\tilde{x}_2^o + 12a\tilde{x}_1^o}{b^2(e^{2aT} - 3e^{-2aT} - 8aT - 4a^2T^2 + 2)} \quad . \tag{4.21}_2$$

The two λ-sensitivity functions $\hat{w}_1(t)$ and $w_1^*(t)$ are shown in Fig. 3.

In both Figs. 1 and 2, non-broken lines show the perturbed trajectories by the λ-variations. It can be clearly be seen that the λ-sensitivity optimal control $u^*(t)$ more successfully achieves the task against the change of system order than the conventional optimal control $\hat{u}(t)$. The detailed variations of $x_1$ at T=1.0 from its desired value $x_1$=0 are listed in Tab. 1 for a few values of λ.

## 5. Conclusions

We have been concerned with the control system containing a small parameter λ which leads to changes of the system order. After the preliminary notion of the λ-sensitivity analysis, the λ-combined system which consists of both the model of a physical system and its sensitivity equation with respect to the λ-parameter is introduced for the purpose of the λ-sensitivity synthesis of optimal control. Here, it is pointed out that the controllability of the λ-combined system plays an important role

in realizing the λ-sensitivity optimal control. We have found out a sufficient condition for the controllability in the case of a linear time-invariant system.

By giving a simple illustrative example of the minimum energy problem with terminal constraints, the advantage of the present method of synthesis over the conventional one is shown.

The authors believe that the introduction of the sensitivity concept into the theory of control system synthesis, especially, optimal control system synthesis fills the so-called gap between the theory and its practice.

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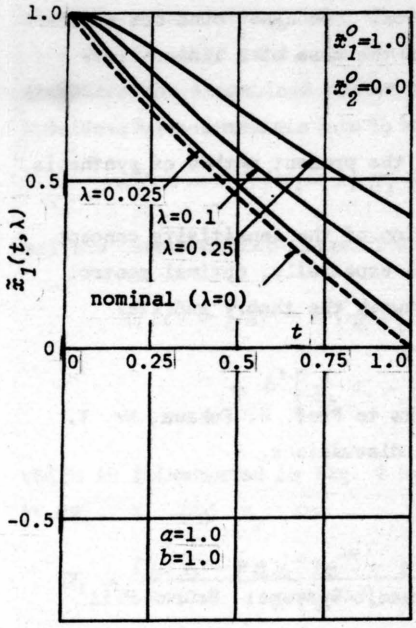


Fig. 1

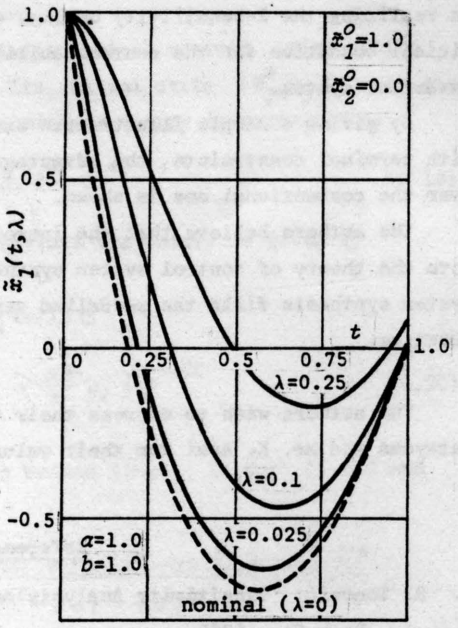


Fig. 2

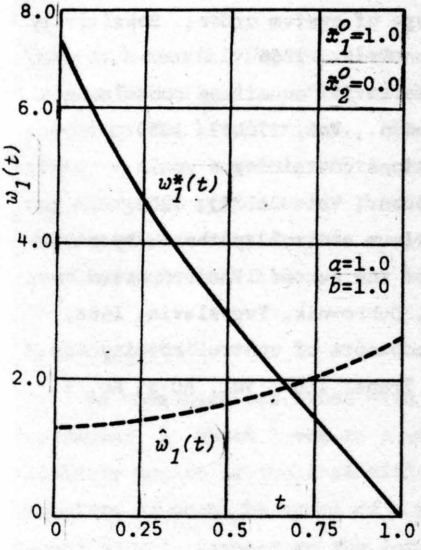


Fig. 3

Fig. 1:
Nominal optimal and perturbed trajectories by the conventional synthesis.

Fig. 2:
Nominal optimal and perturbed trajectories by the sensitivity synthesis.

Fig. 3:
Comparison of the λ-sensitivity functions by the conventional and sensitivity syntheses.

| parameter value | synthesis method | λ | | | | energy consump. |
|---|---|---|---|---|---|---|
| | | 0 | 0.01 | 0.05 | 0.10 | |
| a = -5.0 | c.syn. | 0 | -0.0012 | -0.0055 | -0.0099 | 0.0002 |
| b = 1.0 | s.syn. | 0 | -0.0001 | -0.0008 | -0.0024 | 0.0015 |
| a = -1.0 | c.syn. | 0 | 0.0024 | 0.0126 | 0.0275 | 0.1565 |
| b = 1.0 | s.syn. | 0 | 0.0000 | 0.0014 | 0.0066 | 0.1653 |
| a = 1.0 | c.syn. | 0 | 0.0256 | 0.1145 | 0.2037 | 1.1565 |
| b = 1.0 | s.syn. | 0 | -0.0001 | 0.0008 | 0.0076 | 5.1800 |
| a = 5.0 | c.syn. | 0 | 2.636 | 5.046 | 4.750 | 5.000 |
| b = 1.0 | s.syn. | 0 | 0.0642 | 0.4990 | 0.7834 | 10.046 |

Tab. 1 Comparison of the variations of the terminal constraint $x_1=0$ and the energy consumption.

# UNIFIED APPROACH TO THE SENSITIVITY ANALYSIS OF OPTIMAL CONTROL SYSTEMS

by

Andrzej WIERZBICKI /Poland/[x]

## Introduction

Many papers concerning the sensitivity analysis of optimal control do not present an uniform approach to the problem. Papers of Dorato[6] and Pagurek[12], frequently quoted in literature, are based on an approach, which makes it possible to determine the sensitivity of an ideal optimal control problem, but not to distinguish between sensitivities of optimal control systems of different structures. Other contributions often overlook the fact that the solution of an optimal problem is not determined, unless the end-point conditions of the problem are strictly satisfied[14]. These shortcomints bring about a certain limitation in the number of papers which would present a comparison of sensitivity of optimal control systems of various structures[1,2].

The present paper tries to unify the approaches to the sensitivity analysis of optimal control systems. The analysis is confined to the $\alpha$ -parametric sensitivity[3] of optimal deterministic control. It was necessary to distinguish precisely between the sensitivity of an ideal optimal control problem and the sensitivity of an optimal control system. It was useful to introduce new notions of sensitivity measure and local insensitivity, as well as new definitions of local sensitivity coefficients and global sensitivity indexes. These notions enable us to compare effectively the sensitivity of optimal control systems of various structures; the paper is illustrated by results obtained in sensitivity analyses of several examples.

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Methods of numerical calculations, which are necessary in a general case, are discussed. Variational methods are presented, which help to determine the local sensitivity coefficients; these methods are based on the second variation of the performance functional.

## 1. Sensitivity measure

Let us consider the equation of a real process to be optimally controlled in the form

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, \underline{\alpha}, t) \qquad /1/$$

where $\underline{x}(t) \in R^n$ is the state, $\underline{u}(t) \in \underline{U} \subset R^r$ - the control, $\underline{\alpha} \in \underline{\Omega} \subset R^p$ - the parameters of the process. The performance functional, which should be minimal, can be expressed as

$$Q = f_k(\underline{x}(t_k), \underline{\alpha}) + \int_{t_0}^{t_k} f_0(\underline{x}, \underline{u}, \underline{\alpha}, t)\, dt \qquad /2/$$

The initial conditions $\underline{x}(t_0) = \underline{x}_0$ are given, and some additional conditions of various forms related to the final value $\underline{x}(t_k)$ can be taken into consideration. The functions $\underline{f}, f_k, f_0$ fulfil the usual conditions of continuity as well as differentiability in respect to $\underline{x}$ and - in cases, when it is desirable - in respect to $\underline{u}, \underline{\alpha}, t$.

Without any loss of generality it is possible to assume that the parameters $\underline{\alpha}$ are constant /if they are not, one can alter the form of the functions $\underline{f}, f_0$ as they depend on time $t$ /. The exact value of the vector $\underline{\alpha}$, however, is not known exactly. The optimal control is, therefore, determined by means of any known method using the models of the process and performance

$$\dot{\tilde{\underline{x}}} = \underline{f}(\tilde{\underline{x}}, \tilde{\underline{u}}, \underline{a}, t) \qquad /3/$$

$$\tilde{Q} = f_k(\tilde{\underline{x}}(t_k), \underline{a}) + \int_{t_0}^{t_k} f_0(\tilde{\underline{x}}, \tilde{\underline{u}}, \underline{a}, t)\, dt \qquad /4/$$

where $\underline{a} \in \underline{A} \subset R^p$ are the parameters of the model. It is, therefore, assumed that the functions $\underline{f}, f_k, f_0$ are known exactly; this assumption does not cause any loss of generality because the admissible class of these functions is broad.

The optimal control determined by the model and applied

to the real process depends on the chosen structure of control system. It can be expressed by the functions

$$\hat{\underline{u}}^o = \underline{r}^o(\underline{a},t); \quad \hat{\underline{u}}^1 = \underline{r}^1(\underline{x},\underline{a},t); \quad \hat{\underline{u}}^k = \underline{r}^k(\underline{x},\underline{\alpha},\underline{a},t) \qquad /5/$$

where the upper index "o" refers to the open-loop structure, the index "1" - to the closed-loop structure, the index "k" - to another possible structure of the control system.

The application of the control $\hat{\underline{u}}^o$ or $\hat{\underline{u}}^k$ to the real process results in the process trajectory $\hat{\underline{x}}^o$ or $\hat{\underline{x}}^k$. If the trajectory is defined and fulfils the assumed final conditions, and if the integral /2/ exists, then the performance functional $Q^o(\underline{\alpha},\underline{a})$ or $Q^k(\underline{\alpha},\underline{a})$ is defined as a function of the process parameters $\underline{\alpha}$ and the model parameters $\underline{a}$. As it is known, the performance functional depends on the initial values $\underline{x}_o$ and the final conditions as well, but this dependence will not be considered in this paper.

Even if the integral /2/ exists, the performance functional cannot be defined unless the assumed final conditions are not strictly fulfiled, because an optimal problem becomes a different one after changes in any of its conditions.

If the model parameters are exactly the same as the process parameters, $\underline{a} = \underline{\alpha}$, then the control is strictly optimal and independent of the control system structure. The ideal performance functional is a function of the parameters $\underline{\alpha}$ and can be expressed by

$$\hat{Q}(\underline{\alpha}) = Q^k(\underline{\alpha},\underline{a})\Big|_{\underline{a}=\underline{\alpha}} \qquad /6/$$

The dependence $\hat{Q}(\underline{\alpha})$ is a characteristic feature of the ideal control problem, and not of any control system's structure. There are many cases, when the function $\hat{Q}(\underline{\alpha})$ is defined although the function $Q^o(\underline{\alpha},\underline{a})$ or $Q^k(\underline{\alpha},\underline{a})$ is not defined for any $\underline{a} \neq \underline{\alpha}$. However, if the function $Q^k(\underline{\alpha},\underline{a})$ is defined, then - according to the definition of the optimal control - the following inequality holds

$$Q^k(\underline{\alpha},\underline{a}) \geqslant \hat{Q}(\underline{\alpha}) \qquad /7/$$

The sensitivity measure of an optimal control system

of a given structure is introduced here as the following non-negative function

$$s^k(\underline{\alpha},\underline{a}) = Q^k(\underline{\alpha},\underline{a}) - \hat{Q}(\underline{\alpha}) \qquad /8/$$

or

$$s^k(\underline{\alpha},\underline{a}) = \frac{Q^k(\underline{\alpha},\underline{a})}{\hat{Q}(\underline{\alpha})} - 1 \qquad /9/$$

The last function can be called the relative sensitivity measure. Sometimes it is desirable to consider the relative sensitivity measure in the form $s^k(\underline{\eta},\underline{a})$ where $\underline{\eta} = \left[\frac{a_i}{\alpha_i}\right]'$ is the vector of relative values of the process parameters.

The sensitivity measure is minimal and equal to zero if and only if the control $\hat{\underline{u}}^k$ is strictly optimal. It usually happens only for $\underline{a} = \underline{\alpha}$; there are special cases, however, when it happens for other values of $\underline{a}$. If the sensitivity measure is differentiable and $\underline{a} = \underline{\alpha}$, $\underline{A}$ and $\underline{\Omega}$ being open sets, then its gradients in respect to $\underline{a}$ and $\underline{\alpha}$ are zero-vectors

$$s^k(\underline{\alpha},\underline{a})\Big|_{\underline{a}=\underline{\alpha}} = 0\,; \quad \frac{\partial s^k(\underline{\alpha},\underline{a})}{\partial \underline{a}'}\Big|_{\underline{a}=\underline{\alpha}} = \underline{0}\,; \quad \frac{\partial s^k(\underline{\alpha},\underline{a})}{\partial \underline{\alpha}'}\Big|_{\underline{a}=\underline{\alpha}} = \underline{0} \qquad /10/$$

The sensitivity measure makes it possible to distinguish between control systems structures of different sensitivities, what cannot be obtained by means of the sensitivity coefficient, introduced in [6] and considered in [12], defined by

$$\underline{W}^k(\underline{\alpha}) = \frac{\partial Q^k(\underline{\alpha},\underline{a})}{\partial \underline{\alpha}'}\Big|_{\underline{a}=\underline{\alpha}} \qquad /11/$$

which can be determined by differentiating in respect to $\underline{\alpha}$ the Bellman's equation and then solving the obtained equations /which express the dependence of the coefficient on initial conditions $\underline{x}_0$ /. It is easy to prove - using the notion and properties of the sensitivity measure - that this coefficient is independent of the structure of the control system: assuming that the set $\underline{\Omega}$ is open and the functions $Q^k(\underline{\alpha},\underline{a})$, $\hat{Q}(\underline{\alpha})$ are differentiable, the following equation holds

$$\underline{W}^k(\underline{\alpha}) = \frac{\partial}{\partial \underline{\alpha}'}\left\{\hat{Q}(\underline{\alpha})\cdot\left[1 + s^k(\underline{\alpha},\underline{a})\right]\right\}\Big|_{\underline{a}=\underline{\alpha}} =$$

$$= \frac{\partial \hat{Q}(\underline{\alpha})}{\partial \underline{\alpha}'}\left[1 + s^k(\underline{\alpha},\underline{a})\right]\Big|_{\underline{a}=\underline{\alpha}} + \hat{Q}(\underline{\alpha})\frac{\partial s^k(\underline{\alpha},\underline{a})}{\partial \underline{\alpha}'}\Big|_{\underline{a}=\underline{\alpha}} = \frac{\partial \hat{Q}(\underline{\alpha})}{\partial \underline{\alpha}'} \qquad /12/$$

according to the equations /10/. Thus, the ideal sensitivity coefficient

$$\underline{W}(\underline{\alpha}) = \underline{W}^{k}(\underline{\alpha}) = \frac{\partial \hat{Q}(\underline{\alpha})}{\partial \underline{\alpha}^{t}} \qquad /13/$$

provides only an estimate of the ideal control problem sensitivity, and cannot be used in order to compare the sensitivity of different control system's structures. It can be use for designers of industrial plants /providing, they are optimally controlled/ and not for designers of optimal control systems[x/].

The properties of the functions $Q^{k}(\underline{\alpha},\underline{a})$, $\hat{Q}(\underline{\alpha})$, $\underline{W}(\underline{\alpha})$ are illustrated on Fig.1, which is based on a concrete example - cf.[16].

## 2. Local insensitivity and its conditions

An optimal control system of a given structure can be called locally insensitive, if the function $Q^{k}(\underline{\alpha},\underline{a})$ or the sensitivity measure $S^{k}(\underline{\alpha},\underline{a})$ is defined and a continuous function for all $\underline{a}$ in environment of $\underline{a}=\underline{\alpha}$.

An optimal control problem can be called locally insensitive, if the appropriate open-loop optimal control system is locally insensitive. Many of known optimal control problems are not locally insensitive, because the final conditions are not fulfilled for $\underline{a}\neq\underline{\alpha}$, and the function $Q^{o}(\underline{\alpha},\underline{a})$ is not defined - as it was stressed above. Such optimal control problems can be called fully sensitive.

Under the assumption, that the parameters $\underline{\alpha}$ can influence the functions $\underline{f}$ arbitrarily /and in particular - additively/, it is easy to prove[16] the following necessary condition of the local insensitivity of an optimal control problem:

An optimal control problem is locally insensitive only if the final conditions of the process to be controlled

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x/ There is an exception, however: the ideal sensitivity coefficient can be used in some considerations of optimal multilevel control - cf.11

form a $n$ -dimensional hypersurface in /n+1/ - dimensional space of vectors $[\underline{x}', t]'$. In other words, the function $Q^0(\underline{\alpha}, \underline{a})$ can be defined if only one coordinate or only one-dimensional condition of the final state $\underline{x}(t_k)$ and time $t_k$ is given. The condition of local insensitivity is also sufficient under additional assumptions.

Therefore, all optimal control problems of order greater then one and with given final values $\underline{x}(t_k) = t_k$ - and among them al most all time-optimal problems - are fully sensitive and must be considered as incorrectly set for the sensitivity analysis. It does not mean, however, that closed-loop or more complicated control system's structures for these problems are also fully sensitive[10,15]. An elementary example is given by the optimal control problem of the process $\dot{x} = \alpha u$ with the performance $Q = \int_0^1 u^2 dt$ and with the conditions $x(0) = 0$, $x(1) = 1$. The problem does not fulfil the condition of local insensitivity and is fully sensitive; $Q^0(\alpha, a) = \frac{1}{a^2}$ for $a = \alpha$ and is not defined for $a \neq \alpha$. The closed-loop system is locally insensitive, because $Q^1(\alpha, a) = \frac{1}{a(2\alpha - a)}$ and is defined for $a < 2\alpha$. The application of a special structure, called optimizing feedback[15,16] results in $Q^2(\alpha, a) = \frac{1}{\alpha^2}$ for all $a$; there exists, therefore, a structure in this simple example, which makes the control system fully insensitive. The sensitivity characteristics $S^k(\alpha, a)$ for this example are shown on Fig.2.

## 3. The local insensitivity degree and local sensitivity coefficients

The greatest order $m$ of the derivative $\frac{\partial^m S^k(\underline{\alpha}, \underline{a})}{\partial \alpha_i^m}$ or $\frac{\partial^m S^k(\underline{\alpha}, \underline{a})}{\partial a_i^m}$, by which the derivative is equal to zero for $\underline{a} = \underline{\alpha}$, can be called the local insensitivity degree of the optimal control system in respect to the parameter $\alpha_i$ or $a_i$. The value of the derivative $\frac{\partial^{m+1} S^k(\underline{\alpha}, \underline{a})}{\partial \alpha_i^{m+1}}$ or $\frac{\partial^{m+1} S^k(\underline{\alpha}, \underline{a})}{\partial a_i^{m+1}}$ for $\underline{a} = \underline{\alpha}$ can be called in such cases the local sensitivity coefficient of the system in respect to the chosen parameter,

$$V_{\alpha i}^{k(m+1)} = \left.\frac{\partial^{m+1} S^k(\underline{\alpha}, \underline{a})}{\partial \alpha_i^{m+1}}\right|_{\underline{a}=\underline{\alpha}}; \quad V_{ai}^{k(m+1)} = \left.\frac{\partial^{m+1} S^k(\underline{\alpha}, \underline{a})}{\partial a_i^{m+1}}\right|_{\underline{a}=\underline{\alpha}} \qquad /14/$$

The local insensitivity degree and the local sensitivity coefficients depend on the structure of the optimal control system.

Optimal control systems locally insensitive usually have the first degree of local insensitivity. The sensitivity coefficients are, therefore, usually the second derivatives of the sensitivity measure. However, if a parameter $\alpha_i$ or $a_i$ belongs to the boundary of the set $\underline{\Omega}$ or $\underline{A}$ of its admissible values, a control system can have the zero degree of local insensitivity.

Let us consider for example the well known problem of the time-optimal control of the process $\dot{x}_1 = x_2$, $\dot{x}_2 = \alpha u$, $|u| \leq 1$, $x_1(0) = X_1, x_1(T) = 0, x_2(0) = X_2, x_2(T) = 0$, with the performance $T = \int_0^T 1\,dt$. The condition of local insensitivity is not fulfiled in this problem, and the open-loop control system is fully sensitive. The closed-loop control system is locally insensitive, if the model parameter $a$ is not greater, than the process parameter, $a \leq \alpha$. If $a > \alpha$, then the closed-loop control system is fully sensitive in the sameway as the open-loop system. The set $A$ of admissible parameters $a$ is, therefore, constrained and $a = \alpha$ is the boundary of this set. After some computation one can get the sensitivity measure

$$S^1(\alpha,a) = \begin{cases} \dfrac{(\alpha-a)\sqrt{X_2^2 + 2\alpha X_1 \operatorname{sgn}(X_2|X_2| + 2aX_1)}}{\alpha\sqrt{a}\,(\sqrt{\alpha+a} + \sqrt{2a})}, & a \leq \alpha \\ \infty, & a > \alpha \end{cases} \qquad /15/$$

The insensitivity degree of the system by $a \leq \alpha$ is zero, because $S^1(\alpha,a)\big|_{a=\alpha} = 0$ but $\dfrac{\partial S^1(\alpha,a)}{\partial \alpha}\Big|_{a=\alpha} \neq 0$, $\dfrac{\partial S^1(\alpha,a)}{\partial \underline{a}}\Big|_{a=\alpha} \neq 0$. A sensitivity characteristics of the system is shown in Fig.3.

## 4. Variational approach to the local sensitivity

Let us assume that the functions $\underline{f}, f_0, f_k$ have continuous second order partial and mixed derivatives in respect to $\underline{x}, \underline{u}, \underline{\alpha}$ or $\underline{a}$ and that the optimal control $\hat{\underline{u}}$ as

well as the parameters $\underline{\alpha}, \underline{a}$ belong to the interiors of the sets $\underline{U}, \underline{R}, \underline{A}$ . Furthermore, that there exists a solution of the optimal control problem and that the problem is locally insensitive /for example, the final time $t_k$ only given, which under the above assumptions is sufficient to cause the local insensitivity/. Introducing a hamiltonian

$$H(\underline{\psi}, \underline{x}, \underline{u}, \underline{\alpha}, t) = -f_0(\underline{x}, \underline{u}, \underline{\alpha}, t) + \underline{\psi}' \underline{f}(\underline{x}, \underline{u}, \underline{\alpha}, t) \qquad /16/$$

where $\underline{\psi} \in R^n$ is the costate, we can write the necessary conditions of optimality of the process and the model in the form

$$\dot{\underline{x}} = \frac{\partial H}{\partial \underline{\psi}'}(\underline{\psi}, \underline{x}, \underline{u}, \underline{\alpha}, t); \quad \dot{\underline{\psi}} = -\frac{\partial H}{\partial \underline{x}'}(\underline{\psi}, \underline{x}, \underline{u}, \underline{\alpha}, t); \quad \underline{0} = \frac{\partial H}{\partial \underline{u}'}(\underline{\psi}, \underline{x}, \underline{u}, \underline{\alpha}, t) \quad /17a/$$

$$\dot{\tilde{\underline{x}}} = \frac{\partial H}{\partial \tilde{\underline{\psi}}'}(\tilde{\underline{\psi}}, \tilde{\underline{x}}, \tilde{\underline{u}}, \underline{a}, t); \quad \dot{\tilde{\underline{\psi}}} = -\frac{\partial H}{\partial \tilde{\underline{x}}'}(\tilde{\underline{\psi}}, \tilde{\underline{x}}, \tilde{\underline{u}}, \underline{a}, t); \quad \underline{0} = \frac{\partial H}{\partial \tilde{\underline{u}}'}(\tilde{\underline{\psi}}, \tilde{\underline{x}}, \tilde{\underline{u}}, \underline{a}, t) \quad /17b/$$

Let us assume that $\underline{a} = \underline{\alpha} + \varepsilon \Delta \underline{a}$ where $\varepsilon$ is a sufficiently small number. We have then $\tilde{\underline{x}} = \underline{x} + \varepsilon \delta \underline{x} + \underline{0}(\varepsilon), \tilde{\underline{\psi}} = \underline{\psi} + \varepsilon \delta \underline{\psi} + \underline{0}(\varepsilon)$, $\tilde{\underline{u}} = \underline{u} + \varepsilon \delta \underline{u} + \underline{0}(\varepsilon)$. The variations $\delta \underline{x}, \delta \underline{u}$ fulfil a linear system of differential equations which after some transformations can be written in the form

$$\delta \dot{\underline{x}} = \underline{A}_{11} \delta \underline{x} + \underline{A}_{12} \delta \underline{\psi} + \underline{B}_1 \Delta \underline{a}; \quad \delta \underline{x}(t_0) = \underline{0}$$

$$\delta \dot{\underline{\psi}} = \underline{A}_{21} \delta \underline{x} + \underline{A}_{22} \delta \underline{\psi} + \underline{B}_2 \Delta \underline{a}; \quad \delta \underline{\psi}(t_k) = -\frac{\partial^2 f_k}{\partial \underline{x}' \partial \underline{x}} \delta \underline{x}(t_k) \qquad /18/$$

where

$$\underline{A}_{11} = \frac{\partial^2 H}{\partial \underline{\psi}' \partial \underline{x}} - \frac{\partial^2 H}{\partial \underline{\psi}' \partial \underline{u}} \left[\frac{\partial^2 H}{\partial \underline{u}' \partial \underline{u}}\right]^{-1} \frac{\partial^2 H}{\partial \underline{u}' \partial \underline{x}}$$

$$\underline{A}_{12} = -\frac{\partial^2 H}{\partial \underline{\psi}' \partial \underline{u}} \left[\frac{\partial^2 H}{\partial \underline{u}' \partial \underline{u}}\right]^{-1} \frac{\partial^2 H}{\partial \underline{u}' \partial \underline{\psi}} \qquad /19/$$

$$\underline{A}_{21} = -\frac{\partial^2 H}{\partial \underline{x}' \partial \underline{x}} + \frac{\partial^2 H}{\partial \underline{x}' \partial \underline{u}} \left[\frac{\partial^2 H}{\partial \underline{u}' \partial \underline{u}}\right]^{-1} \frac{\partial^2 H}{\partial \underline{u}' \partial \underline{x}}$$

$$\underline{A}_{22} = -\frac{\partial^2 H}{\partial \underline{x}' \partial \underline{\psi}} + \frac{\partial^2 H}{\partial \underline{x}' \partial \underline{u}} \left[\frac{\partial^2 H}{\partial \underline{u}' \partial \underline{u}}\right]^{-1} \frac{\partial^2 H}{\partial \underline{u}' \partial \underline{\psi}}$$

$$\underline{B}_1 = \frac{\partial^2 H}{\partial \underline{\psi}' \partial \underline{a}} - \frac{\partial^2 H}{\partial \underline{\psi}' \partial \underline{u}} \left[\frac{\partial^2 H}{\partial \underline{u}' \partial \underline{u}}\right]^{-1} \frac{\partial^2 H}{\partial \underline{u}' \partial \underline{a}}$$

$$\underline{B}_2 = -\frac{\partial^2 H}{\partial \underline{x}' \partial \underline{a}} + \frac{\partial^2 H}{\partial \underline{x}' \partial \underline{u}} \left[\frac{\partial^2 H}{\partial \underline{u}' \partial \underline{u}}\right]^{-1} \frac{\partial^2 H}{\partial \underline{u}' \partial \underline{a}}$$

The matrix $\frac{\partial^2 H}{\partial \underline{u}' \partial \underline{u}}$ is not singular, if the solution of

the optimal problem exists and is unique. The variation $\delta\underline{u}$ has the form

$$\delta\underline{u} = -\left[\frac{\partial^2 H}{\partial\underline{u}'\partial\underline{u}}\right]^{-1}\left[\frac{\partial^2 H}{\partial\underline{u}'\partial\underline{\alpha}}\Delta\underline{\alpha} + \frac{\partial^2 H}{\partial\underline{u}'\partial\underline{x}}\delta\underline{x} + \frac{\partial^2 H}{\partial\underline{u}'\partial\underline{\psi}}\delta\underline{\psi}\right] \qquad /20/$$

The optimal control for the model $\tilde{\underline{u}}$ can be applied to the real process as the optimal control in the open-loop system. Hence $\hat{\underline{u}}^o - \hat{\underline{u}} = \tilde{\underline{u}} - \underline{u} = \varepsilon\,\delta\underline{u} + \underline{O}(\varepsilon)$ and $\hat{\underline{x}}^o - \hat{\underline{x}} = \varepsilon\delta\underline{x}^o + \underline{O}(\varepsilon)$ where $\delta\underline{x}^o$ differes from $\delta\underline{x}$ and is the solution of the equation

$$\delta\dot{\underline{x}}^o = \frac{\partial^2 H}{\partial\underline{\psi}'\partial\underline{x}}\delta\underline{x}^o + \frac{\partial^2 H}{\partial\underline{\psi}'\partial\underline{u}}\delta\underline{u}\,; \qquad \delta\underline{x}^o(t_o) = \underline{O} \qquad /21/$$

The sensitivity measure of the open-loop system can be expressed as

$$S^o(\underline{\alpha},\underline{a}) = Q^o(\underline{\alpha},\underline{a}) - \hat{Q}(\underline{\alpha}) =$$
$$= \frac{\varepsilon^2}{2}\left\{\delta\underline{x}^{o\prime}(t_k)\frac{\partial^2 f_k}{\partial\underline{x}'\partial\underline{x}}\delta\underline{x}^o(t_k) - \int_{t_o}^{t_k}\left[\delta\underline{x}^{o\prime}\frac{\partial^2 H}{\partial\underline{x}'\partial\underline{x}}\delta\underline{x}^o + 2\delta\underline{u}'\frac{\partial^2 H}{\partial\underline{u}'\partial\underline{x}}\delta\underline{x}^o + \delta\underline{u}'\frac{\partial^2 H}{\partial\underline{u}'\partial\underline{u}}\delta\underline{u}\right]dt\right\} + O(\varepsilon^2) = \qquad /22/$$
$$= \frac{\varepsilon^2}{2}\Delta\underline{a}'\,\underline{K}\,\Delta\underline{a} + O(\varepsilon^2)$$

The last equality holds because of the linear dependence between $\delta\underline{x}^o, \delta\underline{u}$ and $\Delta\underline{a}$. The matrix $\underline{K}$ can be, therefore, interpreted as the matrix of local sensitivity coefficients $\left.\frac{\partial^2 S^o(\underline{\alpha},\underline{a})}{\partial\underline{a}'\,\partial\underline{a}}\right|_{\underline{a}=\underline{\alpha}}$. The equation /22/ proves, that the open-loop system has at least the first degree of local insensitivity under the assumptions made above.

The application of variational methods to the sensitivity analysis qf closed-loop systems needs more complicated considerations, because $\hat{\underline{u}}^1 - \hat{\underline{u}} \neq \tilde{\underline{u}} - \underline{u}$. For further details see [17].

## 5. Global sensitivity indexes

In order to formulate a global sensitivity index of an optimal control system it is desirable to use probabilistic approach to the problem. It is assumed further, that the parameter $\underline{\alpha}$ is a random variable, although the process and its control are deterministic.

If the probability distribution $p(\underline{\alpha})$ is known, a global sensitivity index can be defined by

$$R_1^k = \min_{\underline{a}\in\underline{A}} \underset{\underline{\alpha}}{E} Q^k(\underline{\alpha},\underline{a}) - \underset{\underline{\alpha}}{E}\hat{Q}(\underline{\alpha}) = \underset{\underline{\alpha}}{E}\left[Q^k(\underline{\alpha},\hat{\underline{a}}) - \hat{Q}(\underline{\alpha})\right] \qquad /23/$$

or

$$r_1^k = \frac{R_1^k}{\underset{\underline{\alpha}}{E}\,\hat{Q}(\underline{\alpha})} \qquad /24/$$

where

$$\underset{\underline{\alpha}}{E}\,Q^k(\underline{\alpha},\underline{a}) = \int_{\underline{\Omega}} Q^k(\underline{\alpha},\underline{a})\,dp(\underline{\alpha}); \quad \underset{\underline{\alpha}}{E}\,\hat{Q}(\underline{\alpha}) = \int_{\underline{\Omega}} \hat{Q}(\underline{\alpha})\,dp(\underline{\alpha}) \qquad /25/$$

The definition of the global sensitivity index is connected with the determination of the best model's parameter $\hat{\underline{a}}$. The global sensitivity index is the difference between the minimum of expected value of performance functional, if a measurement of the realisation of the random variable $\underline{\alpha}$ is impossible, and the expected value of the minimum of performance functional, if the measurement is possible and the control system is strictly optimal for each realisation of $\underline{\alpha}$. The index $R_1^k$ expresses, therefore, expected losses, caused by impossibility of measuring the process parameters; it is a non-negative quantity.

The optimal parameter $\hat{\underline{a}}$ of the model can be determined as the solution of the equation

$$\int_{\underline{\Omega}} \frac{\partial Q^k(\underline{\alpha},\hat{\underline{a}})}{\partial \underline{a}}\,dp(\underline{\alpha}) = \underline{0} \quad \text{or} \quad \int_{\underline{\Omega}} \frac{\partial S^k(\underline{\alpha},\underline{a})}{\partial \underline{a}}\,dp(\underline{\alpha}) = \underline{0} \qquad /26/$$

Fig.4 illustrates an example of global sensitivity analysis and the choice of the optimal parameter $\hat{a}$ under an assumed form of the density $p(\alpha)$.

If the probability distribution $p(\underline{\alpha})$ is not known, but the set $\underline{\Omega}$ is given and bounded, another global sensitivity index can be defined as

$$R_2^k = \min_{\underline{a}\in\underline{A}} \max_{\underline{\alpha}\in\underline{\Omega}} Q^k(\underline{\alpha},\underline{a}) - \min_{\underline{a}\in\underline{A}} \min_{\underline{\alpha}\in\underline{\Omega}} Q^k(\underline{\alpha},\underline{a}) \qquad /27/$$

In most cases an equivalent notation of the index is

$$R_2^k = R_2 = \max_{\underline{\alpha}\in\underline{\Omega}} \hat{Q}(\underline{\alpha}) - \min_{\underline{\alpha}\in\underline{\Omega}} \hat{Q}(\underline{\alpha}) = \hat{Q}(\check{\underline{\alpha}}) - \hat{Q}(\hat{\underline{\alpha}}) \qquad /28/$$

The index $R_2$ does not depend, therefore, on the structure of the optimal control system and is of no use to carry out its synthesis. The value of the best model's parameter $\hat{\underline{a}}$ is here identical with the value of the worst real process es parameter $\check{\underline{\alpha}}$.

## 6. Numerical methods of the sensitivity analysis

A numerical computation of the sensitivity measure is in its essence equivalent to the numerical analysis of an

optimal control system - cf.[16]. The computational effort necessary to determine one value only of the sensitivity measure of an open-loop system, is comparable with the computational effort necessary to determine the open-loop optimal control $\hat{\underline{u}}^0$. In case of the analysis of closed-loop systems the computational effort grows up considerably, if the function $\hat{\underline{u}}^k = \underline{r}^k(\underline{x},\underline{a},t)$ is not known explicitly. A thorough sensitivity analysis needs, therefore, a very considerable effort. Nevertheless sensitivity computations are of much help if we want to design thoroughly an optimal control system for a real process[16].

The computational effort can be suppressed in case of a local sensitivity analysis when variational equations are used as /18, /20/, /21/, /22/ - although the programming effort grows up in this case. The determination of global sensitivity indexes needs a still greater computational effort. Because in most cases the function $Q^k(\underline{\alpha},\underline{a})$ is not known explicitly, the solution of the equation /26/ must be sought by means of the stochastic approximation.

## 7. Structures of optimal control systems and their sensitivity.

The sensitivity of optimal control systems of different structures has not been thoroughly analysed as yet. The author of this paper has analysed many examples [15,16]. It follows from those examples, that we should be careful with heuristic opinion about superiority of some structures /e.g. the closed-loop/ above others /e.g. the open-loop/. Some general conclusions on this matter can be formulated in the following way: if the cost of control dominates in the performance functional, then the open-loop system is of lower sensitivity than the closed-loop; if the cost of the deviations from the desirable state-trajector dominates in the performance functional, then low sensitivity ca be achieved by means of a closed-loop system or a special structu called optimizing feedback, introduced by the author in [15]. The estimation of the dominating component of the performance functional must be carried out with great care. For example, the performance functional in the elementary problem given at the end of section 2 depends on the control $u$ only. However, the closed-loop system /and especially the optimizing feedback/

is less sensitive than the open-loop system. This is due to the strong conditions which must be fulfilled at the end-point of the state trajectory. It can be interpreted by means of introducing a very large costs - fines. for the deviations of the state trajectory from the given end-point.

The sensitivity analysis of adaptive systems or multilevel systems is a more complicated problem. The sensitivity analysis of the two-level system given in Fig.5 makes an interesting example.

It is useful to decompose two controlled processes with common performance functional into two processes with independent performance functionals, introducing coordination variables $\pi_1, \pi_2$ /cf.Fig.5/. The decomposition makes it possible to supress the computational effort, necessary because of the determination of the optimal control in real time. Each partial process is controlled by means of optimizing feedback which is, essentially, a special extremal controller /cf. Fig.5/. The processes are many times repeated. The second level controller determines new coordination variables $\pi_1, \pi_2$ for each repetition on the basis of the last run of the processes, using a coordination algorithm of multilevel control[16]. The relative sensitivity measure of the whole system in the i-th run has the form

$$s^{(i)}(\eta,\alpha) = \frac{\alpha^2(1-\eta^2)^2}{(1+\alpha^2)^2\eta^2}\left[1-(1+\alpha^2\eta^2)\frac{1-\Theta^{(i)}}{1-\eta^2}\right]; \quad \eta = \frac{\alpha}{a} \qquad /28/$$

where only the parameter $\Theta^{(i)}$ depends on the number of runs and is determined by

$$\Theta^{(i)} = \frac{1+\sqrt{1+8\Theta^{(i-1)}}}{4}; \quad \Theta^{(1)} = \frac{1+\sqrt{1+8\eta}}{4} \qquad /29/$$

Because the transformation /29/ is a contraction, $\Theta^{(i)}$ converges to 1 and the sensitivity measure converges to

$$s^{(\infty)}(\eta,\alpha) = \frac{\alpha^2(1-\eta^2)^2}{(1+\alpha^2)^2\eta^2} \qquad /30/$$

The last expression is the same as the sensitivity measure of the system with only one control level. The sensitivity characteristics in successive runs are shown in Fig.6. The sensitivity of the system in the first run.is large; in next runs the sensitivity decreases rapidly. The essential conclusion from

this example is, that the multilevel structure of an optimal control system can have a quite comparable sensitivity with other structures.

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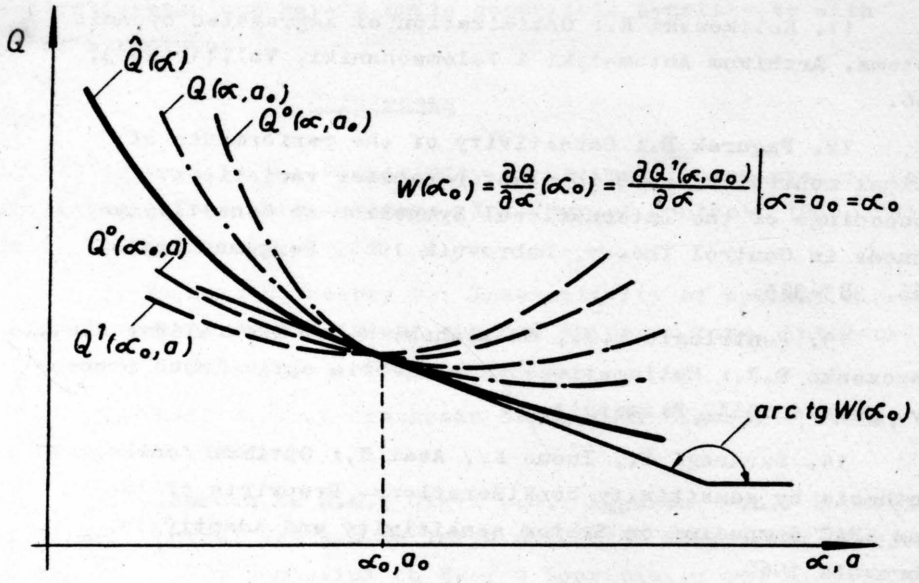


Fig. 1

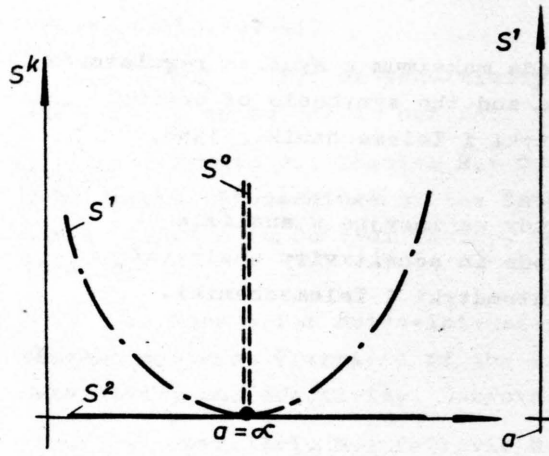


Fig. 2

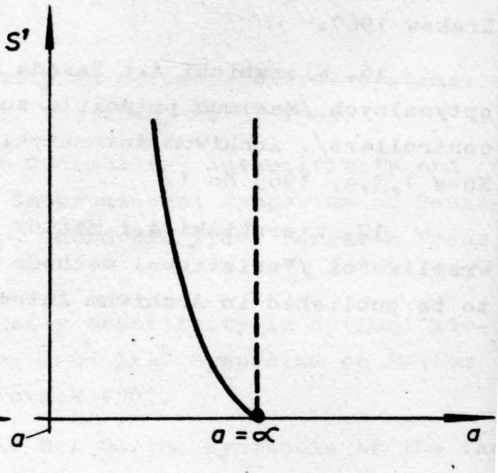


Fig. 3

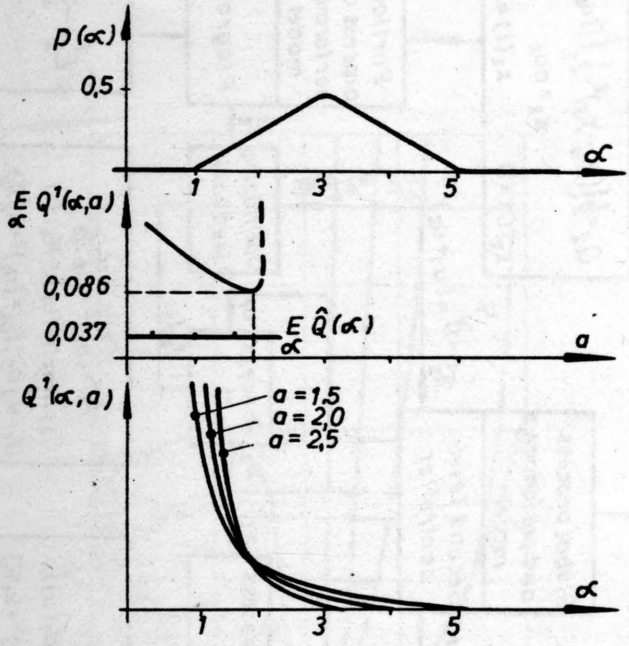


Fig. 4

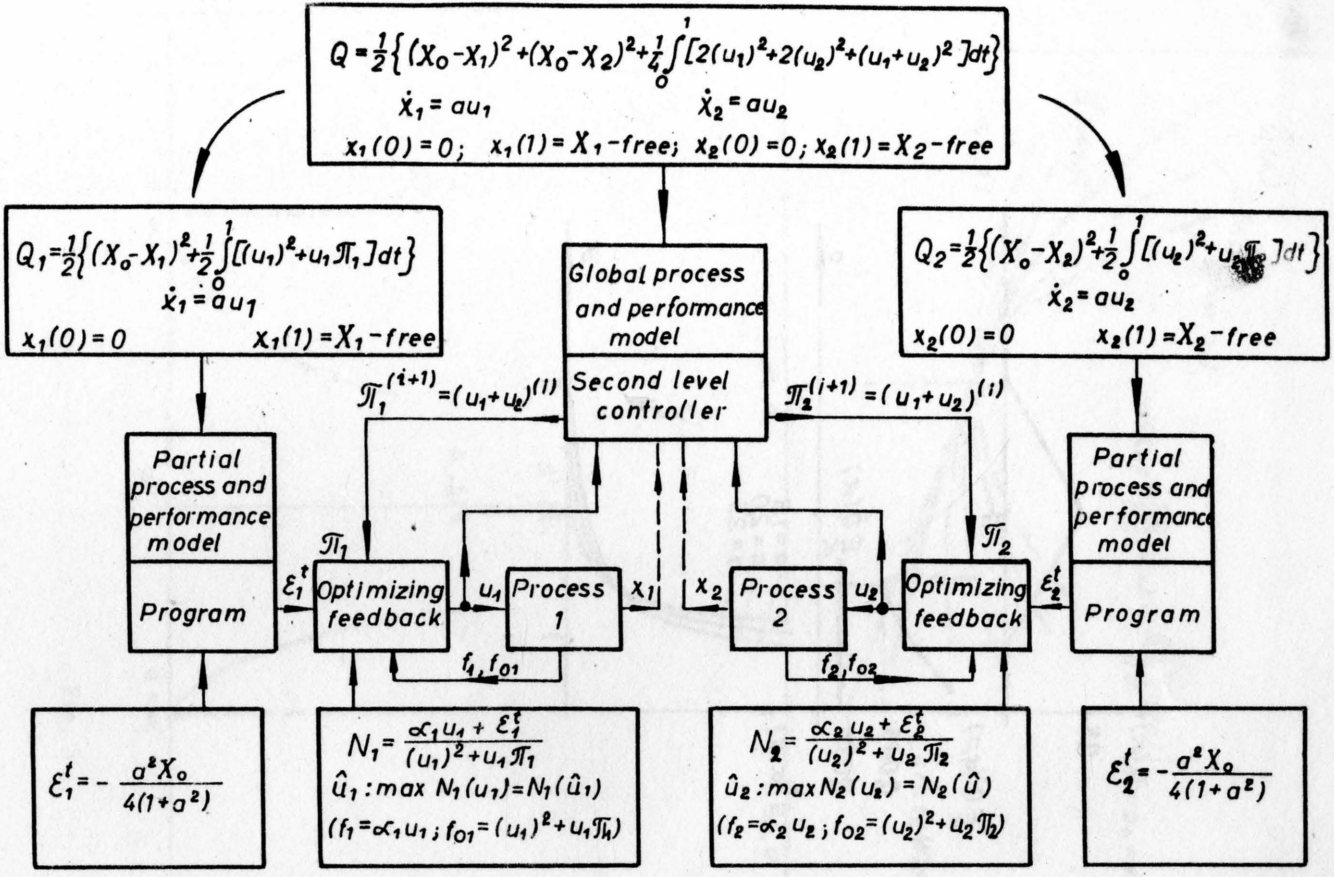
$Q = \frac{1}{2}\{(X_0 - X_1)^2 + (X_0 - X_2)^2 + \frac{1}{4}\int_0^1 [2(u_1)^2 + 2(u_2)^2 + (u_1 + u_2)^2]dt\}$
$\dot{X}_1 = au_1$ $\dot{X}_2 = au_2$
$x_1(0) = 0$; $x_1(1) = X_1$-free; $x_2(0) = 0$; $x_2(1) = X_2$-free
$Q_1 = \frac{1}{2}\{(X_0 - X_1)^2 + \frac{1}{2}\int_0^1 [(u_1)^2 + u_1\Pi_1]dt\}$
$\dot{X}_1 = au_1$
$x_1(0) = 0$ $x_1(1) = X_1$-free
$Q_2 = \frac{1}{2}\{(X_0 - X_2)^2 + \frac{1}{2}\int_0^1 [(u_2)^2 + u_2\Pi_2]dt\}$
$\dot{X}_2 = au_2$
$x_2(0) = 0$ $x_2(1) = X_2$-free
Global process and performance model
Second level controller
$\Pi_1^{(i+1)} = (u_1 + u_2)^{(i)}$
$\Pi_2^{(i+1)} = (u_1 + u_2)^{(i)}$
Partial process and performance model
Program
$\Pi_1$
$\varepsilon_1^t$
Optimizing feedback
$u_1$
Process 1
$x_1$
$x_2$
Process 2
$u_2$
$\Pi_2$
Optimizing feedback
$\varepsilon_2^t$
Partial process and performance model
Program
$f_1, f_{01}$
$f_2, f_{02}$
$\varepsilon_1^t = -\frac{a^2 X_0}{4(1+a^2)}$
$N_1 = \frac{\alpha_1 u_1 + \varepsilon_1^t}{(u_1)^2 + u_1\Pi_1}$
$\hat{u}_1 : \max N_1(u_1) = N_1(\hat{u}_1)$
$(f_1 = \alpha_1 u_1 ; f_{01} = (u_1)^2 + u_1\Pi_1)$
$N_2 = \frac{\alpha_2 u_2 + \varepsilon_2^t}{(u_2)^2 + u_2\Pi_2}$
$\hat{u}_2 : \max N_2(u_2) = N_2(\hat{u})$
$(f_2 = \alpha_2 u_2 ; f_{02} = (u_2)^2 + u_2\Pi_2)$
$\varepsilon_2^t = -\frac{a^2 X_0}{4(1+a^2)}$

Fig. 5

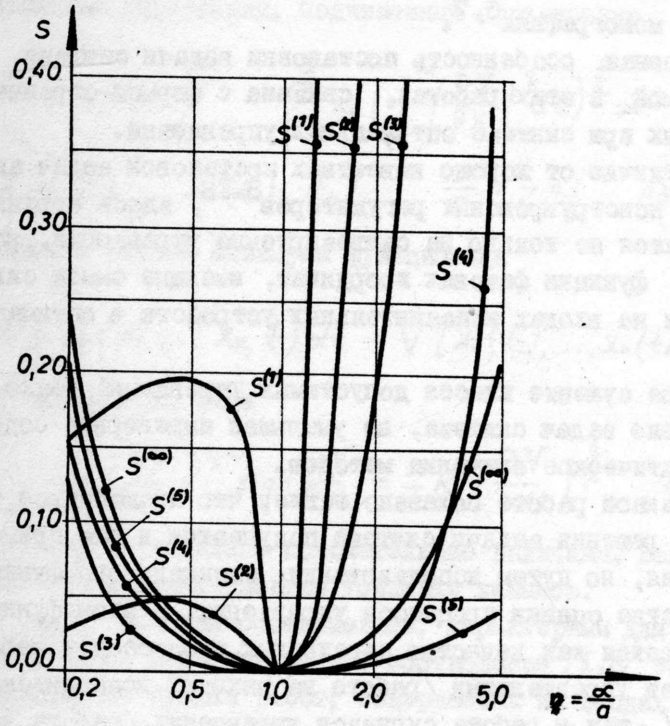
S
0,40
0,30
0,20
0,10
0,00
0,2
0,5
1,0
2,0
5,0
$\rho = \frac{\alpha c}{a}$
$S^{(1)}$
$S^{(2)}$
$S^{(3)}$
$S^{(4)}$
$S^{(5)}$
$S^{(\infty)}$

Fig. 6

# НОВОЕ РЕШЕНИЕ ЗАДАЧИ АНАЛИТИЧЕСКОГО КОНСТРУИРОВАНИЯ СИСТЕМ УПРАВЛЕНИЯ

А.А.Красовский / Москва /

Данная работа содержит развитие и обобщение аналитических методов синтеза систем регулирования, изложенных в статьях[1-5] и монографиях[6,7].

Основная особенность постановки задачи синтеза, рассматриваемой в этих работах, связана с формой ограничений, учитываемых при синтезе оптимальных управлений.

В отличие от хорошо известных постановок задач аналитического конструирования регуляторов[8-13], здесь ограничения накладываются не только на синтезируемые управления, но и на некоторые функции фазовых координат, имеющие смысл сигналов управления на входах исполнительных устройств в оптимальной системе.

Такое сужение класса допустимых управлений резко упрощает решение задач синтеза, не уменьшая инженерное содержание и практическое значение методов.

В данной работе показано также, что аналогичное резкое упрощение решения задачи синтеза получается и без применения ограничения, но путем использования специального функционала в качестве оценки процессов управления. В этом функционале учитывается как качество переходных процессов и работа, совершаемая управлениями /работа на выходах исполнительных устройств/, так и работа сигналов управления /работа на входах исполнительных устройств/ в оптимальной системе.

## § I. Синтез оптимальных управлений нелинейного об"екта при полной степени наблюдаемости[x]

Если дан об"ект

$$\dot{x}_i + F_i(x_1, x_2, \dots, x_n, t) = u_i \qquad /I.I/$$
$$(i = 1, 2, \dots, n)$$

[x] Под полной степенью наблюдаемости понимается возможность измерения и использования для управления всех фазовых координат.

и существует функция $V(x_1, x_2, \dots, x_n, t)$, полная производная которой $\dot{V}$ в силу уравнений неуправляемого об"екта /$u_i = 0$/ равна заданной функции - $Q(x_1, x_2, \dots, x_n, t)$, т.е.

$$\dot{V} = \frac{\partial V}{\partial t} - \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} F_i = -Q \,, \qquad /I.2/$$

то на множестве управлений, подчиненных ограничению

$$\sum_{i=1}^{n} \frac{1}{\kappa_i} \left( \int_{t_1}^{t_2} |u_i|^{\rho} dt \right)^{\frac{2}{\rho}} + \sum_{i=1}^{n} \kappa_i \left( \int_{t_1}^{t_2} \left| \frac{\partial V}{\partial x_i} \right|^{q} dt \right)^{\frac{2}{q}} = \qquad /I.3/$$

$$= C\left[ x_1(t_1), \dots, x_n(t_1), t_1 \right], \quad \frac{1}{\rho} + \frac{1}{q} = 1, \quad \rho \geq 1.$$

оптимальными в смысле минимума функционала

$$I = \int_{t_1}^{t_2} Q(x_1, \dots, x_n, t)\, dt + V\left[ x_1(t_2), \dots, x_n(t_2), t_2 \right]$$

являются управления вида

$$u_i = -\kappa_i \left| \frac{\partial V}{\partial x_i} \right|^{q-1} sign \frac{\partial V}{\partial x_i} = \mp \kappa_i \left( \frac{\partial V}{\partial x_i} \right)^{\frac{q}{\rho}} \qquad /I.4/$$

Здесь $\kappa_i, \rho, q$ - заданные положительные величины. Величины $\kappa_i$ имеют смысл коэффициентов усиления каналов. Величины $\rho, q$ связаны соотношением, характерным для неравенства Гельдера. При $\rho = q = 2$ соотношение /I.3/ соответствует взвешенным суммам работ, совершаемых на входах и выходах исполнительных устройств за время $t_2 - t_1$ :

$$\sum_{i=1}^{n} \frac{1}{\kappa_i} \int_{t_1}^{t_2} u_i^2\, dt + \sum_{i=1}^{n} \kappa_i \int_{t_1}^{t_2} \left( \frac{\partial V}{\partial x_i} \right)^2 dt = C \,,$$

а оптимальные управления пропорциональны компонентам градиента функции $V$ :

$$u_i = -\kappa_i \frac{\partial V}{\partial x_i}$$

Значения $q = 1$, $\rho = \infty$ соответствует заданию взвешенной суммы максимальных значений квадратов управлений и "расхо-

дов" сигналов управлений /величин $\int_{t_1}^{t_2} \left|\frac{\partial V}{\partial x_i}\right| dt$ /.
Оптимальные управления в этом случае являются релейными

$$u_i = -K_i \operatorname{sign} \frac{\partial V}{\partial x_i}$$

При $\rho = 1$, $q = \infty$ ограничивающее условие представляет собой задание взвешенной суммы квадратов "расходов" управлений $\int_{t_1}^{t_2} |u_i| dt$ и квадратов максимальных значений сигналов управлений. Оптимальные управления при этом имеют порог срабатывания / $u_i = 0$ при $\left|\frac{\partial V}{\partial x_i}\right| < 1$ /.

Достаточно простое доказательство приведенной теоремы заключается в следующем. Производная функции $V$ в силу уравнений управляемого / $u_i \neq 0$ / об"екта равна

$$\dot{V} = \frac{\partial V}{\partial t} - \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} F_i + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} u_i = -Q + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} u_i$$

Интегрируя по интервалу $t_2 - t_1$, получаем

$$V[x_1(t_2), \dots, x_n(t_2), t_2] - V[x_1(t_1), \dots, x_n(t_1), t_1] = -\int_{t_1}^{t_2} Q\, dt - \sum_{i=1}^{n} \int_{t_1}^{t_2} \left(\frac{\partial V}{\partial x_i}\right)(-u_i)\, dt$$

и

$$I = V[x_1(t_1), \dots, x_n(t_1), t_1] - \sum_{i=1}^{n} \int_{t_1}^{t_2} \left(\frac{\partial V}{\partial x_i}\right)(-u_i)\, dt \qquad /I.5/$$

Минимуму $I$ соответствует максимум величины

$$\sum_{i=1}^{n} \int_{t_1}^{t_2} \left(\frac{\partial V}{\partial x_i}\right)(-u_i)\, dt \leq \sum_{i=1}^{n} \int_{t_1}^{t_2} \left|\frac{\partial V}{\partial x_i}\right| |u_i|\, dt$$

Согласно неравенству Гельфера

$$\int_{t_1}^{t_2} \left|\frac{\partial V}{\partial x_i}\right| |u_i|\, dt \leq \left(\int_{t_1}^{t_2} \left|\frac{\partial V}{\partial x_i}\right|^q dt\right)^{\frac{1}{q}} \cdot \left(\int_{t_1}^{t_2} |u_i|^p dt\right)^{\frac{1}{p}} \qquad /I.6/$$

Поэтому

$$\sum_{i=1}^{n} \int_{t_1}^{t_2} \left|\frac{\partial V}{\partial x_i}\right| |u_i| dt \le \sum_{i=1}^{n} \gamma_i \zeta_i ,$$

где

$$\gamma_i = \left(\int_{t_1}^{t_2} \left|\frac{\partial V}{\partial x_i}\right|^q dt\right)^{\frac{1}{q}}, \quad \zeta_i = \left(\int_{t_1}^{t_2} |u_i|^p dt\right)^{\frac{1}{p}}$$

Знак равенства здесь имеет место тогда и только тогда, когда $|u_i|^p$ пропорционально $\left|\frac{\partial V}{\partial x_i}\right|^q$. Ограничение /I.3/ при данных обозначениях имеет вид

$$\sum_{i=1}^{n} \frac{\zeta_i^2}{\kappa_i} + \kappa_i \gamma_i^2 = c \qquad /I.7/$$

Максимум величины $\sum_{i=1}^{n} \gamma_i \zeta_i$ при ограничении /I.7/ имеет место при $\zeta_i = \kappa_i \gamma_i$. В этом можно убедиться, решая элементарную задачу на экстремум функции.

Соотношения $|u_i|^p \equiv \left|\frac{\partial V}{\partial x_i}\right|^q$ и $\zeta_i = \kappa_i \gamma_i$, т.е.

$$\left(\int_{t_1}^{t_2} |u_i|^p dt\right)^{\frac{1}{p}} = \kappa_i \left(\int_{t_1}^{t_2} \left|\frac{\partial V}{\partial x_i}\right|^q dt\right)^{\frac{1}{q}}$$

удовлетворяются одновременно при

$$u_i = -\kappa_i \left|\frac{\partial V}{\partial x_i}\right|^{\frac{q}{p}} \operatorname{sign} \frac{\partial V}{\partial x_i} = -\kappa_i \left|\frac{\partial V}{\partial x_i}\right|^{q-1} \operatorname{sign} \frac{\partial V}{\partial x_i}$$

Эти управления придают максимальное возможное значение правой части соотношения /I.6/ и одновременно обеспечивают знак равенства в этом соотношении.

Таким образом управления /I.4/ минимизируют функционал $I$ при ограничении /I.3/.

При $p = q = 2$ оптимальность указанных управлений легко доказывается также с помощью функционального уравнения Беллмана, т.е. на основе обычной процедуры синтеза оптимальных управлений.

Указанные управления являются оптимальными и для того случая, когда ограничивающее условие отсутствует, но исполь-

вуется специальный функционал, учитывающий работу сигналов управления в оптимальной системе. Покажем это, а именно, докажем, что для об"екта /II/ оптимальными в смысле минимума функционала

$$I=\int_{t_1}^{t_2} Q\,dt+\frac{1}{4}\int_{t_1}^{t_2}\sum_{i=1}^{n}\kappa_i\left(\frac{\partial V}{\partial x_i}\right)^2 dt+\int_{t_1}^{t_2}\sum_{i=1}^{n}\frac{u_i^2}{\kappa_i}\,dt+V\left[x_1(t_2),\dots,x_n(t_2),t_2\right] \qquad /I.8/$$

управлениями являются управления вида

$$u_i=-\frac{1}{2}\kappa_i\frac{\partial V}{\partial x_i} \qquad /I.9/$$

где $V$ попрежнему определяется линейным уравнением в частных производных /I.2/. Функционал /I.8/ служит оценкой качества переходных процессов /член $\int_{t_1}^{t_2} Q\,dt$ /, работ управлений /член $\sum_{i=1}^{n}\frac{1}{\kappa_i}\int_{t_1}^{t_2}u_i^2\,dt$ / и сигналов управления в оптимальной системе /член $\sum_{i=1}^{n}\kappa_i\int_{t_1}^{t_2}\left(\frac{\partial V}{\partial x_i}\right)^2 dt$ /, а также текущей величинах отклонений /член $V\left[x_1(t_2),\dots,x_n(t_2),t_2\right]$ /.

Представим $u_i$ в виде

$$u_i=-\frac{1}{2}\kappa_i\frac{\partial V}{\partial x_i}+\delta u_i\ ,$$

где $\delta u_i$ - произвольные вариации управлений.

Полная производная $\dot V$ в силу уравнений управляемого об"екта будет равна

$$\dot V=\frac{\partial V}{\partial t}+\sum_{i=1}^{n}\frac{\partial V}{\partial x_i}\left(-F_i-\frac{1}{2}\kappa_i\frac{\partial V}{\partial x_i}+\delta u_i\right)=-Q-\frac{1}{4}\sum_{i=1}^{n}\kappa_i\left(\frac{\partial V}{\partial x_i}\right)^2+$$

$$+\sum_{i=1}^{n}\frac{\partial V}{\partial x_i}\left(-\frac{1}{4}\kappa_i\frac{\partial V}{\partial x_i}+\delta u_i\right)=-Q-\frac{1}{4}\sum_{i=1}^{n}\kappa_i\left(\frac{\partial V}{\partial x_i}\right)^2+$$

$$+\frac{1}{2}\sum_{i=1}^{n}\frac{\partial V}{\partial x_i}\left(u_i+\delta u_i\right)=-Q-\frac{1}{4}\sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i}\right)^2-\sum_{i=1}^{n}\frac{u_i^2}{\kappa_i}+\sum_{i=1}^{n}\frac{\delta u_i^2}{\kappa_i}\ .$$

Интегрируя по интервалу времени $t_2-t_1$ и учитывая выражение

/I.8/ для функционала $I$ , находим

$$I = V\left[x_1(t_1), \dots, x_n(t_1), t_1\right] + \sum_{i=1}^{n} \frac{1}{\kappa_i} \int_{t_1}^{t_2} \delta u_i^2 \, dt$$

Отсюда непосредственно видно, что минимум $I$ получается при $\delta u_i = 0$, так что управления /I.9/ действительно являются оптимальными. Данное решение единственно.

Если $V$ и $Q$ положительно определенные функции, в этом случае $V$ - функция Ляпунова неуправляемого об"екта, то оптимальные управления /I.4/, /I.9/ обеспечивают устойчивость невозмущенного состояния системы / $x_1 = \dots = x_n = 0$ / при сколь угодно высоких положительных коэффициентах усиления каналов $\kappa_i$ . Это вытекает из соотношения

$$\dot{V} = -Q + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} u_i = -Q - \sum_{i=1}^{n} \kappa_i \left| \frac{\partial V}{\partial x_i} \right|^2 .$$

Для получения оптимальных управлений в явной форме необходимо определить функцию $V$ , удовлетворяющую уравнению /I.2/. Все преимущества перед обычным решением задачи аналитического конструирования обусловлены тем, что уравнение /I.2/ линейно, а в известном решении подобное уравнение нелинейно /содержит член $\frac{1}{4} \sum_{i=1}^{n} \kappa_i \left( \frac{\partial V}{\partial x_i} \right)^2$ /.

Необходимо найти решение /I.2/, не зависящее от граничных условий. Это решение мы будем называть вынужденным. То, что рекомендуется искать именно вынужденное решение, обусловлено следующим. Во-первых, обычно отсутствуют факторы, позволяющие обосновать граничные условия для $V$ . Во-вторых, обычно требуется обеспечить оптимальность переходных процессов вне зависимости от момента их возбуждения, т.е. решение $V$ не должно зависеть от начального значения времени. Этому в общем случае удовлетворяет только "вынужденное" решение.

Для случая линейного об"екта и квадратичного функционала / $Q$ - заданная квадратичная форма/ решение ищется в виде квадратичной формы фазовых координат.

Для коэффициентов этой квадратичной формы получается $\frac{1}{2} n (n+1)$ линейных дифференциальных уравнений[4,7].

Для случая пассивного об"екта как линейного, так и нелинейного в качестве функции $V$ можно использовать полную энергию об"екта $5$, удовлетворяющую в силу закона сохранения энергии уравнению /I.2/ / $Q$ в этом случае диссипативная функция/.

С целью определения оптимальных управлений в более общем случае произвольного нелинейного об"екта вида /I.I/ с аналитическими функциями $F_i$ представим эти функции и заданную функцию $Q$ в виде степенных рядов

$$F_i = \sum_j a_{ij} x_j + \sum_{j,k} b_{ijk} x_j x_k + \sum_{j,k,e} c_{ijke} x_j x_k x_e + \dots \qquad /I.10/$$

$$Q = \sum_{i,j} \beta_{ij} x_i x_j + \sum_{i,j,k} \gamma_{ijk} x_i x_j x_k + \sum_{i,j,k,e} \delta_{ijke} x_i x_j x_k x_e + \dots$$

Здесь коэффициенты об"екта $a_{ij}$, $b_{ijk}, \dots$ и функционала $\beta_{ij}, \gamma_{ijk}, \dots$ являются в общем случае функциями времени.

Суммирование по всем индексам ведется от I до $n$. Коэффициенты не зависят от порядка расположения индексов /для коэффициентов $a_{ij}, b_{ijk}, \dots$ это касается индексов, начиная со второго/.

Функцию $V$ будем искать также в виде степенного ряда:

$$V = \sum_{i,j} A_{ij} x_i x_j + \sum_{i,j,k} B_{ijk} x_i x_j x_k + \sum_{i,j,k,e} C_{ijke} x_i x_j x_k x_e + \dots \qquad /I.11/$$

где коэффициенты $A_{ij}$, $B_{ijk}$, $C_{ijke}, \dots$ /коэффициенты искомых оптимальных управлений/ в общем случае также являются функциями времени.

Подставляя выражения для $\frac{\partial V}{\partial x_i}$, $F_i$, $Q$ в /I.2/ и приравнивая коэффициенты при одинаковых произведениях координат, находим

$$\dot{A}_{ij} - \sum_{p=1}^{n} \left( a_{pi} A_{pj} + a_{pj} A_{pi} \right) = -\beta_{ij}$$

$$(i,j = 1, 2, \dots, n)$$

$$\dot{B}_{ijk} - \sum_{p=1}^{n} (\alpha_{pi} B_{pjk} + \alpha_{pj} B_{pik} + \alpha_{pk} B_{pij}) =$$

$$= -\gamma_{ijk} + \sum_{p=1}^{n} (b_{pjk} A_{pi} + b_{pik} A_{pj} + b_{pij} A_{pk})$$

$$(i,j,k = 1,2,\dots,n)$$

$$\dot{C}_{ijkl} - \sum_{p=1}^{n} (\alpha_{pi} C_{pjkl} + \alpha_{pj} C_{pikl} + \alpha_{pk} C_{pijl} + \alpha_{pl} C_{pijk}) = \quad /I.12/$$

$$= -\delta_{ijkl} + \sum_{p=1}^{n} (b_{pij} B_{pkl} + b_{pik} B_{pjl} + b_{pil} B_{pjk} +$$

$$+ b_{pjk} B_{pil} + b_{pjl} B_{pik} + b_{pkl} B_{pij}) +$$

$$+ \sum_{p=1}^{n} (c_{pjkl} A_{pi} + c_{pikl} A_{pj} + c_{pijl} A_{pk} + c_{pijk} A_{pl})$$

$$(i,j,k,l = 1,2,\dots,n)$$

. . . . . .

Для стационарного нелинейного об"екта и стационарного функционала / $F_i$ , $Q$ не зависит явно от времени/ коэффициенты $A_{ij}, B_{ijk}, C_{ijkl}, \dots$ постоянны и линейные дифференциальные уравнения /I.12/ обращаются в алгебраические линейные уравнения:

$$\sum_{p=1}^{n} (\alpha_{pi} A_{pj} + \alpha_{pj} A_{pi}) = \beta_{ij}$$

$$\sum_{p=1}^{n} (\alpha_{pi} B_{pjk} + \alpha_{pj} B_{pik} + \alpha_{pk} B_{pij}) = \gamma_{ijk} - \sum_{p=1}^{n} (b_{pjk} A_{pi} + b_{pik} A_{pj} + b_{pij} A_{pk}) \quad /I.13/$$

. . . . . .

С учетом того, что коэффициенты $A_{ij}, B_{ijk}, C_{ijkl}, \dots$ не зависят от порядка расстановки индексов, первая группа уравнений /I.12/ и /I.13/ содержит $\frac{n(n+1)}{2}$ неизвестных $A_{ij}$ и уравнений, вторая группа $\frac{n(n+1)(n+2)}{3!}$ неизвестных $B_{ijk}$ и уравнений и т.д.

Первая группа уравнений /I.12/ в матричной форме имеет вид

$$\dot{A} = A\alpha + \alpha^* A - \beta \,, \qquad /I.I4/$$

где $A = \|A_{ij}\|$, $\alpha = \|\alpha_{ij}\|$, $\beta = \|\beta_{ij}\|$ – квадратные матрицы, $\alpha^*$ – транспонированная матрица.

Для стационарного случая соответственно

$$A\alpha + \alpha^* A = \beta \qquad /I.I5/$$

Уравнения /I.I4/, /I.I5/ совпадают с уже известными[I-7] уравнениями для коэффициентов оптимальных управлений линейного об"екта.

Назвая члены $\sum_{j=1}^{n} A_{ij} x_j$ в аргументах оптимальных управлений:

$$\frac{\partial V}{\partial x_i} = 2\sum_{j=1}^{n} A_{ij} x_j + 3\sum_{j,k=1}^{n} B_{ijk} x_j x_k + 4\sum_{j,k,\ell=1}^{n} C_{ijk\ell} x_j x_k x_\ell + \cdots \qquad /I.I6/$$

линейными, формулируем следующий результат.

Линейные члены аргументов оптимальных управлений нелинейного об"екта равны аргументам оптимальных управлений линейного об"екта, совпадающего с первым приближением рассматриваемого нелинейного об"екта.

Из структуры системы алгебраических уравнений /I.I3/ видно, что она может решаться последовательно по группам. Сначала решается первая группа уравнений и определяются коэффициенты $A_{ij}$ линейных членов аргументов оптимальных управлений, затем решается вторая группа уравнений и определяются коэффициенты $B_{ijk}$ квадратичных членов, далее находятся коэффициенты $C_{ijk\ell}$ кубических членов и т.д.

Для случая нестационарного об"екта или изменяющихся во времени коэффициентов заданного функционала $\beta_{ij}, \gamma_{ijk}, \ldots$ необходимо искать решение системы линейных дифференциальных уравнений /I.I2/. При этом следует определять частное решение неоднородных линейных уравнений / "вынужденное" решение/. Кроме приведенных выше соображений это следует, в частности, из того, что при стремлении к нулю скоростей изменения коэффициентов об"екта $a_{ij}, b_{ijk}, \ldots$ и коэффициентов $\beta_{ij}, \gamma_{ijk}, \ldots$ функционала решение должно стремиться к решению алгебраических уравнений /I.I3/.

Решение дифференциальных уравнений /I.I2/, так же как и алгебраических уравнений /I.I3/, может определяться последо-

вательно: сначала решаются уравнения первой группы, затем второй группы и т.д. При достаточно медленном изменении коэффициентов об"екта и функционала удобным способом определения вынужденного решения каждой группы уравнений является метод итераций, в котором в качестве нулевого приближения используются решения соответствующих алгебраических уравнений /I.I3/. Несмотря на полную принципиальную ясность данной процедуры определения оптимальных управлений, практическое определение коэффициентов может вызвать некоторые трудности при высоком порядке об"екта в связи с громоздкостью систем уравнений /I.I2/, /I.I3/. Поэтому целесообразно рассмотреть выражения коэффициентов оптимальных управлений через интегральные оценки весовых функций уравнений первого приближения. Для задачи синтеза оптимальных управлений линейного устойчивого стационарного об"екта подобные выражения получены в работах[I,7]. Они существенно упрощают всю процедуру определения оптимальных управлений.

## § 2. Выражения коэффициентов оптимальных управлений через интегральные оценки весовых функций линейной модели об"екта

Допустим, что невозмущенное состояние неуправляемого об"екта / $u_1 = u_2 = \dots = u_n = 0$ / асимптотически устойчиво по Ляпунову, об"ект и функционал стационарны.

Если имеет место неустойчивость невозмущенного состояния, то заменой фазовых координат, например такой $x_i^* = \exp(-\lambda t)\, x_i$, где $\lambda$ - характеристичное число об"екта, всегда можно привести задачу к случаю устойчивого по Ляпунову об"екта. При этом, разумеется, преобразуются ограничивающие условия.

Весовые функции, соответствующие уравнениям первого приближения, т.е. весовые функции линейной модели об"екта, подчинены уравнениям

$$\dot{W}_i^q + \sum_{p=1}^{n} a_{ip} W_p^q = 0, \quad W_i^q(0) = x_{iq}, \qquad /2.1/$$
$$(i, q = 1, 2, \dots, n)$$

Здесь $W_i^q$ - весовая функция, соответствующая реакции $i$ -го выхода линейной модели на $\delta$ - импульс, поданный на $q$ -й

вход /или на начальное единичное отклонение по $q$ -му выходу/; $\varkappa_{iq}$ - символ Кронекеря:

$$\varkappa_{iq} = \begin{cases} 1 \text{ при } i = q \\ 0 \text{ при } i \neq q \end{cases}$$

Из тождеств

$$\frac{d}{dt}\left(W_i^q W_j^z\right) = \dot{W}_i^q W_j^z + W_i^q \dot{W}_j^z$$

$$\frac{d}{dt}\left(W_i^q W_j^z W_\kappa^s\right) = \dot{W}_i^q W_j^z W_\kappa^s + W_i^q \dot{W}_j^z W_\kappa^s + W_i^q W_j^z \dot{W}_\kappa^s$$

$$\frac{d}{dt}\left(W_i^q W_j^z W_\kappa^s W_\ell^t\right) = \dot{W}_i^q W_j^z W_\kappa^s W_\ell^t + W_i^q \dot{W}_j^z W_\kappa^s W_\ell^t + W_i^q W_j^z \dot{W}_\kappa^s W_\ell^t + W_i^q W_j^z W_\kappa^s \dot{W}_\ell^t.$$

. . . . . .

и уравнений /2.1/ следует

$$\frac{d}{dt}\left(W_i^q W_j^z\right) + \sum_{\rho=1}^{n}\left(a_{i\rho} W_\rho^q W_j^z + a_{j\rho} W_i^q W_\rho^z\right) = 0$$

$$\frac{d}{dt}\left(W_i^q W_j^z W_\kappa^s\right) + \sum_{\rho=1}^{n}\left(a_{i\rho} W_\rho^q W_j^z W_\kappa^s + a_{j\rho} W_\rho^z W_i^q W_\kappa^s + a_{\kappa\rho} W_\rho^s W_i^q W_j^z\right) = 0$$

$$\frac{d}{dt}\left(W_i^q W_j^z W_\kappa^s W_\ell^t\right) + \sum_{\rho=1}^{n}\left(a_{i\rho} W_\rho^q W_j^z W_\kappa^s W_\ell^t + a_{j\rho} W_\rho^z W_i^q W_\kappa^s W_\ell^t + \right. \quad /2.2/$$

$$\left. + a_{\kappa\rho} W_\rho^s W_i^q W_j^z W_\ell^t + a_{\ell\rho} W_\rho^t W_i^q W_j^z W_\kappa^s\right) = 0.$$

. . . . . .

В силу условия устойчивости все весовые функции обращаются в нуль при $t = \infty$. Интегрируя /2.2/ в пределах от 0 до $\infty$ и вводя следующие обозначения для интегральных оценок весовых функций

$$J_{ij}^{qz} = \int_0^\infty W_i^q W_j^z \, dt \; ; \quad J_{ij\kappa}^{qzs} = \int_0^\infty W_i^q W_j^z W_\kappa^s \, dt \; ;$$

$$J_{ij\kappa\ell}^{qzsf} = \int_0^\infty W_i^q W_j^z W_\kappa^s W_\ell^t \, dt \; ; \; \ldots\ldots\ldots$$

получаем

$$\sum_{p=1}^{n}\left(a_{ip}J_{pj}^{qz}+a_{jp}J_{ip}^{qz}\right)=\varkappa_{iq}\varkappa_{jz}$$

$$\sum_{p=1}^{n}\left(a_{ip}J_{pjk}^{qzs}+a_{jp}J_{pik}^{zqs}+a_{kp}J_{pij}^{sqz}\right)=\varkappa_{iq}\varkappa_{jz}\varkappa_{ks} \qquad /2.3/$$

$$\sum_{p=1}^{n}\left(a_{ip}J_{pjke}^{qzsf}+a_{jp}J_{pike}^{zqsf}+a_{kp}J_{pijl}^{sqzf}+a_{lp}J_{pijk}^{fqzs}\right)=\varkappa_{iq}\varkappa_{jz}\varkappa_{ks}\varkappa_{lf}$$

. . . . . .

Сопоставляя структуру левых частей уравнений /I.I3/,/2.3/, отмечаем их идентичность с тем, однако, отличием, что в уравнениях /2.3/ фигурируют коэффициенты транспонированной линейной модели об"екта.

Матрица весовых функций транспонированной модели равна транспонированной матрице весовых функций исходной линейной модели. Поэтому

$$J_{ij}^{*qz}=J_{qz}^{ij};\quad J_{ijk}^{*qzs}=J_{qzs}^{ijk};\quad J_{ijke}^{*qzsf}=J_{qzsf}^{ijke}; \quad \ldots\ldots\ldots \qquad /2.4/$$

где звездочками обозначены интегральные оценки весовых функций транспонированной линейной модели.

Сопоставляя уравнения /2.I/ и /2.3/ и принимая во внимание /2.4/, получаем следующие выражения для коэффициентов оптимальных управлений:

$$A_{ij}=\sum_{q,z}\beta_{qz}J_{qz}^{ij}$$

$$B_{ijk}=\sum_{q,z,s}\left[\gamma_{qzs}-\sum_{p}\left(b_{pzs}A_{pq}+b_{pqs}A_{pz}+b_{pqz}A_{ps}\right)\right]J_{qzs}^{ijk}$$

$$C_{ijke}=\sum_{q,z,s,f}\Big[\delta_{qzsf}-\sum_{p}\big(b_{pqz}B_{psf}+b_{pqs}B_{pzf}+ \qquad /2.5/$$

$$+b_{pqf}B_{pzs}+b_{pzs}B_{pqf}+b_{pzf}B_{pqs}+b_{psf}B_{pqz}\big)-$$

$$-\sum_{p}\left(c_{pzsf}A_{pq}+c_{pqsf}A_{pz}+c_{pqzf}A_{ps}+c_{pqzs}A_{pf}\right)\Big]J_{qzsf}^{ijke}$$

. . . . . .

Первая из этих формул была получена в работе[I]. Эти формулы очень удобны для последовательного определения коэффициентов оптимальных управлений с помощью цифровых вычислительных машин /ЦВМ/, а также аналоговых вычислительных машин /АВМ/ в сочетании с ручным счетом или вычислениями на ЦВМ.

Действительно, интегральные оценки весовых функций устойчивой линейной модели об"екта легко могут определяться на ЦВМ или АВМ. Для получения коэффициентов линейных членов аргументов оптимальных управлений достаточно найти интегральные квадратичные оценки весовых функций.
Для получения коэффициентов квадратичных членов аргументов управлений - кубические интегральные оценки весовых функций и т.д.

После этого согласно формулам /2.5/ дальнейшие вычисления сводятся к операциям умножения и сложения.

Для некоторых случаев практически возможно аналитическое определение интегральных оценок весовых функций /принципиальная возможность такого определения имеется всегда/ и определение оптимальных коэффициентов в общем буквенном виде. Об этом свидетельствует приводимый ниже пример.

Интегральные оценки весовых функций линейной модели с фиксированными коэффициентами могут с успехом использоваться и при определении оптимальных управлений для нестационарного нелинейного об"екта, если при решении соответствующих уравнений /I.I2/ применить метод итераций /предполагается, что коэффициенты об"екта и функционала изменяются достаточно медленно/.

Действительно, нулевые приближения для искомых коэффициентов при методе итераций определяются той же системой алгебраических линейных уравнений /I.I3/, что и в стационарном случае, с той разницей, что коэффициенты $a_{ij}, b_{ijk}, \ldots$ $\beta_{ij}, \gamma_{ijk}, \ldots$ здесь медленно меняющиеся функции времени. Таким образом нулевые приближения коэффициентов оптимальных управлений в любой заданный момент времени могут быть определены по формулам /2.5/, где все коэффициенты об"екта и функционала взяты для данного момента времени, а интегральные

оценки вычислены для фиксированных значений $a_{ij}$, соответствующих данному моменту времени.

Поправки первых приближений согласно /I.9/ определятся уравнениями типа /I.I0/, где в правых частях будут фигурировать производные нулевых приближений, поправки второго приближения аналогичным образом выражаются через производные поправок первого приближения и т.д. Если для поправок $\nu$ -го приближения ввести верхний индекс / $\nu$ /, то согласно /2.5/ эти поправки определятся формулами

$$A_{ij}^{(\nu+1)} = \sum_{q,z} \dot{A}_{qz}^{(\nu)} J_{qz}^{ij} \qquad /2.6/$$

$$B_{ijk}^{(\nu+1)} = \sum_{q,z,s} \left[ \dot{B}_{qzs}^{(\nu)} - \sum_{\rho} \left( b_{\rho zs} A_{\rho q}^{(\nu+1)} + b_{\rho qs} A_{\rho z}^{(\nu+1)} + b_{\rho qz} A_{\rho s}^{(\nu+1)} \right) \right] J_{q}$$

. . . . . .

Здесь, как и раньше, интегральные оценки вычисляются для фиксированных /"замороженных"/ значений коэффициентов об"екта, соответствующих заданному моменту времени. Производные по времени могут определяться численным путем по значениям, вычисленным для достаточно близких моментов времени.

Итак, для нестационарного нелинейного об"екта программы изменения во времени коэффициентов оптимальных управлений могут быть определены путем нахождения интегральных оценок весовых функций линейной модели первого приближения с "замороженными" коэффициентами и операций умножения, сложения, численного дифференцирования.

Необходимые операции могут производиться как на ЦВМ, так и на АВМ в сочетании с ЦВМ или ручным счетом.

Пример.

Рассмотрим задачу синтеза оптимальных управлений для случая, когда уравнения об"екта имеют вид

$$\dot{x}_i - \lambda_i x_i + \sum_{j,k} b_{ijk} x_j x_k + \sum_{j,k,\ell} c_{ijk\ell} x_j x_k x_\ell + \ldots = u_i \qquad /2.7/$$
$$(i = 1, 2, \ldots n)$$

Здесь $\lambda_i$ либо отрицательные действительные числа, либо комплексные числа с отрицательными действительными частями. Для линейной модели этого об"екта весовые функции равны

$$W_i^q = \begin{cases} 0 \text{ при } i \neq q \\ \exp(\lambda_i t) \text{ при } i = q \end{cases}$$

Поэтому

$$\gamma_{ij}^{ij} = -\frac{1}{\lambda_i + \lambda_j}, \quad \gamma_{ijk}^{ijk} = -\frac{1}{\lambda_i + \lambda_j + \lambda_k}, \quad \gamma_{ijke}^{ijke} = -\frac{1}{\lambda_i + \lambda_j + \lambda_k + \lambda_e}, \ldots\ldots$$

все другие интегральные оценки с неодинаковыми наборами верхних и нижних индексов равны нулю.

Формулы /2.5/ коэффициентов оптимальных управлений в данном случае существенно упрощаются:

$$A_{ij} = -\frac{B_{ij}}{\lambda_i + \lambda_j} \qquad /2.8/$$

$$B_{ijk} = -\frac{\gamma_{ijk} - \sum_{p=1}^{n} (b_{pjk} A_{pi} + b_{pik} A_{pj} + b_{pij} A_{pk})}{\lambda_i + \lambda_j + \lambda_k}$$

$$C_{ijke} = -\frac{1}{\lambda_i + \lambda_j + \lambda_k + \lambda_e} \Big[ \delta_{ijke} - \sum_{p=1}^{n} (b_{pij} B_{pke} + b_{pik} B_{pje} + b_{pie} B_{pjk} +$$

$$+ b_{pjk} B_{pie} + b_{pje} B_{pik} + b_{pke} B_{pij}) - \sum_{p=1}^{n} (c_{pjke} A_{pi} + c_{pike} A_{pj} + c_{pije} A_{pk} + c_{pijk} A_{pe}) \Big]$$

. . . . . .

Если подынтегральное выражение функционала задано в виде канонической квадратичной формы:

$$Q = \sum_{i=1}^{n} \beta_{ii} x_i^2,$$

т.е. $\beta_{ij} = 0$ при $j \neq 0$, $\gamma_{ijk} = 0$, $\delta_{ijke} = 0$, .....

то происходит дальнейшее упрощение выражений /2.8/ оптимальных коэффициентов:

$$A_{ii} = -\frac{\beta_{ii}}{2\lambda_i}, \quad A_{ij} = 0 \text{ при } j \neq i,$$

$$B_{ijk} = b_{ijk} \frac{A_{ii} + A_{jj} + A_{kk}}{\lambda_i + \lambda_j + \lambda_k}$$

$$C_{ijke} = \frac{1}{\lambda_i + \lambda_j + \lambda_k + \lambda_e} \Big[ \sum_{p=1}^{n} (b_{pij} B_{pke} + b_{pik} B_{pje} + b_{pie} B_{pjk} +$$

$$+ b_{pjk} B_{pie} + b_{pje} B_{pik} + b_{pke} B_{pij}) + c_{ijke} (A_{ii} + A_{jj} + A_{kk} + A_{ee}) \Big]$$

. . . . . .

Если квадратичные члены в характеристиках об"екта об"екта отсутствуют / $B_{ijk} = 0$/, то аргументы оптимальных управлений содержат кроме линейных только кубические и более высокие члены. Из формул /2.5/ видно, что эта закономерность является общей: при квадратичном заданном функционале минимальный порядок нелинейных членов в аргументах оптимальных управлений совпадает с минимальным порядком нелинейных членов в уравнениях об"екта. Общим свойством является также определенная симметрия оптимальных управлений: $j$ -ая координата в линейной части аргумента $i$ -го управления входит с тем же коэффициентом, что и $i$ -ая координата в $j$ -м управлении / $A_{ij} = A_{ji}$ /; произведение координат $x_j y_k$ в $i$ -м управлении входит с тем же коэффициентом, что произведение координат $y_i y_k$ в $j$ -м управлении / $B_{ijk} = B_{jik}$ / и т.д.

Практическое применение данных методов аналитического конструирования показывает достаточно высокую их эффективность. Некоторая потеря "оптимальности", связанная с ограничениями /I.3/, или специальной формой /I.8/ функционала окупается как простотой решения, так и простотой самих оптимальных управлений[x], обуславливающей относительную простоту реализации. Кроме того, ограничения типа /I.3/ и функционалы вида /I.8/ не являются искусственными, а имеют, как упоминалось выше, вполне определенный технический смысл.

Данные методы могут быть обобщены на случай, когда не все фазовые координаты доступны для непосредственного измерения /так называемая неполная степень наблюдаемости[7] /.

Может быть осуществлено обобщение и на случай действия шумов[14].

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x/ Это относится к линейным членам управлений.

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# ЛИНЕЙНЫЕ И НЕЛИНЕЙНЫЕ ОПТИМАЛЬНЫЕ РЕШЕНИЯ И СТРУКТУРЫ В ЗАДАЧЕ ЛЕТОВА И КАЛМАНА ПО ОПТИМАЛЬНОМУ СИНТЕЗУ ЛИНЕЙНЫХ ОБЪЕКТОВ

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## Введение

В настоящей работе рассматриваются некоторые подходы к решению задачи Летова[1] определения обратной связи для оптимальных по интегральным выпуклым оценкам на ограниченном отрезке времени, в том числе и задачу об оптимальном быстродействии. Решение задачи доведено до составления нелинейных инвариантных структур, с выделением нелинейных регуляторов по схеме, представленной на рис. I. При этом открывается возможность обобщения обратной оптимальной задачи Калмана[2] на нелинейные системы автоматического управления. В этом направлении, однако, намечаются лишь первые шаги.

Первая часть работы посвящена оптимальным системам по быстродействию, для которых, следуя Болтянскому[3] составлен алгоритм оптимального нелинейного синтеза по схеме Летова (рис. I.) Во второй части приводится нелинейный синтез для оптимальной задачи с интегральным выпуклым критерием качества. При этом, следуя В.М.Попову[4] обобщаются некоторые соотношения между уравнением Риккати и системой сопряженных уравнений для оптимальных систем.

## I. Алгоритм оптимального синтеза по быстродействию

Рассмотрим прикладную задачу оптимизации перехода линейного объекта

$$\dot{x} = A(t)x + b(t)u \qquad (1.1)$$

из начального положения $x(t_0) = x_0$ в конечное $x(t_1) = x_1$.

Здесь $A(t)$ - квадратная $n \times n$ действительная матрица, абсолютно интегрируемая на любом конечном интервале времени; $b(t)$ - $n$-мерный вектор, абсолютно интегрируемый на любом конечном интервале времени; $u(t)$ - скалярная функция времени, кусочно непрерывная и принимающая значения в замкнутой области управления, содержащей нуль-управление. Для упрощения, область управления будет

$$|u(t)| \leq 1 \qquad (1.2)$$

Известно, что принцип максимума Л.С.Понтрягина[5] даёт необходимую и достаточную информацию для полного решения данной задачи. Именно, окончательное решение задачи сводится к интегрированию системы уравнений

$$\dot{x} = A(t)x + b(t)\,\mathrm{sign}\, b'(t)y(t) \qquad (1.3)$$

$$\dot{y} = -A'(t)y \qquad (1.4)$$

при конечных условиях

$$x(t_0) = x_0\,, \quad x(t_1) = x_1 \qquad (1.5)$$

Для решения такой задачи предложены многочисленные методы, в основном приближенные, которые позволяют определять начальный вектор $y(t_0) = y_0$, обеспечивающий попадание оптимальной траектории в точку $x_1 = 0$[6,7].

Здесь предлагается замкнутый алгоритм построения оптимального синтеза $u = u(x)$. При этом получаются полные информации о топологии пространства состояний $X$, о числе переключений, а также о нелинейном регуляторе, который оптимизирует переходный режим объета.

Предлагаемый алгоритм исходит из того, что область управляемости пространства состояний разбивается на многообразия по критерию числа переключений, необходимых для оптимального перехода из любой точки $x_0$ данного многообразия в конечную точку $x_1 = 0$. Для всех многообразий даётся уравнения оптимального управления и оптимальной траектории как в начальном, так и в переходном режимах.

Прежде всего отметим конечную точку как нуль-мерное многообразие

$$V_0 = \{0\} \qquad (1.6)$$

в которое попадают все оптимальные траектории.

Попадание на многообразие (I.6) возможно в данной задаче лишь по двум путям, определяемым уравнениями

$$\dot{x} = A(t)x \pm b(t) \tag{I.7}$$

при конечных условиях

$$x(\tau_1) = \xi_1 , \quad x(t_1) = 0 \tag{I.8}$$

Решения уравнений (I.7) при (I.8) дают

$$\xi_1 = \mp \int_{\tau_1}^{t_1} \Phi^{-1}(\tau, \tau_1) b(\tau) d\tau \tag{I.9}$$

где $\Phi(\tau, \tau_1)$ - переходная матрица однородного уравнения, принимающая начальное значение $\Phi(\tau_1, \tau_1) = E$ - единичная матрица.

С другой стороны из решения уравнения (I.4) , т.е.

$$y(t) = \Psi(t, \tau_1) y(\tau_1) , \quad \Psi(\tau_1, \tau_1) = E \tag{I.10}$$

получается линейная форма относительно компонентов вектора $y(\tau_1)$ :

$$b'(t) y(t) = b'(t) \Psi(t, \tau_1) y(\tau_1) \tag{I.11}$$

Множество точек $\xi_1 = \xi_1(\tau_1)$, $\tau_1 \leq t_1$ , из которых можно попасть на многообразие (I.6) по траекториям уравнений (I.7) без переключений, т.е. точки, для которых можно найти такой начальный вектор $y(\tau_1)$, $-\infty < \tau_1 \leq t_1$ , чтобы во всем интервале $[\tau_1, t_1]$ линейная форма (I.11) не обратилась в ноль и отвечала знаку из уравнения (I.7) в соответствии с уравнением (I.3) , образуют одно-размерное многообразие $V_1 = V_1^+ \cup V_1^-$ :

$$V_1^{\pm} = \left\{ \xi_1 \,|\, \xi_1(\tau_1) = \mp \int_{\tau_1}^{t_1} \Phi^{-1}(\tau, \tau_1) b(\tau) d\tau \,;\; b'(t) y(t) \geq 0, \forall t \in [\tau_1; t_1] \right\} \tag{I.12}$$

В этом определении верхние знаки отвечают ветви $V_1^+$ , а нижние - $V_1^-$.

Так как многообразие $V_1$ зависит от одного параметра $\tau_1$ , геометрически оно представляет собой одномерное множество, т.е. кривую в $n$-мерном пространстве. Сразу видно, что $V_0$ входит в $V_1$ как точка при $\tau_1 = t_1$.

Пусть задано начальное состояние объект $x(t_0)=x_0 \in V_1$, т.е. найдется такое $\tau_1 = t_0$, чтобы $\xi_1 = x_0$ при соблюдении верхних или нижних знаков в формальном определении (I.12). Тогда оптимальная траектория имеет вид

$$x^*(t) = \Phi(t, t_0)\left[x_0 \pm \int_{t_0}^{t} \Phi^{-1}(\tau, t_0) b(\tau) d\tau\right] \qquad (I.13)$$

Пусть найден также вектор $y(t_0)$, который удовлетворяет соответствующим условиям (I.12) и образуем решение сопряженной системы

$$y^*(t) = \Psi(t, t_0) y^*(t_0) \qquad (I.14)$$

Из уравнений (I.13) и (I.14), учитывая известное соотношение между матрицами сопряженных систем

$$\Phi(t, t_0)\,\Psi'(t, t_0) = E \qquad (I.15)$$

может быть составлено соотношение

$$y^*(t) = y^*(x^*(t), t) \qquad (I.16)$$

которое приводит к обыкновенным системам релейного типа с дополнительными нелинейностями. Такие соотношения, а также соответствующие структуры оптимального управления легко составляются для автономных оптимизируемых объектов, как видно будет ниже.

Для определения вектора $y(t_0)$ можно считать, что при $t_0 = \theta_1$ происходит переключение, которое выводит состояние объекта $x$ на многообразие $V_1$.

Попадание изображающей точки, движущейся по оптимальной траектории, на $V_1$ возможно только из многообразия $V_2$ при одном переключении. Отмечая две ветви $V_2^+$ и $V_2^-$ этого многообразия, которые продолжаются непрерывно, после переключения, на $V_1^+$ и соответственно $V_1^-$, можно составить следующее формальное определение:

$$V_2 = V_2^+ \cup V_2^-;\quad V_2^{\pm} = \left\{ \xi_2 \,\middle|\, \xi_2(\tau_2) = \pm \int_{\tau_2}^{\theta_1} \Phi^{-1}(\tau, \tau_2) b(\tau) d\tau + \Phi^{-1}(\theta_1, \tau_2)\,\xi_1^{\pm},\ \xi_1^{\pm} \in V_1^{\pm} \right\};$$

$$b'(\theta_1)\, y(\theta_1) = 0,\quad \tau_2 \le \theta_1 \le t_1;\quad b'(t)\, y(t) \lneq 0,\ \forall t \in [\tau_2, \theta_1)\} \qquad (I.17)$$

Оно показывает, что многообразие $V_2$ представляет собой поверхность в $n$-мерном пространстве состояний, определяемое двумя параметрами $\tau_2, \theta_1$. При $\tau_2 = \theta_1$ получается кривая $V_1$ на многообразии $V_2$.

Повторяя тот же самый приём, что и выше, получаются все многообразия до $V_{n1}$ :

$$V_{n1} = V_{n1}^{+} \cup V_{n1}^{-} \; ; \; V_{n1}^{\pm} = \Big\{ \xi_n^{\pm} \,/\, \xi_n^{\pm}(\tau_n) = \pm(-1)^n \int_{\tau_n}^{\theta_{n-1}} \Phi^{-1}(\tau, \tau_n) b(\tau) d\tau + \Phi^{-1}(\theta_{n-1}, \tau_n) \xi_{n-1}^{\pm} ,$$

$$\xi_{n-1}^{\pm} \in V_{n-1}^{\pm} \; ; \; b'(\theta_{n-1}) y(\theta_{n-1}) = 0 , \; \tau_n \leq \theta_{n-1} \leq \theta_{n-2} \; ; (-1)^n b'(t) y(t) \leqslant 0 , \; \forall t \in [\tau_n, \theta_{n-1}) \Big\} \tag{I.18}$$

которая представляет собой $n$ -мерное множество в пространстве состояний, определяемое параметрами $\theta_1$, $\theta_2$, ..., $\theta_{n-1}$, $\tau_n$. Ясно, что при $\tau_n = \theta_{n-1}$ на многообразие $V_{n1}$ выделяется многообразие $V_{n-1}$, при $\theta_{n-1} = \theta_{n-2}$ на многообразие $V_{n-1}$ выделяется $V_{n-2}$ и т.д.

Покажем теперь как можем пользоваться этими многообразиями. Пусть, например, $x_0 \in V_{n1}$. Тогда $\tau_n = t_0$, а следовательно $\xi_n = x_0$. Если $\theta_{n-1}$, $\theta_{n-2}$, ..., $\theta_1$ известно, то однозначно определяются точки перехода оптимальной траектории $x^*$ с одного многообразия на следующее в порядке убывания размерности : $\xi_{n-1}$, $\xi_{n-2}$, ..., $\xi_1$, а на последнем участке определяется и $t_1$ из условия $x^*(t_1) = 0$.

Для определения моментов переключения $\theta_{n-1}$, $\theta_{n-2}$, ..., $\theta_1$, $t_1$ составляется система $n$ уравнений

$$\xi_n^{\pm}(\tau_n) = \pm(-1)^n \int_{t_0}^{\theta_{n-1}} \Phi^{-1}(\tau, t_0) b(\tau) d\tau \pm (-1)^{n-1} \Phi(\theta_{n-1}, t_0) \int_{\theta_{n-1}}^{\theta_{n-2}} \Phi^{-1}(\tau, \theta_{n-1}) b(\tau) d\tau \pm \ldots$$

$$\mp \Phi(\theta_{n-1}, t_0) \Phi(\theta_{n-2}, \theta_{n-1}) \ldots \Phi(\theta_1, \theta_2) \int_{\theta_1}^{t_1} \Phi^{-1}(\tau, \theta_1) b(\tau) d\tau = x_0 \tag{I.19}$$

которая получается непосредственно из первых равенств формальных определений всех многообразий $V_{n1}$, $V_{n-1}$, ..., $V_1$, после исключения векторов $\xi_{n-1}$, ..., $\xi_1$, входящих туда линейно.

Решение системы уравнений (I.19) должно удовлетворять условиям :

$$\text{а)} \quad t_0 \leq \theta_{n-1} \leq \theta_{n-2} \leq \ldots \leq \theta_1 \leq t_1 \tag{I.20}$$

вытекающем из правильного порядка прохождения оптимальной траектории $x^*(t)$, $t \in [t_0, t_1]$ от $x_0 \in V_{n1}$ через многообразия $V_{n1}$, $V_{n-1}$, ..., $V_1$ до $x_1 = 0$.

Теперь можно определить направление вектора $y^*(t_0)$ из уравнений переключения

$$b'(\theta_i) \Psi(\theta_i, t_0) y^*(t_0) = 0 ; \; i = 1, 2, \ldots, n-1 \tag{I.21}$$

которые представляют собой систему $n-1$-го однородного управления относительно его $n$ компонентов. После определения направления

вектора $y^*(t_0)$ строится соответствующее решение

$$y^*(t) = \Psi(t, t_0) y^*(t_0) \qquad (I.22)$$

и проверяются если:

б) нет других корней уравнения $b'(t) y^*(t) = 0$ кроме тех, которые входили в уравнения (I.21) ;

в) знак линейной формы $b'(t) y^*(t)$ меняется в правильном направлении в моменты переключения.

Если все эти условия выполнены при строгом соответствии последовательностей ветвей многообразий, то оптимальное управление найдено и можно сразу составить оптимальный синтез. Более того, можно составить соотношения вида (I.16) , а также соответствующую нелинейную структурную схему оптимального регулятора.

Уравнения (I.21) остаются в силе и в том случае, когда $x_0 \in V_j$ , $j \leq n-1$ . При этом $t_0 = \theta_{n-1} = \theta_{n-2} = \cdots = \theta_j$ , $1 \leq j \leq n-1$ будет кратным корнем уравнения (I.19) , а первые $n-j$ уравнения из (I.21) сводятся к производным соответствующей линейной формы по $t$ в точку $t = t_0$ приравняемым нулю.

Если все условия а) , б) и в) выполнены во всей области управляемости $X_u = \{x_0 / \forall x_0 \exists u \in [-1, 1] \; x_0 \longrightarrow x_1$ за конечное время$\}$ , то многообразие распространяется на всю область управляемости: $V_{n1} = X_u$ . В частности, для автономных объектов ($A$ и $b$ - постоянные) такое положение сводится к теореме об $n$ интервалах[8], причем область управляемости $X_u$ совпадает со всем пространством $X$ в том случае, если все собственные числа матрицы $A$ имеют отрицательные действительные части.

Однако, во многих случаях область $V_{n1}$ занимает только часть области управляемости $X_u$ . В другой части $X_u \setminus V_{n1}$ указанные выше условия не выполняются. Именно, уравнение

$$b'(t) y^*(t) = 0 \qquad (I.23)$$

в условии б) имеет и другие корни, кроме тех, которые получены из уравнений (I.19) . Тогда продолжается разбиение области $X_u$ на многообразия $V_{n1}$ , $V_{n2}$ , $V_{n3}$ , $V_{n4}, \ldots$, представляющие собой $n$ - мерные множества начальных положений $x_0 \in X_u$, для которых оптимальное управление содержит $n-1$ , $n$ , $n+1$, $n+2$, ... переключения. Такие многообразия определяются одно за другим попятным интегрированием уравнений (I.3) и (I.4) при условиях (I.5) , задаваясь числом пе-

реключений. При этом используются все составленные выше многообразия.

Например, для $V_{n2}$ имеем

$$V_{n2}=V_{n2}^{+}\cup V_{n2}^{-};\ V_{n2}^{\pm}=\Big\{\xi_{n2}\ \Big|\ \xi_{n2}(\tau_{n2})=\pm(-1)^{n+1}\int_{\tau_{n2}}^{\theta_{n1}}\Phi^{-1}(\tau,\tau_{n2})\,b(\tau)\,d\tau+\Phi^{-1}(\theta_{n1},\tau_{n2})\,\xi_{n1}^{\pm},$$

$$\xi_{n1}^{\pm}\in V_{n1}^{\pm};\ b'(\theta_{n1})\,y(\theta_m)=0,\ \tau_{n2}\le\theta_{n1}\le\theta_{n-1};(-1)^{n+1}b'(t)\,y(t)\le 0,\ \forall t\in[\tau_{n2},\theta_{n1})\Big\} \qquad (I.24)$$

Из равенства $\xi_{n2}^{\pm}=x_0$ получается система $n$ уравнений, а из уравнений I.21, к которым прибавляется еще одно уравнение для $\theta_{n1}$ из определения I.24, после исключения вектора $y(t_0)$, получается

$$\det\left[\Psi'(\theta_1,t_0)b(\theta_1)\quad\Psi'(\theta_2,t_0)b(\theta_2)\quad\cdots\quad\Psi'(\theta_{n1},t_0)b(\theta_{n1})\right]=0 \qquad (I.25)$$

Итак, получено $n+1$ уравнение для определения моментов переключения $t_1,\theta_1,\theta_2,\ldots,\theta_{n-1},\theta_{n1}$. Если решение этой системы удовлетворяет расширенные на $V_{n2}$ условия вида а), б) и в), то действительно $x_0\in V_{n2}$ и задача оптимального синтеза полностью решена. В противном случае вводится еще одно переключение и повторяются вычисления в предположении $x_0\in V_{n3}$. Этот процесс продолжается до тех пор, пока условия а), б) и в) не выполнены целиком.

Последовательные многообразия $V_{n1}, V_{n2}, V_{n3}, \ldots$ разделены между собой гиперповерхностями $n-1$-ой размерности, которые получаются точечным преобразованием многообразия $V_{n-1}$ в процессе попятного интегрирования уравнений движения. Такие гиперповерхности, за исключением $V_{n-1}$, являются клетками второго рода[3].

Между положительными и отрицательными подмножествами многообразий $V_{n1}, V_{n2}, V_{n3}, \ldots$ имеются гиперповерхности особых траекторий, которые попадают на поверхности переключения под нулевым углом.

Логическую схему изложенного алгоритма составим в предположении, что матрицы **A** и **b** – постоянные. Известно, что для любой действительной постоянной квадратной матрицы A существует такая квадратная неособая матрица T, что[9]

$$A=T^{-1}KT,\quad K=\begin{bmatrix}\ddots & & & \\ & \delta_i & \omega_i & \\ & -\omega_i & \delta_i & \\ & & & \ddots\ \delta_j\ \ddots\end{bmatrix}\begin{matrix}\\ \\ i=1,2,\ldots,m\\ j=2m+1,\ldots,n\end{matrix} \qquad (I.26)$$

где $\delta_i$, $\omega_i$ – действительные и мнимые части комплексных сопряженных собственных чисел матрицы $A$, $\delta_j$, – действительные собственные числа матрицы $A$. Так как матрица T осуществляет аффинное неособое преобразование пространства состояний $X=\{x\}$ в некоторое

пространство $Z=\{Tx\}$, которое имеет ту же качественную структуру, что и $X$, но только в искаженном виде, будем считать, что такое преобразование произведено и матрица $A$ имеет каноническую форму $A=K$.

Для упрщнения алгоритма будем различать две части : 1) $x_0\in V_{n_1}$, считая, что $V_{n_1}$ включает в себя все многообразия $V_k$, $k=0, 1, \dots, n-1$ и 2) $x_0\notin V_{n_1}$. Если все собственные числа матрицы $A$ действительны, то в силу теоремы об $n$ интервалах первая часть исчерпывает все возможности, ибо $V_{n_1}=X_u$.

Вычислительный аппарат для первой части состоит из уравнений

$$\pm\left[e^{-K t_1}-2e^{-K\theta_1}+2e^{-K\theta_2}-\dots+(-1)^{n-1}2e^{-K\theta_{n-1}}+(-1)^n E\right]K^{-1}b=x_0 \qquad (I.27)$$

к которым сводятся уравнения (I.19) для автономного объекта и которые полностью определяют моменты переключения $\theta_{n-1}$, $\theta_{n-2},\dots$, $\theta_1$, $t_1$, $(t_0=0)$, а также из формулы, которая дает сразу значение начального вектора $y(0)$ при $y_\ell(0)=1$:

$$y(0)=\left[\left(e^{-K\theta_{n-1}}b \quad e^{-K\theta_{n-2}}b \quad \dots \quad e^{-K\theta_1}b \quad I_\ell\right)'\right]^{-1}I_\ell \qquad (I.28)$$

где $I_\ell$ - вектор с нулевыми элементами, за исключением $\ell$-го элемента, равного единице. Если два или больше чисел $\theta_{n-1}, \theta_{n-2}, \theta_{n-3},\dots$ равны между собой, то соответствующие им столбцы транспонированной матрицы в формуле (I.28) умножаются на $K^0=E, K, K^2$ и т.д. Учитывая формулу (I.28), вектор $y^*(t)$ определяется сразу в виде

$$y^*(t)=\left\{\left[e^{K(t-\theta_{n-1})}b \quad e^{K(t-\theta_{n-2})}b \quad \dots \quad e^{K(t-\theta_1)}b \quad e^{Kt}I_\ell\right]'\right\}^{-1}I_\ell \qquad (I.29)$$

Проверка результатов производится в следующем порядке:

$$\begin{aligned}
&\text{а)}\quad 0\le\theta_{n-1}\le\theta_{n-2}\le\dots\le\theta_1\le t_1, \quad \text{все действительны}\\
&\text{б)}\quad b'y^*(t)\neq 0,\ \forall t\neq\theta_k,\ t\in[0,t_1],\ k=1,2,\dots,n\\
&\text{в)}\quad \left\{\begin{matrix}(-1)^{k-1}b'y^*(t)>0,\ \forall x_0\in V_k^+,\ t\in(\theta_k,\theta_{k-1})\\ (-1)^{k-1}b'y^*(t)<0,\ \forall x_0\in V_k^-,\ t\in(\theta_k,\theta_{k-1})\end{matrix}\right\}\ \theta_0=t_1,\ \theta_n=t_0=0
\end{aligned} \qquad (I.30)$$

Если все эти условия выполнены, то однозначно определяются точки переключения $\xi_k$ на оптимальной траектории $x^*(t)$ по формуле

$$\xi_k^*=\pm(-1)^k\left[E-2e^{-K\theta_{k-1}}+2e^{-K\theta_{k-2}}-\dots+(-1)^{k-1}2e^{-K\theta_1}+(-1)^k2e^{-Kt_1}\right]K^{-1}b \qquad (I.31)$$

$$k=1,2,3,\dots,n$$

На каждом участке $[\theta_k, \theta_{k-1}]$ оптимальная траектория имеет вид

$$x^*(t) = e^{K(t-\theta_k)}[\xi_k \pm (-1)^{k-1} K^{-1} b] \mp (-1)^{k-1} K^{-1} b \qquad (I.32)$$

а сопряженный вектор определяется формулой

$$y^*(t) = e^{-K'(t-\theta_k)} \eta_k \qquad (I.33)$$

где обозначено $\eta_k = y^*(\theta_k)$. Уравнения (I.32) можно решать относительно $n$ ненулевых элементов матрицы $e^{K(t-\theta_k)}$ как функции $x^*(t)$, причем в силу линейности решение единственно. Дальше, от матрицы $e^{K(t-\theta_k)}$ можно переходить к матрице $e^{-K'(t-\theta_k)}$. В результате этого, уравнение (I.33) принимает вид

$$y^* = y^*(x^*) \qquad (I.34)$$

при том единственно. Например, в случае действительных собственных чисел матрицы $A$, т.е. при $m = 0$ в канонической матрице (I.26),

$$y_j^* = \frac{\xi_k^j \pm (-1)^{k-1} \frac{b^j}{\delta_j}}{x^{j*} \pm (-1)^{k-1} \frac{b^j}{\delta_j}} \eta_{kj}, \quad x^* \in V_k$$
$$j = 1, 2, \ldots, n; \; k = 1, 2, \ldots, n-1 \qquad (I.35)$$

где принято $x = [x^j]$, $y = [y_j]$, $b = [b^j]$, $j = 1, 2, \ldots, n$. Теперь можем образовать линейную форму

$$b'y^* = \sum_{j=1}^{n} b^j y_j^*(x^{j*}) \qquad (I.36)$$

которая замыкает цепь оптимального по быстродействию управления в виде системы с двумя нелинейностями (рис.2). Такая структура отличается от других известных схем[10] тем, что вычислительное устройство не входит в цепь управления.

На этом первая часть алгоритма закончена. Вторая часть вступает в действие лишь в том случае, если матрица $A$ имеет собственные комплексные числа и при прохождении первой части программы условия (I.30) не выполняются.

Для второй части алгоритма уравнения (I.27) заменяется уравнениями

$$\pm\left[e^{-K\theta_1} - 2e^{-K\theta_2} + \ldots + (-1)^{n+N-1} 2e^{-K\theta_{n+N-1}} + (-1)^{n+N} E\right] K^{-1} b = x_0 \qquad (I.37)$$

в которые входят $n+N$ неизвестных $\theta_{n+N-1}, \theta_{n+N-2}, \ldots, \theta_1, \theta_0 = t_1$. Для полного определения этих неизвестных к $n$ уравнениям (I.37) прибавляются $N$ уравнения вида (I.25), т.е.

$$\det \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & e^{-\delta_i \theta_k}\cos\omega_i\theta_k & e^{-\delta_i\theta_k}\sin\omega_i\theta_k & \cdots & e^{-\delta_j\theta_k} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & e^{-\delta_i\theta_{n-1+q}}\cos\omega_i\theta_{n-1+q} & e^{-\delta_i\theta_{n-1+q}}\sin\omega_i\theta_{n-1+q} & \cdots & e^{-\delta_j\theta_{n-1+q}} & \cdots \end{bmatrix} = 0 \qquad (I.38)$$

$i=1,2,\ldots,m;\ j=2m+1,\ldots,n;\ k=0,1,2,\ldots,n-1;\ q=1,2,\ldots,N$

Формулы (I.28) и (I.29) остаются в силе и для второй части алгоритма. Проверка результатов производится при помощи условий вида (I.30), расширенных на все $n+N$ корней уравнения (I.37) и (I.38). Повторяя применение алгоритма при $N=1,2,\ldots$, найдется такое целое число $N$, при котором все эти условия выполнены. Тем самым найдено число переключений $n+N-1$ и решение завершается при помощи расширенных формул вида (I.31) - (I.34).

Описанный алгоритм дается в виде логической схемы на рис.8

Все приведенные выше построения для одномерного управления обобщаются без трудностей на случай многомерного управления линейного объекта, а также на некоторые классы нелинейных объектов.

## 2. Нелинейный синтез оптимальной системы по интегральным выпуклым критериям

Рассмотрим задачу оптимизации управления $u(x)$ линейным объектом

$$\dot{x} = A(t)x + B(t)u \qquad (2.1)$$

где $A(t)$ и $B(t)$ - непрерывные действительные матрицы размерностей $n \times n$ и $n \times m$, определенные на известном интервале времени $[t_0, t_1]$. Задано начальное состояние

$$x(t_0) = x_0 \qquad (2.2)$$

из которого $m$-мерное управление $u$ переводит объект в конечное состояние

$$x(t_1) = x_1 \qquad (2.3)$$

При этом конечный момент времени $t_1$ будем считать фиксированным.

Критерием качества управления будет

$$I(u) = \frac{1}{2}\left[ f(x(t_1)) + \int_{t_0}^{t_1} (x'G(t)x + u'H(t)u)\,dt \right] \qquad (2.4)$$

где $f(x(t_1))$ - заданная непрерывная функция, $G(t)$ и $H(t)$ - непрерывные действительные квадратные симметричные матрицы размерностей $n \times n$ и $m \times m$ на интервале $[t_0, t_1]$. Кроме того, предполагается, что $G(t)$ положительно-полуопределенная, а $H(t)$ - положительно-определенная матрицы.

Управление $u(t)$ будем считать допустимым, если оно обеспечивает перемещение объекта из положения $x_0$ в положение $x_1$, при ограниченном значении $I(u)$ и оптимальным, если функционал $I(u)$ принимает минимальное (или максимальное) значение.

В работе[11] даны основные теоремы о существовании и единственности оптимального управления. В частности доказано, что единственное решение системы уравнений

$$\dot{x} = A(t)x - B(t)H^{-1}(t)B'(t)y$$
$$\dot{y} = -G(t)x - A'(t)y \qquad (2.5)$$

при граничных условиях (2.2) и

$$y(t_1) = \operatorname{grad} f(x(t_1)) \qquad (2.6)$$

определяет оптимальную траекторию $x^*(t)$ и соответствующее оптимальное управление

$$u^*(t) = -H^{-1}(t)B'(t)y^*(t) \qquad (2.7)$$

При некоторых частных видах функции $f(x)$, оптимальное управление (2.7) определяется формулой

$$u^* = -H^{-1}(t)B'(t)R(t)x^* \qquad (2.8)$$

где $R(t)$ - решение матричного управления Риккати

$$\dot{R}(t) + R(t)B(t)H^{-1}(t)B'(t)R(t) + R(t)A'(t) + A(t)R(t) + G(t) = 0 \qquad (2.9)$$

при граничном условии, которое зависит от функции $f(x)$. Например, при $f(x) = x'Fx$, где $F$ - квадратная положительно полуопределенная матрица, граничное условие имеет вид[11,12]

$$R(t_1) = F \qquad (2.10)$$

Если же конечное состояние $x_1$ должно находиться на заданной выпуклой гладком множестве

$$g_s(x_1) = 0;\ s = 1, 2, \ldots, p;\ p \le n \qquad (2.11)$$

то граничные условия определяются формулой[13,14]

$$y(t_1) = -\lambda \operatorname{grad} f(x_1) - \sum_{s=1}^{p} \mu_s \operatorname{grad} g_s(x_1) \qquad (2.12)$$

где $\lambda$ и $\mu_s$ - неотрицательные числа, одно из которых равно единице. Например, при $\lambda = 1$, уравнения (2.11) и (2.12) полностью определяют вектор $y(t_1)$ и множители $\mu_s$.

В частности, предполагая $f = 0$, если уравнения (2.11) линейны:

$$x^s(t_1) = 0;\ s = 1, 2, \ldots, p;\ p \le n$$

то получается

$$E_1 x_1 = 0$$
$$E_2 y_1 = 0 \qquad (2.13)$$

где $E_1$ – диагональная матрица, у которой первые $p$ элементы равны единице, а последние $n-p$ – нулю, $E_2 = E - E_1$ .

При $E_1 = 0$ конечное состояние (2.3) свободно и оптимальное управление получается в виде (2.8) , причем уравнение Риккати (2.9) интегрируется при условии (2.10) , где $F = 0$ . Для интегрирования уравнения Риккати можно применять метод последовательных приближений[15]. Покажем, что матрицу $R(t)$ можно получать также путем интегрирования линейной системы сопряженных уравнений.

Пусть $\Phi(t)$ и $\Psi(t)$ обозначают матрицы перехода для уравнений (2.5) , т.е.

$$\dot{\Phi} = A(t)\Phi - B(t)H^{-1}(t)B'(t)\Psi$$
$$\dot{\Psi} = -G(t)\Phi - A'(t)\Psi \qquad (2.14)$$

Общее решение такой системы имеет вид

$$\Phi(t) = \Phi_1(t)C_1 + \Phi_2(t)C_2$$
$$\Psi(t) = \Psi_1(t)C_1 + \Psi_2(t)C_2 \qquad (2.15)$$

где $\Phi_i(t)$, $\Psi_i(t)$ , $i = 1,2$ – линейно – независимые частные решения уравнений 2.14 , а $C_1, C_2$ – квадратные матрицы действительных постоянных интегрирования.

Решение системы уравнений (2.5) можно теперь представить в виде

$$x = \Phi_1(t)D_1 + \Phi_2(t)D_2$$
$$y = \Psi_1(t)D_1 + \Psi_2(t)D_2 \qquad (2.16)$$

где $D_1$ и $D_2$ – векторы, которые однозначно определяются из заданных конечных условий, например (2.2) и (2.13) .

Введем в рассмотрение матрицу

$$R(t) = \Psi(t)\Phi^{-1}(t) \qquad (2.17)$$

Легко проверить, что если $\Phi(t)$ и $\Psi(t)$ представляют собой решение системы уравнений (2.14) , то матрица $R(t)$ определяется уравнением Риккати (2.9) , причем граничные условия для $R(t)$ полностью определены граничными условиями для $\Phi(t)$ и $\Psi(t)$. Обратный переход приводит к уравнению

$$\Phi^{-1}[\dot{\Phi} - A\Phi + BH^{-1}B'\Psi] + \Psi^{-1}[\dot{\Psi} + G\Phi + A'\Psi] = 0 \qquad (2.18)$$

которое равносильно системе (2.14) , если матрицы $\Phi(t)$ и $\Psi(t)$ неособые

при всех $t\in[t_0, t_1]$. Действительно, матрица $R(t)$ не меняется, если матрицы $\Phi(t)$ и $\Psi(t)$ умножить на произвольную постоянную неособую матрицу C, причем уравнение (2.18) остается в силе. Следовательно, матрицы $\Phi(t)$ и $\Psi(t)$ представляют собой решение системы линейных уравнений (2.14), причем они определены с точностью до произвольного постоянного неособого множителя C. Таким образом, в решении (2.15) матрицу $C_1$, например, можно заменить единичной матрицей E. Тогда матрица $C_2$ полностью определена граничным условием для матрицы $R(t)$, в частности (2.10).

Описание матрицы $R(t)$ при помощи системы линейных сопряженных уравнений (2.14) позволяет определять ее как функцию векторов $x^*$, $y^*$ и времени $t$ (в неавтономном случае). Приводим способ составления такой функции для автономного случая.[17,18]

Пусть $Q$ представляет собой квадратную неособую матрицу, которая ставит в соответствие матрицу однородной системы (2.14) с квазидиагональной матрицей

$$\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} = Q \begin{bmatrix} A & -BH^{-1}B' \\ -G & -A' \end{bmatrix} Q^{-1}$$

где $K$ имеет вид (1.26). Известно,[19] что такая матрица составлена из собственных векторов матрицы системы (2.14). При этом частные решения системы $\Phi_i(t), \Psi_i(t)$ имеют вид

$$\Phi_1(t) = Q_{11} e^{K(t-t_0)} \qquad \Phi_2(t) = Q_{12} e^{-K'(t-t_0)}$$
$$\Psi_1(t) = Q_{21} e^{K(t-t_0)} \qquad \Psi_2(t) = Q_{22} e^{-K'(t-t_0)} \qquad (2.19)$$

где $Q_{ij}$ – квадратные подматрицы матрицы $Q$:

$$Q = \begin{bmatrix} \overbrace{Q_{11}}^{n} & \overbrace{Q_{12}}^{n} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{matrix} \}n \\ \}n \end{matrix} \qquad (2.20)$$

Теперь из уравнений (2.16) однозначно определяются матрицы $e^{K(t-t_0)}$, $e^{-K'(t-t_0)}$ как функции векторов $x^*$, $y^*$ и начальных условий $x_0$:

$$e^{K(t-t_0)} = L(x^*, y^*, x_0), \quad e^{-K'(t-t_0)} = L'^{-1}(x^*, y^*, x_0) \qquad (2.21)$$

Далее, соотношение

$$LL^{-1} = E$$

даёт возможность определить вектор $y^*$ как функцию $x^*$, $x_0$ и $t_1$:

$$y^* = y^*(x^*, x_0, t_1) \qquad (2.22)$$

причем такое определение единственно если требовать выполнение конечных условий. Соотношение (2.22) вместе с формулой (2.7) определяет нелинейный инвариантный синтез для рассматриваемой задачи в виде структурной схемы, представленной на рис. 4.

Очевидно, если оптимальный синтез можно привести к линейному закону (2.8) , то матрица $R(t)$ представима в виде функции от $x^*$ и $x_0$.

Полученные результаты легко распространяются на более общий случай критерия качества, когда под интеграл входит производная по времени от управления. [8] При этом получается нелинейный регулятор с интегрирующим звеном, работающим по схеме рис. 5.

В частном случае, при $t_1 = \infty$, нелинейные регуляторы становятся линейными и совпадают с известными результатами Летова-Калмана.

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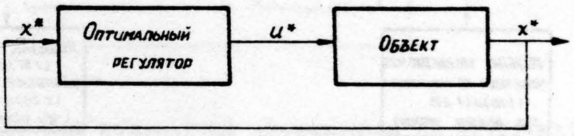


Рис. 1. Общий вид оптимальной системы управления

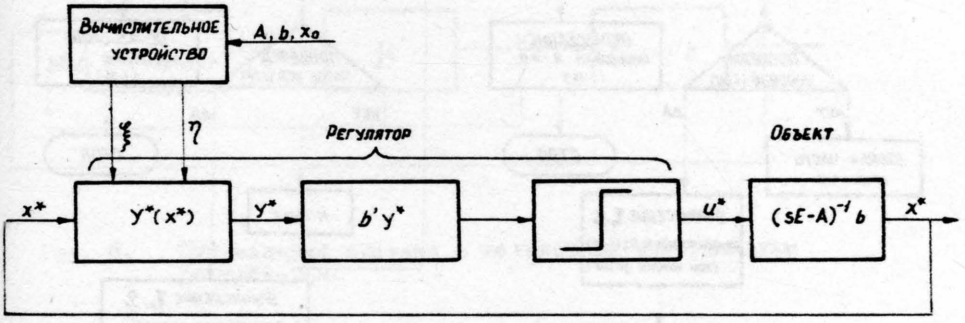


Рис. 2. Оптимальная по быстродействию система с двумя нелинейностями в регуляторе

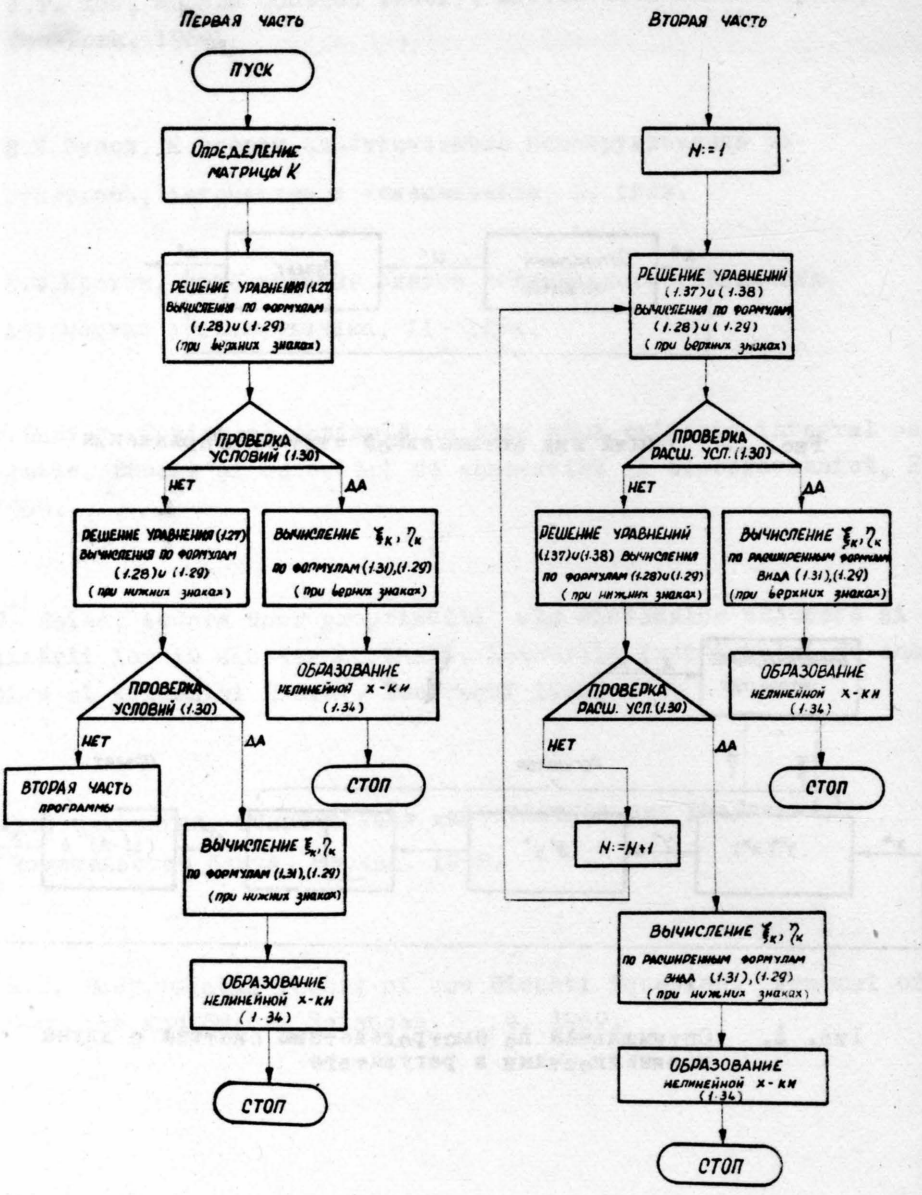


Рис. 3. Логическая схема работы вычислительного устройства для оптимизации быстродействия.

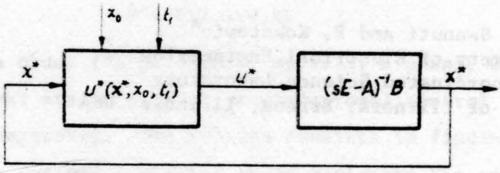


Рис. 4. Инвариантная нелинейная оптимальная по интегрально-квадратичному критерию система

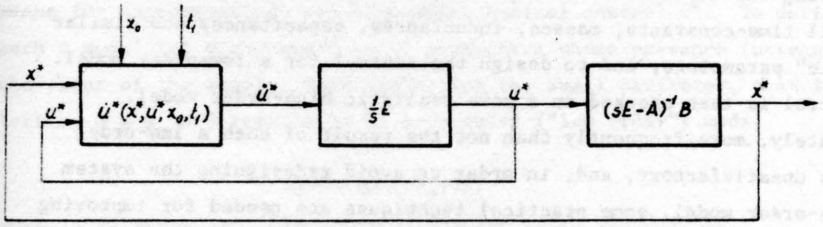


Рис. 5. Оптимальная система с нелинейным интегральным регулятором.

# SINGULAR PERTURBATION METHOD FOR NEAR OPTIMUM DESIGN OF HIGH-ORDER NONLINEAR SYSTEMS

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## INTRODUCTION

Realistic descriptions of physical plants to be controlled usually result in high-order mathematical models which make the application of existing synthesis procedures impractical. This is especially true in synthesis of optimal controls where computational difficulties in solving the two-point boundary value problems become enormous if the order of the model is high. Therefore it is a practical necessity to find a method for reducing the model order. In practice a common approach is to neglect some small time-constants, masses, inductances, capacitances and similar "parasitic" parameters, and to design the control for a low-order model. This control is then checked on a more realistic high-order model. Unfortunately, more frequently than not the result of such a low-order design is unsatisfactory, and, in order to avoid redesigning the system on a high-order model, some practical techniques are needed for improving the low-order design. In [1] an optimal control oriented method for reducing the order of mathematical models has been developed by which the desired improvement of a low-order design is achieved by using low-order models only.

The singular perturbation method proposed in [1] assumes that a control optimal for a high-order model is a continuous and differentiable function of a small parameter whose presence increases the order of the model. In this paper the required continuity and differentiability properties of high-order optimal control are proved for a class of nonlinear systems and further extensions and applications of the method are given. A second order design of a nonlinear fifth-order system illustrates the practical applicability and the computational simplicity of the method.

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## HIGH-ORDER AND LOW-ORDER MODELS

In this section appropriate high-order and low-order descriptions of a plant are defined. Let

$$\dot{y} = F(y,u,\alpha,t) \tag{1}$$

be the (n+m)-th order ("high-order") model of a plant, where y is the (n+m)-dimensional state, u is the r-dimensional control and $\alpha$ is an m-dimensional parameter. The problem consists in finding a control $u^*$ ("high-order optimal control") which minimizes the performance index

$$J = \int_{t_o}^{T} W(y,u,t)\,dt, \tag{2}$$

where the initial state $y(t_o) = y_o$ is fixed and the final state y(T) is free. It is assumed that the order n+m of (1) is too high for the optimization problem to be practically solvable by existing numerical methods. Therefore, a low-order model is sought which will provide a means for approximating the high-order optimal control $u^*$. To derive such a model let $\alpha$ represent small parameters whose presence increases the order of the model. Then neglecting the small parameter, that is letting $\alpha = \alpha_o = 0$ results in an n-th order ("low order") model

$$\dot{\underline{x}} = f(\underline{x},\underline{u},\alpha_o,t), \tag{3}$$

where $\underline{x}$ is the n-dimensional ("low-order") state and $\underline{u}$ is the r-dimensional control.

In view of the specific role of $\alpha$ the equation (1) can be rewritten in a form convenient for the application of the singular perturbation theory [1,2]. Consider the i-th component of (1), $\dot{y}_i = F_i(y,u,\alpha,t)$, and assume that for $\alpha_i = 0$ this differential equation reduces to an algebraic or transcendental equation. This happens when $\alpha_i = 0$ is a pole of some multiplicity k of the function $F_i$, $F_i = \dfrac{h_i(y,u,\alpha,t)}{\alpha_i^k}$. For simplicity express all small parameters $\alpha_i$, i = 1,...,m, as some analytic functions of nonnegative scalar $\lambda$, and let these functions have zero at $\lambda = 0$. Then (1) can be put into the form

$$\dot{x} = g(x,z,u,\lambda,t), \tag{4'}$$

$$\lambda\dot{z} = G(x,z,u,\lambda,t), \tag{4"}$$

where x and z are respectively the n-dimensional and the m-dimensional

parts of the high-order state y, and (2) can be rewritten as follows

$$J = \int_{t_o}^{T} V(x,z,u,t)\,dt. \tag{5}$$

Finally, the low-order model (3) is obtained from (4) by letting $\lambda = 0$,

$$\dot{\underline{x}} = g(\underline{x},\underline{z},\underline{u},0,t), \tag{6'}$$

$$0 = G(\underline{x},\underline{z},\underline{u},0,t). \tag{6''}$$

and substituting into (6') the solution $\underline{z} = \varphi(\underline{x},\underline{u},t)$ of (6''),

$$\dot{\underline{x}} = g(\underline{x},\varphi(\underline{x},\underline{u},t),\underline{u},0,t) \equiv f(\underline{x},\underline{u},\alpha_o,t). \tag{7}$$

It can be noted that the initial condition for the low-order model (7) must be n-dimensional and that $\underline{z}$ at $t = t_o$ may not coincide with $z_o$. In the sequel a design based on the high-order model (4) will be called the high-order design and a design based on the low-order model (7) will be called the low-order design.

## STATEMENT OF THE PROBLEM

A direct application of a low-order design to a high-order plant may result in a performance far from optimum [1] or may even make the system unstable [3]. In [1] an approximation method has been proposed which represents a compromise between a simple, but unsatisfactory low-order design, and an optimal but computationally involved high-order design. The method proposed in [1] consists of approximating the high-order-optimal control $u^*(t,\lambda)$ by its truncated MacLaurin series in $\lambda$,

$$\hat{u} = u^*(t,0) + \frac{\partial u^*}{\partial \lambda}(t,\lambda)\Big|_{\lambda=0} \lambda. \tag{8}$$

The method in [1] is based on the assumption that the optimal control $u^*(t,\lambda)$ is continuous and differentiable with respect to $\lambda$ at $\lambda = 0$.

The problem in this paper is to reexamine the assumption made in [1] and to prove, for a class of nonlinear systems, the continuity and differentiability properties of $u^*(t,\lambda)$ at $\lambda = 0$. An algorithm for calculating $\frac{\partial u^*}{\partial \lambda}(t,\lambda)$ at $\lambda = 0$ is also to be developed. All calculations are to be performed on a low-order (n-th order) model only, while approximation must be valid for a high-order ((n+m)-th order) model. In this sense the control $\hat{u}$ is "optimally sensitive" with respect to the change in order.

In this paper systems nonlinear in x, but linear in z and u are considered, that is

$$\dot{x} = g(x,z,u,\lambda,t) \equiv g_1 + B_1 z + C_1 u, \tag{9'}$$

$$\lambda \dot{z} \equiv G(x,z,u,\lambda,t) \equiv G_1 + B_2 z + C_2 u, \tag{9''}$$

where $g_1 = g_1(x,\lambda,t)$ and $G_1 = G_1(x,\lambda,t)$ are n and m dimensional vectors, and $B_1 = B_1(x,\lambda,t)$, $B_2 = B_2(x,\lambda,t)$, $C_1 = C_1(x,\lambda,t)$ and $C_2 = C_2(x,\lambda,t)$ are nxm, mxm, nxr, and mxr dimensional matrices, respectively. It is assumed that $g_1$, $B_1$, and $C_1$ have continuous second partial derivatives and $G_1$, $B_2$, and $C_2$ have continuous third partial derivatives in a region $\Gamma$ of the space of variables x, t, and with respect to $\lambda$ for $\lambda \in [0,\lambda_{max}]$. It is also assumed that the performance index (5) is quadratic in u,

$$J = \int_{t_o}^{T} V(x,u,t)dt = \frac{1}{2}\int_{t_o}^{T} [Q(x,t) + u'R(x,t)u]dt, \tag{10}$$

where rxr dimensional matrix R(x,t) is positive definite in the region $\Gamma$ and $R^{-1}(x,t)$ and Q(x,t) are assumed to have continuous second partial derivatives in the region $\Gamma$.

## CONTINUITY AND DIFFERENTIABILITY OF $u^*(t,\lambda)$ at $\lambda = 0$

Let p and $\hat{q}$ be the adjoint variables corresponding to x and z, and let $q = \frac{\hat{q}}{\lambda}$. Then a necessary condition for the optimal control problem (9) and (10) is

$$\dot{x} = g_1 + B_1 z + C_1 u, \tag{11'}$$

$$\lambda \dot{z} = G_1 + B_2 z + C_2 u, \tag{11''}$$

$$\dot{p} = \nabla_x V - g_x' p - G_x' q, \tag{12'}$$

$$\lambda \dot{q} = -g_z' p - G_z' q, \tag{12''}$$

$$u = R^{-1}(C_1' p + C_2' q), \tag{13}$$

with the boundary values

$$x(t_o) = x_o; \quad z(t_o) = z_o; \quad p(T) = 0; \quad q(T) = 0. \tag{14}$$

Equations (11), (12), and (13) represent the high-order canonic equations. The low-order canonic equations are obtained by letting $\lambda = 0$,

$$\underline{\dot{x}} = \underline{g}_1 + \underline{B}_1 \underline{z} + \underline{C}_1 \underline{R}^{-1}(\underline{C}_1' \underline{p} + \underline{C}_2' \underline{q}), \tag{15'}$$

$$0 = \underline{G}_1 + \underline{B}_2 \underline{z} + \underline{C}_2 \underline{R}^{-1}(\underline{C}_1' \underline{p} + \underline{C}_2' \underline{q}), \tag{15''}$$

$$\underline{\dot{p}} = \nabla_{\underline{x}} V - g_{\underline{x}}' \underline{p} - G_{\underline{x}}' \underline{q}, \tag{16'}$$

$$0 = -\underline{B}_1' \underline{p} - \underline{B}_2' \underline{q}, \tag{16''}$$

$$\underline{u} = \underline{R}^{-1}(\underline{C}_1' \underline{p} + \underline{C}_2' \underline{q}) \tag{17}$$

with the boundary values

$$\underline{x}(t_o) = x_o; \quad \underline{p}(T) = 0. \tag{18}$$

The continuity and differentiability properties of the solution of (11), (12), (13), and (14) are established by the following:

Theorem: If the solution $\underline{x}^*(t)$ of the low-order equations (15) and (16) along with the condition (18) lies in the region $\Gamma$, and if the matrix $B_2(x,0,t)$ has all eigenvalues with negative real parts in the region $\Gamma$ then as $\lambda$ tends to zero the high-order optimal functions $x^*(t,\lambda)$, $z^*(t,\lambda)$, $p^*(t,\lambda)$, $q^*(t,\lambda)$, and $u^*(t,\lambda)$ tend to the low-order optimal functions $\underline{x}^*(t)$, $\underline{z}^*(t)$, $\underline{p}^*(t)$, $\underline{q}^*(t)$, and $\underline{u}^*(t)$ for all t, $t_o < t < T$. Furthermore, they are differentiable with respect to $\lambda$ at $\lambda = 0$.

This theorem is proved by applying a more general theorem due to Tupčiev [2].

Let

$$X = \begin{bmatrix} x \\ p \end{bmatrix}; \quad Z = \begin{bmatrix} z \\ q \end{bmatrix};$$

$$\Psi = \Psi(X,Z,\lambda,t) = \begin{bmatrix} g_1 + B_1 z + C_1 R^{-1}(C_1' p + C_2' q) \\ \nabla_x V - g_x' p - G_x' q \end{bmatrix};$$

$$\Phi = \Phi(X,Z,\lambda,t) = \begin{bmatrix} G_1 + B_2 z + C_2 R^{-1}(C_1' p + C_2' q) \\ - B_1' p - B_2' q \end{bmatrix}.$$

Equations (11) and (12) can be given the form,

$$\dot{X} = \Psi(X,Z,\lambda,t),$$

$$\lambda \dot{Z} = \Phi(X,Z,\lambda,t).$$

The partial derivative of $\Phi$ with respect to Z evaluated at $\lambda = 0$ is

$$\Phi_Z = \begin{bmatrix} B_2(x,0,t) & C_2(x,0,t)R^{-1}(x,0,t)C_2'(x,0,t) \\ 0 & -B_2'(x,0,t) \end{bmatrix}.$$

To show that Tupčiev's hypothesis is satisfied, a matrix $S = \begin{bmatrix} I & E(x,t) \\ 0 & I \end{bmatrix}$ where $E(x,t)$ is a mxm matrix and I is an mxm identity matrix is sought such that the following condition is satisfied:

$$S^{-1}\Phi_Z S = \begin{bmatrix} B_2(x,0,t) & 0 \\ 0 & -B_2'(x,0,t) \end{bmatrix}. \tag{19}$$

This condition (19) is satisfied if and only if $E(x,t)$ satisfies the linear matrix equation,

$$B_2(x,0,t)\ E(x,t) + E(x,t)\ B_2'(x,0,t) = -C_2(x,0,t)\ R^{-1}(x,0,t)\ C_2'(x,0,t). \tag{20}$$

Since the matrix $B_2(x,0,t)$ has all eigenvalues with negative real parts in the region $\Gamma$, the matrices $B_2(x,0,t)$ and $-B_2'(x,0,t)$ have no eigenvalues in common and hence equation (20) has a unique solution for $E(x,t)$ [4].

Now in view of Tupčiev's theorem [2], the solution $x^*(t,\lambda)$, $z^*(t,\lambda)$, $p^*(t,\lambda)$, and $q^*(t,\lambda)$ of (11), (12), and (14) tend as $\lambda$ tends to zero to the solution $\underline{x}^*(t)$, $\underline{z}^*(t)$, $\underline{p}^*(t)$, and $\underline{q}^*(t)$ of (15), (16), and (18), and further $x^*(t,\lambda)$, $z^*(t,\lambda)$, $p^*(t,\lambda)$, and $q^*(t,\lambda)$ are differentiable with respect to $\lambda$ at $\lambda=0$ for all $t$, $t_o < t < T$. This fact proves the above theorem.

An important extension of the above theorem is possible when functions g and G (9) are analytic in all of their arguments. It can be shown that in this case the functions $x^*(t,\lambda)$, $z^*(t,\lambda)$, $p^*(t,\lambda)$, $q^*(t,\lambda)$, and $u^*(t,\lambda)$ are analytic in $\lambda$ at $\lambda=0$, that is that they can be expanded in an infinite MacLaurin series in $\lambda$.

## OBTAINING $\frac{\partial u^*(t,\lambda)}{\partial \lambda}$ AT $\lambda=0$

In this section it is shown that the second term of the truncated series (8) can be obtained by solving a low-order matrix Riccati equation along with two low-order vector linear equations.

Introducing at $\lambda=0$ the sensitivity functions $\omega = \frac{\partial \underline{x}^*}{\partial \lambda}$, $\sigma = \frac{\partial \underline{z}^*}{\partial \lambda}$, $\pi = \frac{\partial p^*}{\partial \lambda}$, $\eta = \frac{\partial q^*}{\partial \lambda}$, $\nu = \frac{\partial u^*}{\partial \lambda}$, differentiating (11), (12), and (13) with respect to $\lambda$ and letting $\lambda=0$, the low-order sensitivity equations are obtained,

$$\dot{\omega} = g_{\underline{x}}\omega + \underline{B}_1\sigma + \underline{C}_1\nu + g_{\underline{\lambda}}, \tag{21'}$$

$$0 = G_{\underline{x}}\omega + \underline{B}_2\sigma + \underline{C}_2\nu + G_{\underline{\lambda}} - \dot{\underline{z}}^*, \tag{21''}$$

$$\dot{\pi} = -H_{\underline{xx}}\omega - H_{\underline{zx}}\sigma - H_{\underline{px}}\pi - H_{\underline{qx}}\eta - H_{\underline{ux}}\nu - H_{\underline{\lambda x}}, \tag{22'}$$

$$0 = -H_{\underline{xz}}\omega - H_{\underline{pz}}\pi - H_{\underline{qz}}\eta - H_{\underline{\lambda z}} - \dot{\underline{q}}^*, \tag{22''}$$

$$0 = H_{\underline{xu}}\omega + H_{\underline{pu}}\pi + H_{\underline{qu}}\eta + H_{\underline{uu}}\nu + H_{\underline{\lambda u}}, \tag{23}$$

where H is the Hamiltonian, $H = -\frac{1}{2}(Q+u'Ru) + p'(g_1 + B_1 z + C_1 u) + q'(G_1 + B_2 z + C_2 u)$.

The determination of boundary conditions for $\omega$ and $\pi$ deserves a separate discussion [1]. If the sensitivities $\omega$ and $\pi$ are calculated for some $\lambda > 0$, that is if no change in equation order occurs, the boundary conditions for the corresponding sensitivity equations are zero. This is evident since the initial conditions $x_o$ and $z_o$ and final conditions $p(T) = 0$ and $q(T) = 0$ do not depend on $\lambda$. The situation is different when $\lambda$ changes from $\lambda > 0$ to $\lambda = 0$. Then the model order reduces to n and the initial condition $z_o$ and the final condition $q(T)$ are ignored in the low-order equations (15) and (16). This change indicates that in the low-order model $\omega$ and $\pi$ may have non-zero boundary values. It can be shown [1] that the final value $\pi(T)$ is zero since the performance index (10) does not explicitly contain the variable z, while the initial value $\omega(t_o)$ is given by

$$\omega(t_o) = \int_{t_o}^{\infty} B_1(x_o,0,t_o)(z(\tau) - \varphi(x_o,\underline{u}^*(t_o),t_o))d\tau, \qquad (24)$$

where $\varphi(x_o,\underline{u}^*(t_o),t_o) = -B_2^{-1}(x_o,0,t_o)(G_1(x_o,0,t_o) + C_2(x_o,0,t_o)\underline{u}^*(t_o))$, and the variable $z(\tau)$ is the solution of the equation $\frac{dz}{d\tau} = G_1(x_o,0,t_o) + C_2(x_o,0,t_o)\underline{u}^*(t_o) + B_2(x_o,0,t_o)z(\tau)$ with the initial condition $z(\tau)\big|_{\tau=t_o} = z_o$.

Since the matrices $H_{qz} = B_2'$ and $H_{uu} = -R$ are nonsingular, the equations (21''), (22''), and (23) can be solved for $\sigma$, $\eta$, and $\nu$, so that (21') and (22') can be rewritten in the form

$$\dot{\omega} = A_1\omega + A_2\pi + \delta_1, \qquad (25')$$

$$\dot{\pi} = A_3\omega - A_1'\pi + \delta_2, \qquad (25'')$$

where the expressions for the matrices $A_1$, $A_2$, $A_3$ and the vectors $\delta_1$ and $\delta_2$ can be obtained from (21), (22), and (23). Linear equation (25) with the initial condition $\omega(t_o)$ and with the final condition $\pi(T) = 0$ represents a well known two-point boundary value problem [5]. With

$$\pi = -M\omega + \gamma \qquad (26)$$

this problem is reducible to a final value problem for the nxn dimensional symmetric matrix Riccati equation

$$\dot{M} = -MA_1 - A_1'M + MA_2M - A_3 \; ; \quad M(T) = 0, \qquad (27)$$

and an n-dimensional linear vector equation

$$\dot{\gamma} = (MA_2 - A_1')\gamma + M\delta_1 + \delta_2; \ \gamma(T) = 0. \tag{28}$$

Once the matrix M and the vector $\gamma$ are found, the substitution of (26) into (25') allows the vector $\omega$ to be obtained as the solution of the equation

$$\dot{\omega} = (A_1 - A_2M)\omega + A_2\gamma + \delta_1 \tag{29}$$

for the initial condition given by (24).

Then in view of (23), the control sensitivity $\nu$ can be evaluated from

$$\nu = \underline{R}^{-1}((D_2 - D_1M)\omega + D_1\gamma + H_{\lambda\underline{u}} - H_{\underline{qu}}(\underline{B}_2')^{-1}(H_{\lambda\underline{z}} + \underline{\dot{q}}^*)), \tag{30}$$

where the expressions for the matrices $D_1$ and $D_2$ can be easily obtained from (21), (22), and (23).

Hence, the obtaining of $\frac{\partial u^*}{\partial \lambda}(t,\lambda)$ at $\lambda=0$ consists of the following steps:

1. Evaluation of the integral (24).
2. Solving of nxn dimensional symmetric Riccati equation (27).
3. Solving of n dimensional linear equations (28) and (29).

Since the evaluation of the integral (24) is straightforward, it can be seen from (27), (28), and (29) that the above singular perturbation method requires for an (n+m)th order plant essentially the same amount of computation as the neighboring optimum [6,7,8] design for an n-th order plant.

A Design Example:

A speed-tracking system is shown in Fig. 1. Usually the time constants of the amplidyne excitation winding $T_1$, amplidyne transverse winding $T_4$, and the generator excitation winding $T_2$ are small compared to the time constant of motor excitation winding $T_3$ and the mechanical time constant. Thus $T_1$, $T_2$, and $T_4$ are considered as small parameters and expressed as functions of $\lambda$, $T_1 = \lambda$, $T_2 = \lambda$, $T_4 = 0.5\lambda$. The high-order (n+m = 5) equation (4) is

$$\dot{x}_1 = (b_2x_2z_1 - b_1x_1),$$

$$\dot{x}_2 = (-c_1x_2 + b_3z_3),$$

$$\lambda \dot{z}_1 = (-2+\lambda b_1+\lambda c_1) b_7 x_1 x_2 - (b_2 x_2^2 z_1 + b_3 x_1 z_3)\lambda b_7 - 2z_1 + 2b_6 z_2,$$

$$\lambda \dot{z}_2 = -z_2 + b_5 u_1,$$

$$\lambda \dot{z}_3 = -z_3 + b_4 u_2, \tag{31}$$

where $x_1$, $x_2$, $z_1$ are respectively angular velocity, excitation current and armature current of the motor, $z_2$, $z_3$ are excitation currents of the amplidyne and generator respectively. $u_1$ and $u_2$ are control voltages. $b_i$, i=1 to 7 are gain coefficients of various components and $c_1 = \frac{1}{T_3}$. The values of various constants are $b_1 = b_2 = b_3 = 1$, $b_4 = b_5 = .05$, $b_6 = 20$, $b_7 = .1$, $c_1 = 0.5$, $x_1(o) = -100$, $x_2(o) = 5$, $z_1(o) = -20$, $z_2(o) = -2.5$, $z_3(o) = 2.5$, and $\lambda = .1$.

It is desired that the speed $x_1$ follows a given trajectory $x_d$,

$$x_d = \begin{cases} -100 + \frac{400}{3} t & \text{for } 0 \le t \le 1.5, \\ 100 & \text{for } 1.5 \le t \le 5. \end{cases}$$

It is further desired that the excitation current $x_2$ and the control $u_2$ remain close to their nominal values 5 and 50 respectively.

The method developed in this paper is applied to obtain the approximation (8) of the optimal controls $u_1^* = u_1^*(t,\lambda)$ and $u_2^* = u_2^*(t,\lambda)$ which minimize the chosen performance index

$$J = \tfrac{1}{2}\int_o^5 ((x_1-x_d)^2 + 0.25(x_2-5)^2 + 0.025(u_1^2 + (u_2-50)^2))dt.$$

For $\lambda=0$ the high-order equation (31) reduces to the low-order (n=2) equation,

$$\dot{\underline{x}}_1 = b_2(b_5 b_6 \underline{u}_1 \underline{x}_2 - b_7 \underline{x}_1 \underline{x}_2^2) - b_1 \underline{x}_1,$$

$$\dot{\underline{x}}_2 = (-c_1 \underline{x}_2 + b_3 b_4 \underline{u}_2).$$

The variables $\underline{z}_1$, $\underline{z}_2$, and $\underline{z}_3$ are given by

$$\underline{z}_1 = (b_5 b_6 \underline{u}_1 - b_7 \underline{x}_1 \underline{x}_2), \quad \underline{z}_2 = b_5 \underline{u}_1, \quad \underline{z}_3 = b_4 \underline{u}_2.$$

The high-order optimal design based on the system equation (31) requires the solution of a tenth order two point boundary value problem, whereas the singular perturbation method requires the solution of only a fourth order boundary value problem as in low-order design. The additional amount of computation needed for steps 1, 2, and 3 for the evaluation of the second terms $\frac{\partial u_1^*}{\partial \lambda}$ and $\frac{\partial u_2^*}{\partial \lambda}$ of the optimally sensi-

tive control (8) is negligible compared to the solution of boundary value problem. Such small additional computation resulted in a considerable improvement of the performance, as it can be seen in Fig. 2. For as large values of $\lambda$ as 0.4 (in the problem only $\lambda = 0.1$ is required), the performance of the system with the optimally sensitive control $u_1^*(t,0) + \frac{\partial u_1^*(t,\lambda)}{\partial \lambda} \cdot \lambda$, $u_2^*(t,0) + \frac{\partial u_2^*(t,\lambda)}{\partial \lambda} \cdot \lambda$ is close to the performance achieved with the high-order optimal control $u_1^*(t,\lambda)$, $u_2^*(t,\lambda)$ and for the order of magnitude better than the performance achieved with low-order optimal control $u_1^*(t,0)$, $u_2^*(t,0)$.

## CONCLUSION

For a class of nonlinear systems it is proved that the optimal control $u^*(t,\lambda)$ is continuous and differentiable with respect to a small parameter $\lambda$ which cahnges the order of the plant controlled. This result made it possible to find a near optimum control for a high-order plant by using a low-order model. This means that a $2(n+m)$-dimensional boundary value problem is approximately solved by a $2n$-dimensional boundary value problem and a final value problem for an nxn dimensional matrix Riccati equation. Thus computation required for a high-order plant is equivalent to that needed for a neighboring optimum design for a low-order plant.

A second-order near optimum design of a fifth-order nonlinear system illustrates applicability and computational simplicity of the method developed.

It is expected that in future research a result similar to the theorem proved here can be obtained for a broader class of nonlinear systems.

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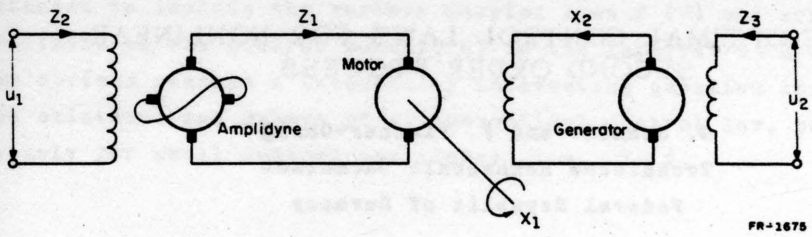


Fig. 1. Schematic diagram of a speed-tracking servo.

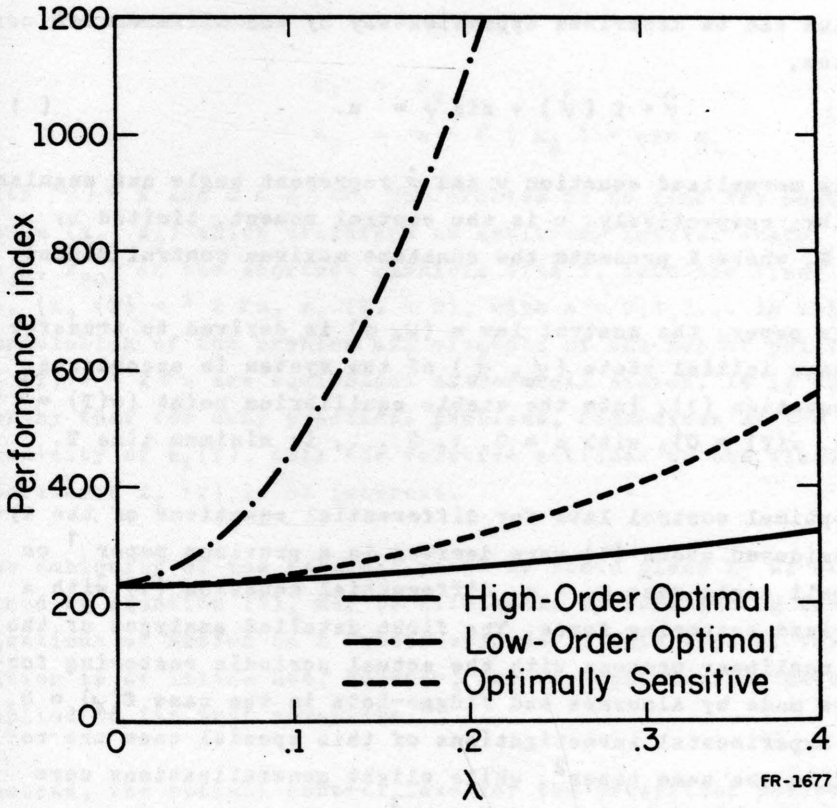


Fig. 2. Performance index for the three designs.

# TIME-OPTIMAL CONTROL LAWS FOR NONLINEAR SECOND ORDER PROCESS

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## I. Introduction

It is a well-known mechanical fact that the pitch motion of a satellite in circular orbit and the vibration of a mathematical pendulum can be described approximately by the differential equation,

$$\ddot{\varphi} + f(\dot{\varphi}) + \sin\varphi = u. \qquad (1)$$

In this normalized equation $\varphi$ and $\dot{\varphi}$ represent angle and angular velocity, respectively. u is the control moment, limited by $|u| \leq K$, where K presents the constant maximum control moment.

In this paper, the control law $u(\varphi, \dot{\varphi})$ is derived to transfer any known initial state $(\varphi_0, \dot{\varphi}_0)$ of the system in accordance with equation (1), into the stable equilibrium point ($\varphi(T) = \pm 2\pi n$, $\dot{\varphi}(T) = 0$), with $n = 0, 1, 2 \ldots$, in minimum time T.

Time-optimal control laws for differential equations of the type considered above (1) were derived in a previous paper [1] on the small angle case $\varphi$, i.e. differential equation (1) with a linearized restoring force. The first detailed analyses of the above nonlinear process with the actual periodic restoring force were made by Almuzara and Flügge-Lotz in the case $f(\dot{\varphi}) = 0$, $K > 1$. Experimental investigations of this special case are reported in the same paper[2], while slight generalizations were further described by Lee and Marcus[3].

It is intended through this paper to present a complete survey on time-optimal control laws for plants according to equation (1). Earlier investigations of this subject were only partial. In the scope of this paper, however investigations have been

extended to include the various damping laws f ($\dot{\varphi}$) and arbitrary limits on the control moment, u, in the interval $0 < K < \infty$. For obvious reasons a technically interesting question concerns the existence and nature of a time-optimal control law, particularly for small control amplitudes, i.e., $K < 1$.

## II. Theoretical Considerations

### 1. General Problems

For further investigations equation (1) should be transformed into the state form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u - f(x_2) - \sin x_1 \end{aligned} \qquad (2)$$

with $|u| \leq K$ and $0 < K < \infty$. The problem is to find the control law $u(x_1, x_2)$ which transfers an arbitrary initial state $(x_{10}, x_{20})$ in the shortest possible time T, into the final state, $(x_1(T) = \pm 2\pi n, x_2(T) = 0)$, with $n = 0,1,2\ldots$ In this formulation of the problem all elements of the set of points $x_1(T) = \pm 2\pi n$ are equivalent as terminal states. It is noteworthy that for many practical problems, regardless of the periodicity of $x_1(t)$, only the relative attitude to the final coordinate $x_1(T)$ is of interest.

The ambiguity of the terminal point in state plane $R^2$ as defined by equation (2), may be eliminated by formulating the equations of motion on a "state cylinder". This type of formulation is of little use, however, for the optimization method applied to its best advantage.

Instead, the optimal control laws for the prescribed periodical terminal set are simply derived through investigation of the results for optimal control of the same system, in which, however, a non-periodical terminal state ($x_1(T) = 0$, $x_2(T) = 0$) in $R^2$ is used. The main step in this type of derivation is the construction of so-called indifference curves on the basis of the field of isochrones. These indifference curves limit in $R^2$

periodically distributed equivalent subregions, as will become clearer later.

## II.2. Conditions given by the Maximum Principle

With Pontrjagin's Maximum Principle[1] one derives from:

$$\max_{|u|\leq K} H = \max_{|u|\leq K} \left[1 + p_1 x_2 - p_2 f(x_2) - p_2 \sin x_1 + p_2 u\right] \quad (3)$$

for the case of non-singular control the following "maximal control"

$$u(t) = K \operatorname{sign} p_2(t) \quad (4)$$

where $p_1$ and $p_2$ satisfy the co-state equations:

$$\dot{p}_1 = p_2 \cos x_1$$
$$\dot{p}_2 = -p_1 + p_2 \frac{\partial f(x_2)}{\partial x_2} \quad (5)$$

Further investigation shows an analytical solution, to the two-point boundary value problem as given in equation (2),(4), and (5), cannot be completely achieved. Therefore the well-known method of backward time integration is utilized on an analog computer. Application of this procedure requires an essential assumption. For non-zero boundary values of the co-state system, equation (5), the zeros of the solution component $p_2(t)$ must be isolated and of finite number (non-singular control). This assumption is validated by the application of a comparison theorem[4] for second order differential equations. Thus, from the condition $p_2(t) = 0$ the switching curves for the control u $(x_1, x_2)$ in $R^2$ can be constructed. However these must be considered only as possible switching curves, since the Maximum Principle defines the necessary conditions for optimality.

## II .3. Determination of true switching curves

These definitions are included to clarify further investigations:
Isochrones: The locus in the state space $R^2$ of all initial points $(x_{10}, x_{20})$ that can be brought to the terminal state $(x_1(T), x_2(T))$, in the same time T, provided that all possible switching

curves found through application of the Maximum Principle are taken into account.

Indifference curves: The locus in the state space $R^2$ of all initial points $(x_{10}, x_{20})$ that can be brought to the terminal state $(x_1(T), x_2(T))$ on two different paths in the same minimum time.

From the latter definition derives the correlation, indifference curves separate regions in the state plane where the control variable $u(x_1, x_2)$ has different signs.

The basic method applied in this work for the construction of indifference curves and true switching curves is explained in detail for the special case $f(x_2) = 0$, $K > 1$, $x_1(T) = x_2(T) = 0$, formerly treated by Almuzara and Flügge-Lotz[2]. The results are illustrated in fig. 1.

Backwards time integration produces these possibilities for switching curves: the zero trajectory $S^+$, $S^-$ and the connected curve of $S^i_{Ra}$, $S^i_{Rb}$ and the curve of $S^i_{La}$, $S^i_{Lb}$ each approaching infinity.

As a result of the symmetry of equation (2) in $x_1$ and $x_2$ only a half-plane is considered in fig. 1 and the following discussions.

The field of isochrones for the switching curves shown in fig. 1 proves that for certain initial states $(x_{10}, x_{20})$ an ambiguity exists with respect to time T in reaching the terminal point. If in these states only the trajectories reaching the terminal point in minimal time are admitted, then $S^i_{Ra}$, $S^i_{La}$ and $S^+$, $S^-$ prove to be the only true switching curves.

Furthermore, the self-intersection of the isochrones set for a certain time T results in further ambiguities. That is, the terminal point can be reached in the same minimal time, however with different switching sequences of the control input.

The locus of such self intersections results in the indifference curves $G^i_R$, $G^i_L$ which connect the true switching curves $S^i_{Ra}$, $S^i_{La}$. These findings coincide with analytical investigations[2].

So far only the terminal point case ($x_1(T) = 0$, $x_2(T) = 0$) has been considered. However, as explained earlier in this discussion, the more specific point set ($x_1(T) = \pm 2\pi \cdot n$, $x_2(T) = 0$) is admitted as the terminal state. Along the same lines a field of isochrones can be constructed for this case. It can be shown that, generally, further parts of originally derived true switching curves disappear and another, new indifference curve H appears.

For example, in the case ($f(x_2) = 0$, $K > 1$) only the zero trajectory remains a switching curve. This switching curve and its corresponding indifference curve are illustrated in the representation of the unrolled "state cylinder" section in fig. 2.

## III. General Results

The methods as discussed in Section II, with respect to a special case of equation (2), are now applied to the solution of the time-optimal control problem for other cases of equation (2). The following three different cases will be considered.

1. undamped system: $f(x_2) = 0$
2. linear damping law: $f(x_2) = \alpha x_2$
3. quadratic damping law: $f(x_2) = \beta x_2 |x_2|$

The magnitude limit K of the control variable u is supposed to be arbitrary, i.e. $0 < K < \infty$. The following investigation will show that qualitatively different switching laws result according to the magnitude of K. However, certain parallels can be drawn to the time-optimal switching laws for systems with linearized restoring force, i.e.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u - f(x_2) - x_1, \end{aligned} \qquad (6)$$

The essential results of the investigations are plotted in figs. 3 - 7. A detailed discussion for case 1, $f(x_2) = 0$, is given. Cases 2 and 3 produce qualitatively similar results.

## III.1 Undamped system: $f(x_2) = 0$

### 1.1 High control amplitudes, $K \geq 1$

Under this condition, the switching curves are valid as sketched in figures 2 and 4a respectively. These switching curves are pieced together exclusively from parts of zero trajectories which satisfy the equation:

$$S^+ \cup S^-: x_2 = -\operatorname{sign} x_1 \sqrt{2(K\,|x_1| + \cos x_1 - 1} \qquad (7)$$

Only one analytically indeterminable indifference curve exists, which lies between the switching curves. Thus, the number of switchings of the control variable for this case is at most one.

### 1.2 Low control amplitudes, $0 < K < 1$

For the maximum control amplitude $K < 1$ there are qualitatively different switching curves. These curves are greatly determined by the absolute value of K, while for the amplitude $K \geq 1$ no qualitative changes in the behavior of the switching curve take place.

Most important for $K < 1$ is that the switching curves consist of switching arcs forming a series connection. As K decreases the number of arcs increases inversely. See figures 4b to 4d. Here the "points of convergence" of the arcs lie on the $x_1$-axis.

The evolution of true switching curves influenced by K will now be discussed in greater detail. If K is reduced from $K = 1$, in addition to the zero trajectories $S^+$, $S^-$ the curves $T^o_{Ra}$, $T^o_{Rb}$, $T^o_{La}$, $T^o_{Lb}$ arise as possible switching curves. The field of isochrones shows finally that only parts of $T^o_{Ra}$, $T^o_{La}$ and a restpiece of the zero trajectories $S^+$, $S^-$ do indeed form true switching curves, figure 4b. Nevertheless, the components $S^-$ and $T^o_{Ra}$ or $S^+$ and $T^o_{La}$ respectively are not connected. The uniqueness of the switching law is accomplished by the occurrence of new, additional components of the indifference curve, $H^o_R$, $H^o_L$. A typical situation is sketched in figure 3, an enlarged section of figure 4b.

Further reduction of K to $K_1$ leads to the case in which the zero trajectory $S^-$ intersects the negative $x_1$-axis at the point $x_{11}$ and branches off in a sort of separatrix. (All considerations are restricted to the left half-plane due to existing symmetry). Furthermore, the possible switching curve $T^0_{Ra} \cup T^0_{Rb}$ terminates at the same point $x_{11}$. The true switching curves, however, are still separated as illustrated in fig. 3.

If K is diminished still further, i.e. $K < K_1$, true switching curves converge in a point $x_1 > x_{11}$ thus forming a connected switching curve, as in figure 4 c.

Now the fundamental evolutionary tendency of switching curves is described, as it continues similarly with further diminuation of K. For example, in fig. 4d, $T^1_{Ra} \cup T^1_{Rb}$ arise as further possible switching curves, as in fig. 4b, $T^0_{Ra} \cup T^0_{Rb}$. Besides, from the same example and the form $T^0_{Ra}$, a new switching arc appears. Finally, for $K = K_2$, this new arc $T^0_{Ra}$ intersects $T^1_{Ra}$ at the point $x_{12}$ on the $x_1$-axis. The construction of true switching curves and indifference curves, as described before, follows from the field of isochrones.

From the differential equation of the trajectories in $R^2$ as it follows from equation (2),

$$\frac{dx_2}{dx_1} = \frac{u - \sin x_1}{x_2} \qquad (8)$$

a general contingency can be derived between $K_i$ and $x_{1i}$. Since the separatrix branches off at the point ($x_1 = x_{1i}$, $x_2 = 0$) into two parts with finite gradient $dx_2 / dx_1$, the following corralary may be obtained from equation (8):

$$-K_i - \sin x_{1i} = 0, \quad i = 1,2,3... \qquad (9)$$

Thus, intersection points $x_{1i}$ of the separatrix lie in the interval $( -\pi , -\frac{\pi}{2} )$ for $0 < K_i < 1$.
From this fact follows, on the other hand, that in the limiting case ( $K \longrightarrow 0$) there lies an unlimited number of seamless connected switching arcs along the $x_1$-axis. Consequently, the maximal number of switchings of the control of variable is (i + 1).

## III.2. Linear damping law: $f(x_2) = \alpha \cdot x_2, \alpha > 0$

The first differentiation of cases ($\alpha \geq 2$) is obtained from consideration of the differential equation for the costate variable $p_2$, as it follows from equation (5),

$$\ddot{p}_2 - \alpha \dot{p}_2 + p_2 \cos x_1 = 0 \qquad (10)$$

Through variable transformation, i.e. $p_2 = v \cdot \exp(\alpha/2)t$, the last equation can be converted into:

$$\ddot{v} + v\left(-\frac{\alpha^2}{4} + \cos x_1(t)\right) = 0 \qquad (11)$$

From a comparison theorem[4] for estimating the solutions of differential equations, it is found that for $\alpha \geq 2$ the intermediate variable $v(t)$ and $p_2(t)$, each has at most one zero. Thus, independent of K there exists at most one switching in the case of $\alpha \geq 2$, as in the optimal switching law. Further analytic results are difficult to find, since a closed analytic solution of the differential equation (2) with $f(x_2) = \alpha x_2$ has not to date been obtained.

## 2.1. Overdamped system: $\alpha \geq 2$; K arbitrary

Switching and indifference curves resemble qualitatively those curves of figure 4a. That is, only the zero trajectories are possible and true switching curves at the same time, figure 5a. According to equation (11) at most one switching of the optimal control function is possible.

## 2.2 Underdamped system: $\alpha < 2$; K arbitrary

In analogy to the above undamped system, different values of K result in a qualitatively similar sequence of cases, as illustrated in figures 5b - 5d. Corresponding cases for the various systems are noted in table 1:

| | switching curves: | | |
|---|---|---|---|
| $f(x_2)=$ | $S^- \cup S^+$ | $T^0_{Ra} \cup S^- \cup S^+ \cup T^0_{La}$ | $(\bigcup_{p \le i} T^p_{Ra}) \cup S^- \cup S^+ \cup (\bigcup_{p \le i} T^p_{La})$ $i=1,2,3..;p=0,1,2..$ |
| 0 | $K \ge 1$ | $1 > K \ge K_1$ | $K_i > K \ge K_{i+1}$ |
| $\alpha x_2 \vert x_2 \vert$ | $K \ge K_o(\alpha)$ | $K_o(\alpha) > K \ge K_1(\alpha)$ | $K_i(\alpha) > K \ge K_{i+1}(\alpha)$ |
| $\beta x_2 \vert x_2 \vert$ | $K \ge K_o(\beta)$ | $K_o(\beta) > K \ge K_1(\beta)$ | $K_i(\beta) > K \ge K_{i+1}(\beta)$ |

Table: 1

The interpretation of the results may proceed as in III.1., the undamped system $f(x_2) = 0$. Equation (9) holds especially valid. The functional dependence of $K_o(\alpha)$ and $K_i(\alpha)$ was experimentally found; see figure 6. According to the number of switching arcs, there is a maximum of (i + 1) switchings of the optimal control function.

## III.3. Quadratic damping law: $f(x_2) = \beta x_2 \vert x_2 \vert, \beta > 0$.

In contrast to the linear damping law, III.2., no fundamental differentiation of cases exists with respect to the damping coefficient $\beta$. As for the undamped and linearly damped systems with $0 < \alpha < 2$, qualitatively similar results are found. See table 1 for comparison. The fundamental functional dependence of $K_o(\beta)$ and $K_i(\beta)$, $\beta > 0$ and arbitrary, is illustrated in figure 7. A detailed plot of switching and indifference curves is not necessary here. Instead, an equation for the trajectories can be formulated. It follows from:

$$\frac{dx_2}{dx_1} = \frac{u - \sin x_1}{x_2} - 2\beta \vert x_2 \vert \qquad (12)$$

for the switching curve $S^+ \cup S^-$, the resultant of the zero trajectory components:

$$S^+ \cup S^- : \qquad (13)$$

$$x_2 = -\operatorname{sign} x_1 \sqrt{\frac{K}{\beta}\left(e^{2\beta \vert x_1 \vert} - 1\right) + \frac{2}{1+4\beta^2}\left(\cos x_1 + 2\beta \cdot \sin \vert x_1 \vert - e^{2\beta \vert x_1 \vert}\right)}.$$

Furthermore, from equation (9), (12), and (13), the dependence $K_i(\beta)$ can be computed. As in previously described cases, the maximal number of optimal control function switchings is (i + 1).

## IV. Conclusions

From the investigation of the time-optimal control of a system according to equation (2), whereas the magnitude of the control variable u is limited to K, the following conclusions can be drawn:

(1) For any initial state $(x_{10}, x_{20})$, except for initial points on indifference curves, there exists a unique time-optimal feedback control law $u(x_1, x_2)$ for the periodically distributed terminal state $(x_1(T) = \pm 2\pi n, x_2(T) = 0)$.

(2) Optimal control function u(t) is a piecewise constant function of magnitude K. The number of switchings is finite. The switching law (control law) is uniquely determined by the switching curves in the state plane.

(3) It is correlative to compare the results of this investigation with the well-known results for time-optimal control systems with linearized restoring force as described by equation (6):

   a) The shapes of switching and indifference curves are basically different in linear and non-linear cases. Thus, the number of switchings of the optimal control function, needed to transfer the same initial state to a final state (e.g. origin), is commonly reduced to a few switchings in the nonlinear case.

   b) For most initial states, the optimal time needed to reach the origin of the state plane is smaller for systems in accordance with a nonlinear periodical restoring force.

   c) The switching laws for the processes with linear and nonlinear restoring force coincide approximately at the terminal state, i.e. near the origin of the state plane.This is especially applicable to small values of the control magnitude K.

(4) Finally, these investigations of the exact nonlinear process result in many cases in switching laws u $(x_1, x_2)$ simpler with respect to practical realizability. On the other hand, the results prove that an investigation of a linearized model of the actually nonlinear process can be of conditional value only. Besides, the simpler switching curves, in the nonlinear system case, indicate that the performance of the control system will not be significantly altered when the time-optimal control law is replaced by a simple suboptimal realization of the switching and indifference curves. Analytical and experimental studies verify this assumption.

(5) Similar results as reported here, can be given for other terminal states, e.g. the time-optimal control law for a transfer of any initial point to the unstable equilibrium point $(x_1(T) = \pi \pm 2\pi n,\ x_2(T) = 0)$.
Furthermore, the same type of methods and arguments were used to obtain optimal control laws for the nonlinear process, when the criterion for evaluating system performance is minimal fuel consumption (time integral over absolute value of control moment u) or minimal fuel consumption-plus time. These results are significant if cold gas systems are assumed to generate the control moment u.

Acknowledgements

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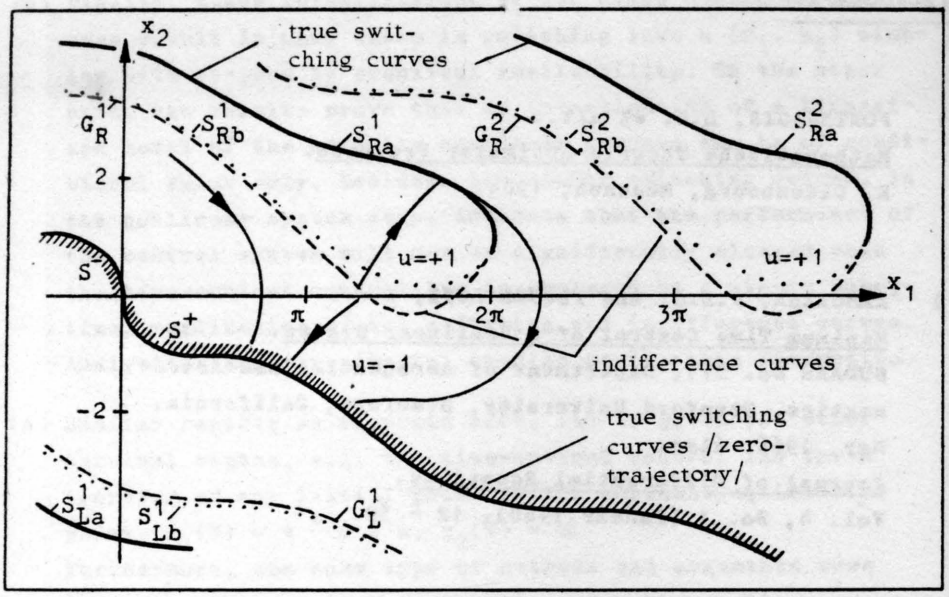


Fig. 1: Switching and indifference curves for $f/x_2/=0$, K=1, with terminal state $x_1/T/ = x_2/T/ = 0$

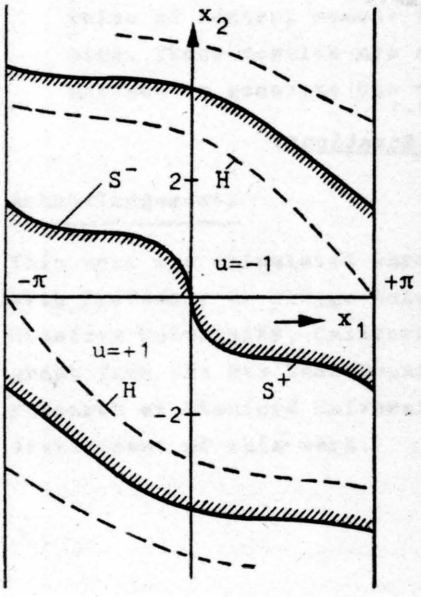


Fig. 2: Switching and indifference curves for K=1, $f/x_2/ = 0$, with terminal state $x_1/T/ = \pm 2\pi n$, $x_2/T/ = 0$

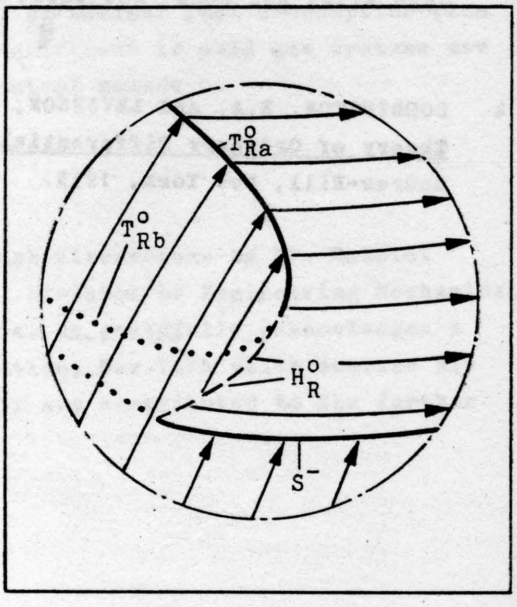


Fig. 3: Enlarged section of figure 4b

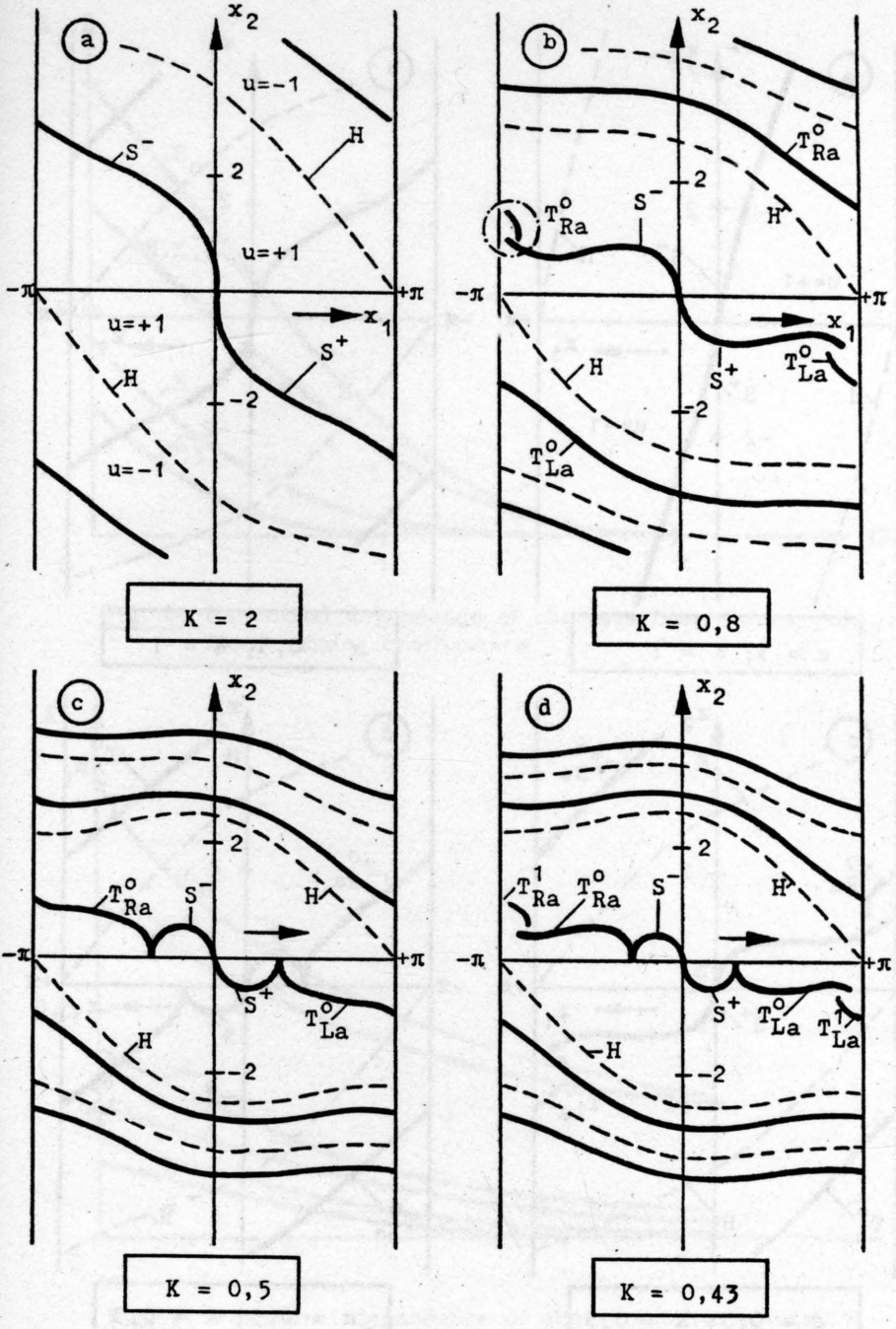


Fig. 4: Switching and indifference curves for the undamped system, $f\,/x_2/ = 0$

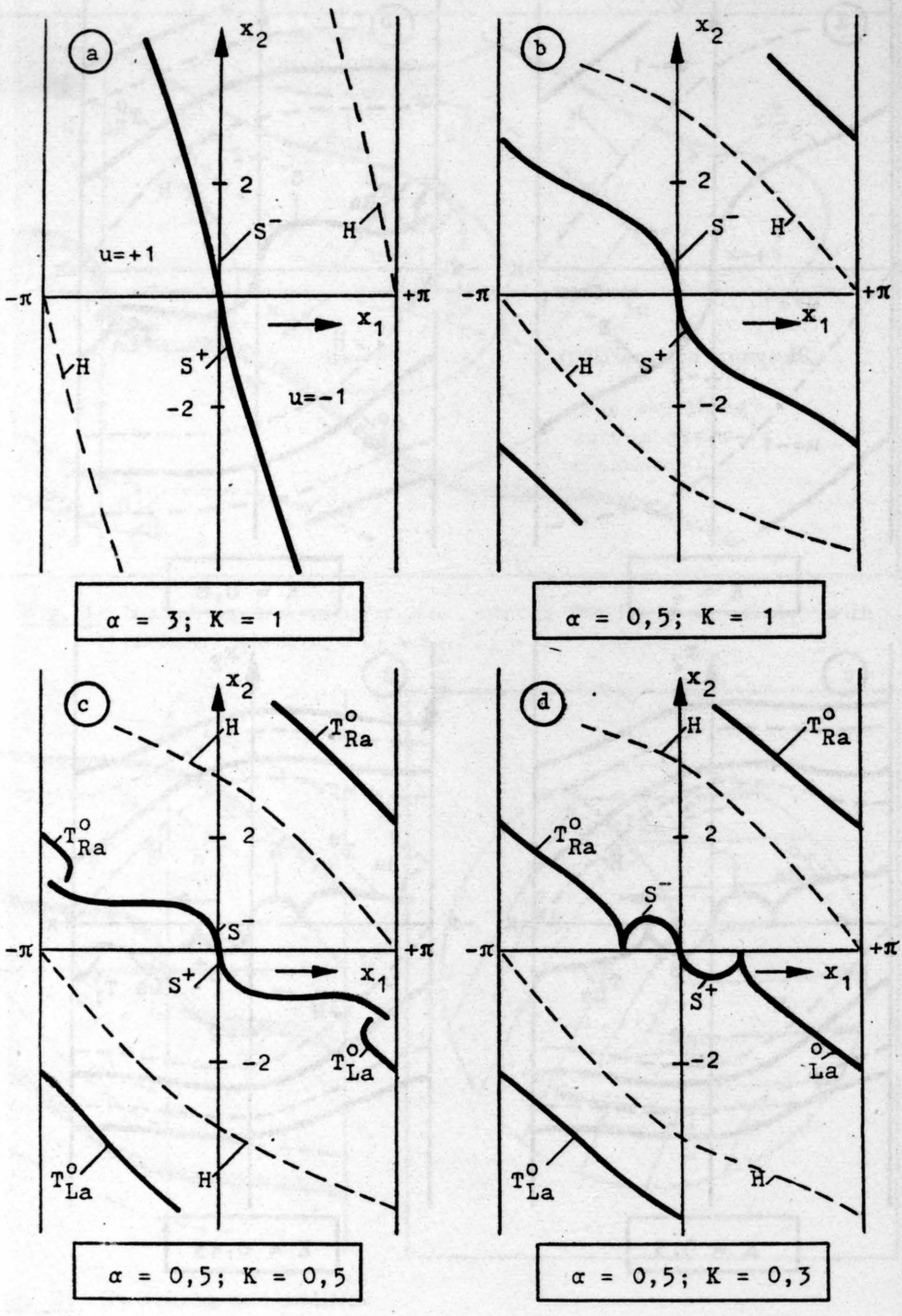


Fig. 5: Switching and indifference curves for system with linear damping law, $f/x_2/ = \alpha x_2$

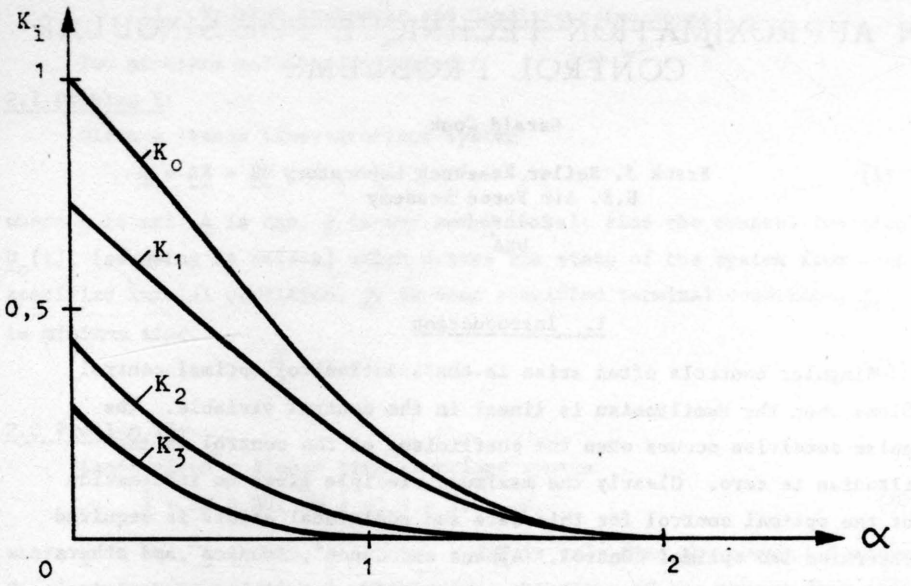


Fig. 6: Functional dependence of characteristic values of K from damping coefficient $\alpha$

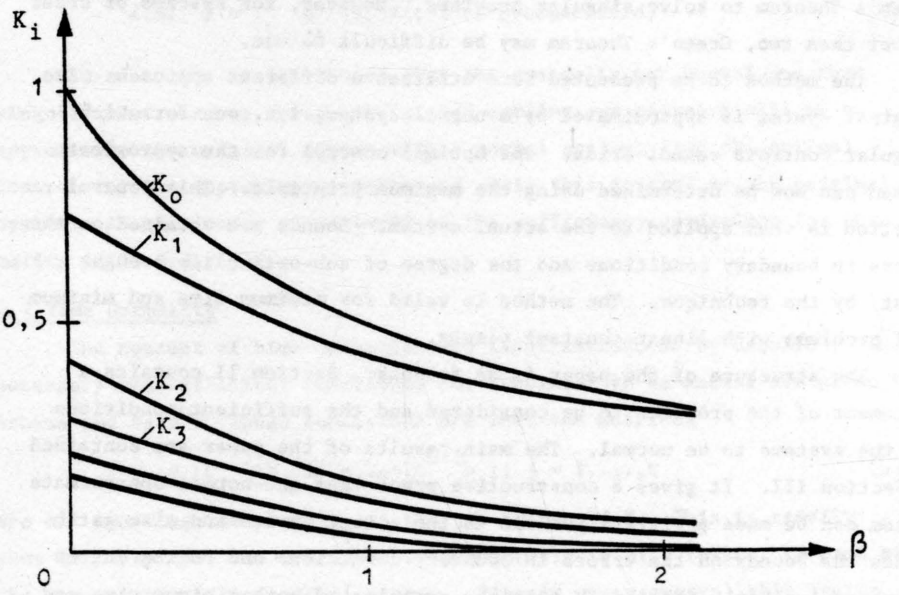


Fig. 7: Functional dependence of characteristic values of K from damping coefficient $\beta$

# AN APPROXIMATION TECHNIQUE FOR SINGULAR CONTROL PROBLEMS

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## I. Introduction

Singular controls often arise in the solutions of optimal control problems when the Hamiltonian is linear in the control variable. The singular condition occurs when the coefficient of the control in the Hamiltonian is zero. Clearly the maximum principle gives no information about the optimal control for this case and additional effort is required to determine the optimal control. Athans and Canon[1], Johnson[2] and others have done work in this area utilizing the maximum principle and deriving additional necessary conditions for an optimal solution. Specific problems have been solved. Miele[3] and later Hermes and Haynes[4] have used Green's Theorem to solve singular problems. However, for systems of order higher than two, Green's Theorem may be difficult to use.

The method to be presented here utilizes a different approach. The original system is approximated by a normal system, i.e. one for which singular controls cannot arise. The optimal control for the approximate system can now be determined using the maximum principle. This control function is then applied to the actual system. Bounds are obtained on the errors in boundary conditions and the degree of sub-optimality brought about by the technique. The method is valid for minimum time and minimum fuel problems with linear constant plants.

The structure of the paper is as follows: Section II contains a statement of the problems to be considered and the sufficient conditions for the systems to be normal. The main results of the paper are contained in Section III. It gives a constructive proof that the normal approximate system can be made arbitrarily close to the actual system and also establishes the bounds on the errors in boundary conditions and on the degree of sub-optimality. Section IV presents examples of both minimum time and minimum fuel problems. Section V points out related areas for possible future research.

## II. Problem Statement and Conditions for Normality

Two problems will be considered.

### 2.1 Problem I:

Given a linear time-invariant system

$$\dot{\underline{X}} = \underline{\underline{A}}\underline{X} + \underline{\underline{B}}\underline{U} \; ; \; |u_j| \leq 1 \; j = 1,\ldots,r \tag{1}$$

where $\underline{x}$ is nx1, $\underline{A}$ is nxn, $\underline{B}$ is nxr and $\underline{U}$ is rx1; find the control function $\underline{U}_o(t)$, (assuming it exists) which drives the state of the system from some specified initial condition, $\underline{\theta}$, to some specified terminal condition, $\underline{\phi}$, in minimum time.

$$\text{i.e., } J = \int_o^T dt; \; T \text{ is free.} \tag{2}$$

### 2.2 Problem II:

Again given a linear time-invariant system

$$\dot{\underline{X}} = \underline{\underline{A}}\underline{X} + \underline{\underline{B}}\underline{U}; \; |u_j| \leq 1 \; j = 1,\ldots,r \tag{3}$$

where $\underline{x}$ is nx1, $\underline{A}$ is nxn, $\underline{B}$ is nxr, and $\underline{U}$ is rx1; find the control function $\underline{U}_o(t)$, (assuming it exists) which drives the state of the system from some specified initial condition, $\underline{\theta}$, to some specified terminal condition, $\underline{\phi}$, in fixed time and at the same time minimizes the amount of fuel used.

$$\text{i.e., } J = \int_o^T \sum_{i=i}^{r} |u_i| dt; \; T \text{ is prespecified.} \tag{4}$$

In each case it is assumed that the system is not normal and that singular controls may arise. As stated earlier our approach will be to approximate the original system with a normal system, find the optimal control for the approximate system and apply this control to the original system. At this point a statement of the sufficiency conditions for normality is in order.

### 2.3 Time Normality

The concept of time normality was first introduced by Lasalle[5]. The necessary and sufficient conditions for Problem I to be normal are given in Athans and Falb[6]. These conditions are that the matrices

$$\underline{G}_j = [\underline{b}_j \vdots \underline{Ab}_j \vdots \underline{A^2 b}_j \vdots \cdots \vdots \underline{A^{n-1} b}_j]; \; j = 1,\ldots,r \tag{5}$$

are all non-singular where $\underline{b}_j$ is the jth column of $\underline{B}$. This is readily seen to imply that the system must be controllable with respect to each of the components of the control vector. Clearly every controllable system is not time normal.

### 2.4 Fuel Normality

The conditions for fuel normality were established by Athans[7]. The

sufficient conditions for Problem II to be normal are that

i) the matrices $\underline{G}_j$ defined by (5) must all be non-singular and

ii) the matrix $\underline{A}$ must have no zero eigenvalues.

These conditions imply not only that the system must be controllable with respect to each component of the control vector, but also that the system must not have an integrating capability.

## III. Generating the Approximate System and Bounding Its Accuracy

### 3.1 The Approximate System

3.11 Theorem 1.

Any system of the form (1) can be approximated arbitrarily closely by a time-normal or fuel-normal system.

Proof: The time-normal approximation will be shown first. The proof is constructive. Part of the development will be similar to that of Gilbert[8]. We first approximate the matrix $\underline{A}$ by one which has distinct eigenvalues. From Bellman[9] we have that for any square matrix $\underline{A}$ there is another matrix $\hat{\underline{A}}$ which has distinct eigenvalues and which also has the property

$$\|\underline{A} - \hat{\underline{A}}\| \leq \epsilon \tag{6}$$

where

$$\|\underline{C}\| \triangleq \sum_{i,j}^{N} |c_{ij}| \tag{7}$$

and $\epsilon$ is arbitrarily small.

Let us denote

$$\underline{E} = \underline{A} - \hat{\underline{A}} . \tag{8}$$

The first approximation of (1) is then

$$\dot{\hat{\underline{X}}} = (\underline{A} + \underline{E})\,\hat{\underline{X}} + \underline{B}\underline{U} . \tag{9}$$

We now diagonalize the matrix $\underline{A} + \underline{E}$,

$$\underline{P}^{-1}\,\dot{\hat{\underline{X}}} = \underline{P}^{-1}(\underline{A} + \underline{E})\,\underline{P}\underline{P}^{-1}\,\hat{\underline{X}} + \underline{P}^{-1}\,\underline{B}\underline{U}. \tag{10}$$

Defining $\underline{P}^{-1}\,(\underline{A}+\underline{E})\,\underline{P} = \underline{\Lambda}$; (11) $\underline{P}^{-1}\,\hat{\underline{X}} = \underline{Y}$, (12)

and $\underline{P}^{-1}\,\underline{B} = \underline{D}$, (13)

we have $\dot{\underline{Y}} = \underline{\Lambda}\,\underline{Y} + \underline{D}\,\underline{U}$ . (14)

Partitioning $\underline{D}$ into columns,

$$\underline{D} = [\underline{d}_1 \vdots \underline{d}_2 \vdots \cdots \vdots \underline{d}_r], \tag{15}$$

we obtain $\dot{\underline{Y}} = \underline{\Lambda}\,\underline{Y} + \underline{d}_1 u_1 + \underline{d}_2 u_2 + \cdots + \underline{d}_r u_r$. (16)

It is easy to show that normality is independent of coordinates. Therefore

we can test for normality in the normal coordinates, (16).

Forming the $\underline{G}_j$'s of (5) for the system of (16) yields

$$\underline{G}_j = [\underline{d}_j \vdots \underline{\Lambda}\,\underline{d}_j \vdots \underline{\Lambda}^2\underline{d}_j \vdots \cdots \vdots \underline{\Lambda}^{n-1}\underline{d}_j] \tag{17}$$

and thus

$$\det [\underline{G}_j] = [\prod_{i\neq j}^{n} d_{ij}] \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{vmatrix} . \tag{18}$$

The Vandermonde determinant is non-zero for distinct $\lambda_1$'s which we have insured. Thus the condition for controllability with respect to $u_j$ is that the $d_{ij}$'s be non-zero for all i. For time normality which is controllability with respect to each component of the control vector, we then have the following requirement:

$$d_{ij} \neq 0;\ i = 1, \ldots, n \tag{19}$$

$$j = 1, \ldots, r .$$

Let us add to the matrix $\underline{D}$ a matrix $\underline{F}$ such that the matrix

$$\underline{H} = \underline{D} + \underline{F} \tag{20}$$

has no zero elements. Note that the elements of $\underline{F}$ can be arbitrarily small.

The approximate system is now

$$\dot{\hat{\underline{y}}} = \underline{\Lambda}\,\hat{\underline{y}} + (\underline{D} + \underline{F})\,\underline{u}. \tag{21}$$

This system is time normal. In the original coordinates (21) reads

$$\dot{\hat{\underline{x}}} = (\underline{A} + \underline{E})\,\hat{\underline{x}} + (\underline{B} + \underline{P}\,\underline{F})\,\underline{U} \tag{22}$$

where the elements of $\underline{E}$ and $\underline{F}$ can be made arbitrarily small.

Recalling the conditions for fuel normality, we see that the system of (21) is fuel normal if all of the diagonal elements of $\underline{\Lambda}$ are non-zero. These elements are distinct; therefore, at most one of them could be zero. A matrix $\underline{L}$ can be added to $\underline{\Lambda}$ such that

$$\hat{\underline{\Lambda}} = \underline{L} + \underline{\Lambda} \tag{23}$$

has no zero diagonal elements. Only one element of $\underline{L}$ need be non-zero and this element can be arbitrarily small. It must be such that the diagonal elements of $\hat{\underline{\Lambda}}$ are distinct. In the original coordinates, the fuel normal system is

$$\dot{\hat{\underline{x}}} = (\underline{A} + \underline{E} + \underline{P}\,\underline{L}\,\underline{P}^{-1})\,\hat{\underline{x}} + (\underline{B} + \underline{P}\,\underline{F})\,\underline{U} \tag{24}$$

where the elements of $\underline{E}$, $\underline{L}$ and $\underline{F}$ can be made arbitrarily small.

This completes the constructive proof that any linear, time-invariant system can be approximated arbitrarily closely by a time-normal or a fuel-normal system. In practice, for specific problems, one may find simpler ways of achieving the normal system. The examples in Section IV will demonstrate this.

## 3.2 Errors in Boundary Conditions

An obvious question which now arises is the following. Given that one has approximated the original system with one which is normal and determined the optimal control, $\underline{U}_o(t)$, for this system; what errors in boundary conditions will result if $\underline{U}_o(t)$ is applied to the original system? The answer is given in the following theorem.

### 3.21 Theorem II.

If $\underline{U}(t)$ drives the system of (24) from $\underline{\theta}$ to $\underline{\Phi}$ in time T, this same control function will drive the system of (3) from $\underline{\theta}$ to $\underline{\Phi} + \underline{\delta}(T)$ in time T where $\underline{\delta}(T)$ has the following property:

$$|\underline{\delta}(T)| \leq [\int_o^T |(\underline{E} + \underline{PLP}^{-1})\,\tilde{\underline{X}}(t) + \underline{PFU}(t)|dt]\, e^{K_1 T} \tag{25}$$

where $|\underline{Z}| \triangleq \max |Z_i|;\ i = 1, \ldots n$

and $K_1 \triangleq \underset{i=1,\ldots,n}{\text{MAX}} \sum_{j=1}^{n} |a_{ij}|$ .

Proof: Subtracting equation (3) from (24) yields the following equation:

$$\mathring{\underline{X}} - \mathring{\underline{X}} = \underline{A}\,(\tilde{\underline{X}} - \underline{X}) + (\underline{E} + \underline{PLP}^{-1})\,\tilde{\underline{X}} + \underline{PFU}(t). \tag{26}$$

Defining $\underline{\delta}(t) = \tilde{\underline{X}}(t) - \underline{X}(t)$

we have $$\mathring{\underline{\delta}}(t) = \underline{A}\,\underline{\delta}(t) + (\underline{E} + \underline{PLP}^{-1})\tilde{\underline{X}} + \underline{PFU}(t). \tag{27}$$

Making use of a lemma from Hartman[10] yields

$$|\underline{\delta}(t)| \leq [\int_o^T |(\underline{E} + \underline{PLP}^{-1})\,\tilde{\underline{X}}(t) + \underline{PFU}(t)|dt]\, e^{K_1 T}\ . \tag{28}$$

$K_1$ must have the property

$$K_1 |\underline{y}| \geq |\underline{A}\,\underline{y}| \quad \forall\ \underline{y}\ . \tag{29}$$

$K_1 = \underset{i=1,\ldots,n}{\text{MAX}} \sum_{j=1}^{n} |a_{ij}|$ is easily shown to be the least value satisfying (29). Q.E.D.

### 3.22 Definitions

Since the elements of $\underline{E}$, $\underline{L}$ and $\underline{F}$ can be made arbitrarily small it is convenient to make the following definitions:

$$\epsilon\underline{E}^* = \underline{E},$$

$$\epsilon\underline{L}^* = \underline{L},$$

and $$\epsilon\underline{F}^* = \underline{F}.$$

### 3.23 Comment

Equation (28) now reads

$$|\underline{\delta}(T)| < \epsilon\, [\int_o^T |(\underline{E}^* + \underline{PL}^*\underline{P}^{-1})\,\underline{X}(t) + \underline{PF}^*\underline{U}(t)|dt]\, e^{K_1 T}. \tag{30}$$

Note that $|\underline{\delta}(T)|$ is of order $\epsilon$ which may be made arbitrarily small. Therefore the error in boundary conditions can be made arbitrarily small also. The expression for $|\delta(t)|$ can be evaluated since $\underline{\tilde{X}}(t)$ and $\underline{U}(t)$ will be known from the solution of the approximate system. Also note that by taking $\underline{L}^* = 0$ in (30) one obtains the desired bound for the time-normal approximation.

3.24 Definition

Denote by $\Sigma_o(\underline{\theta}, T)$ the set of states which can be reached by the original system in time T starting at $\underline{\theta}$ and utilizing admissible controls. Denote by $\Sigma_A(\underline{\theta}, T)$ the analogous set for the approximate system. Let $\partial\Sigma_o(\underline{\theta}, T)$ and $\partial\Sigma_A(\underline{\theta}, T)$ represent the respective boundaries of the sets.

3.25 Corollary

The boundaries, $\partial\Sigma_o(\underline{\theta}, T)$ and $\partial\Sigma_A(\underline{\theta}, T)$, of the reachable sets almost coincide, i.e.

$$\forall\, \tilde{X}\in\partial\Sigma_A,\ \exists\, X\in\partial\Sigma_o \ni |\underline{\tilde{X}} - \underline{X}| = 0(\epsilon) \tag{31}$$

and

$$\forall\, X\in\partial\Sigma_o,\ \exists\, \tilde{X}\in\partial\Sigma_A \ni |\underline{\tilde{X}} - X| = 0(\epsilon)\ . \tag{32}$$

Proof: Two cases can occur. Either

(i) one set contains the other (see figure 1.a) or

(ii) neither set contains the other (see figure 1.b).

For case (i), (31) follows from (30) (e.g., apply to the original system the same control which drove the approximate system to $\partial\Sigma_A$). To prove (32), for an arbitrary point $\underline{X}_o \in \partial\Sigma_o$, construct the support hyperplane and its normal. (From Neustadt[11], $\Sigma_o$ and $\Sigma_A$ are convex and compact). Call $\underline{X}_A$ the point where the normal intersects $\partial\Sigma_A$. By convexity the closest member of $\partial\Sigma_o$ to $\underline{X}_A$ is $\underline{X}_o$. Again from (30) we know that this distance is of order $\epsilon$. Since $\underline{X}_o$ was arbitrary (32) follows. If $\Sigma_o \supseteq \Sigma_A$, the argument is identical and requires only one additional inequality,

$$|\underline{X}(T) - \underline{\tilde{X}}(T)| \le \epsilon\, [\int_o^T (-\underline{E}^* - \underline{PL}^*\underline{P})\, \underline{X}(t) - \underline{PF}^*\underline{U}(t)\,|dt]\, e^{K_2 T} \tag{33}$$

where

$$K_2 \triangleq \max_{i=1,\ \ldots,\ n} \sum_{j=1}^{n} |a_{i_j} + e_{i_j} + (\underline{PLP}^{-1})_{i_j}|.$$

(The proof of (33) is analogous to the proof of (25).)

Case (ii) is proven by using the arguments for case (i) piecewise.

Q.E.D.

3.3 Suboptimality

Another important criterion by which any approximation technique must be judged is the additional cost above the true optimum incurred when the technique is used. A bound in this additional cost will be obtained by

first bounding the true optimum cost of the original system.

3.31 Definitions

Let $\underline{U}_o(t)$ represent the optimal control which drives the original system from $\underline{\theta}$ to $\underline{\Phi}$ and minimizes the cost,

$J_o \triangleq$ the minimum cost for the original system,

$|\underline{\delta}(T, \underline{U}_o(t))| \triangleq$ the solution of equation (33) where $\underline{U}(t)=\underline{U}_o(t)$,

$\partial C \triangleq$ the boundary of a hypercube centered about $\underline{\Phi}$ with dimension $2|\underline{\delta}(T, \underline{U}_o(t))|$,

$\Delta \tilde{J} \triangleq$ the minimum cost required to drive the approximate system from a point on $\partial C$ to $\underline{\Phi}$,

$\tilde{\underline{U}}_o(t) \triangleq$ the optimal control which drives the approximate system from $\underline{\theta}$ to $\underline{\Phi}$ and minimizes the cost,

$\tilde{J}_o \triangleq$ the minimum cost for the approximate system,

$|\underline{\delta}(T, \tilde{\underline{U}}_o(t))| \triangleq$ the solution of equation (30) when $\underline{U}(t)=\tilde{\underline{U}}_o(t)$,

$\partial\tilde{C} \triangleq$ the boundary of a hypercube centered about $\underline{\Phi}$ with dimension $2|\underline{\delta}(T, \tilde{\underline{U}}_o(t))|$, and

$\Delta J \triangleq$ the minimum cost required to drive the original system from a point on $\partial\tilde{C}$ to $\underline{\Phi}$.

3.32 Theorem III

The optimum cost of the original system is bounded by the following inequality:

$$\tilde{J}_o + \Delta J_M \geq J_o \geq \tilde{J}_o - \Delta \tilde{J}_M \tag{34}$$

where

$$\Delta J_M = \underset{\underline{X}\in\partial\tilde{C}}{MAX} \Delta J$$

and

$$\Delta \tilde{J}_M = \underset{\tilde{\underline{X}}\in\partial C}{MAX} \Delta \tilde{J} .$$

Proof: Consider the trajectories in figure 2.a. From (33) $\tilde{\underline{X}}(T)$ lies within or on $\partial C$. Call $T_\partial$ the time at which $\underline{X}(t)$ intersects $\partial C$. Clearly $T_\partial \leq T$ and

$$\tilde{J}(\underline{U}_o(t), T_\partial) \leq \tilde{J}(\underline{U}_o(t), T) = J_o(\underline{U}_o(t), T). \tag{35}$$

(The right hand equality would not necessarily hold if the cost functional depended on the state). Denote by $\tilde{J}_{\partial_o}$ the minimum cost to drive the approximate system from $\underline{\theta}$ to $\partial C$. Call $\tilde{\underline{X}}_c$ the point of intersection. By definition

$$\tilde{J}_{\partial_o} \leq \tilde{J}(\underline{U}_o(t), T_\partial). \tag{36}$$

Also by definition

$$\tilde{J}_{\partial_o} + \Delta J\,(\underline{X}_c) \geq \tilde{J}_o \,. \tag{37}$$

Combining (35), (36) and (37) and utilizing the definition of $\Delta \tilde{J}_m$ yields

$$J_o \geq \tilde{J}_o - \Delta \tilde{J}_m \,,$$

the right hand side of (34).

Consider now the trajectories in figure 2.b. From (30) $\underline{X}(T)$ lies within or on $\partial\tilde{C}$. Call $T_\partial$ the time at which $\underline{X}(t)$ intersects $\partial\tilde{C}$. Clearly $T_\partial \leq T$ and

$$J(\tilde{U}_o(t),\ T_\partial) \leq J\,(\tilde{U}_o(t),\ T) = \tilde{J}_o(\tilde{U}_o(t),\ T). \tag{38}$$

Denote by $J_{\partial_o}$ the minimum cost to drive the original system from $\underline{\varrho}$ to $\partial\tilde{C}$. Call $\underline{X}_c$ the point of intersection. By definition

$$J_{\partial_o} \leq \tilde{J}\,(\underline{\tilde{U}}_o(t),\ T_\partial). \tag{39}$$

Also by definition

$$J_{\partial_o} + \Delta J(\underline{X}_c) \geq J_o \,. \tag{40}$$

Combining (38), (39) and (40) and utilizing the definition of $\Delta J_m$ yields

$$\tilde{J}_o + \Delta J_m \geq J_o \,,$$

the left hand side of (34). Q.E.D.

3.33 Comments

$\Delta J_m$ can be evaluated since from $\underline{\tilde{U}}_o(t)$ and $\underline{\tilde{X}}(t)$, $\partial\tilde{C}$ can be determined. We still have the problem that we are working with the original (non-normal) system; however, for the bound we can use any control (not necessarily optimal). Of course the closer the control to optimal the tighter will be the bound.

For $\Delta \tilde{J}_m$, we are working with the approximate system, however, the $\partial C$ is a function of $\underline{U}_o(t)$ which is not known. The most that we can say here is that the dimension of $\partial C$ is of order $\epsilon$, and for particular systems one may be able to show that $\Delta \tilde{J}_m$ is proportional to a power or root of $\epsilon$.

In most cases, $\Delta J_m$ and $\Delta \tilde{J}_m$ will approach zero as a $\epsilon$ approaches zero and as stated earlier one can make $\epsilon$ arbitrarily small and still maintain normality.

## IV. Examples

### 4.1 Example I

The attitude control of a spinning spacecraft is the first problem to be presented. Assume that the spacecraft is symmetric about one of its

axes and is spinning at a constant velocity about this axis. Assume that four non-spinning thrusters are mounted at ninety degree intervals on a bearing supported platform at one end of the spacecraft. These thrusters act in the plane normal to the axis of symmetry. Let $x_1$ represent the normalized angular velocity about an axis through two of the thrusters, $x_3$ the normalized angular momentum about the same axis and $u_1$ the thrust along this axis. Likewise let $x_2$ and $x_4$ represent the normalized velocity and momentum respectively about an axis through the other two thrusters and $u_2$ the thrust along this axis. For small angular exursion about these axes, the following differential equations hold.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{41}$$

Also assume that the thrusters are bounded in magnitude such that

$$|u_i| \leq 1 \; ; \; i = 1, 2 \; .$$

Let the objective be to transfer the system from some initial state to some final state in minimum time. Applying the test for time normality as given in paragraph 2.3 one readily sees that the problem is not normal and that singular controls may arise. This is apparent from the fact that $u_1$ does not influence $x_4$ nor does $u_2$ influence $x_3$. However, if we modify the $\underline{A}$ matrix so that

$$\hat{\underline{A}} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon \\ 0 & 0 & \epsilon & 0 \end{bmatrix} , \tag{42}$$

we now have a time-normal problem for arbitrarily small but non-zero $\epsilon$. Here we see that the constructive procedure outlined in Section III which assured us that the system could be made normal is by no means the only way to do so.

4.2 Example II

The differential equations of the second example are given below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad |u| \leq 1 \; . \tag{43}$$

This set of equations can be used to represent a variety of physical situations, e.g. a centrifuge on ideal bearings being torqued about its axis

of symmetry. Here $x_1$ would represent normalized angular position, $x_2$ normalized angular velocity and u the normalized torque. Another example is a satellite being perturbed slight about its orbit. Here $x_1$ would represent the normalized position of the satellite along the direction of thrust measured from the position where the satellite would be if no perturbing thrust had occurred. Likewise $x_2$ would represent a normalized relative velocity measured along the same axis and u the normalized thrust. Equation (43) would hold over a small arc. Single-axis control of a non-spinning spacecraft could also be described by equation (43).

Let us suppose it is desired to transfer a system described by equation (43) from some initial state to some final state in a specified time while minimizing the amount of fuel consumed. Applying the test for fuel normality as given in paragraph 2.4, we see that the problem is not normal and that singular controls may arise. All states are controlled by u, but the system has integrating capabilities.

By modifying the $\underline{A}$ matrix to

$$\hat{\underline{A}} = \begin{bmatrix} \epsilon & 1 \\ 0 & \epsilon \end{bmatrix}, \tag{44}$$

we now have a fuel-normal problem for arbitrarily small but non-zero $\epsilon$. For numerical results take as initial conditions, $X_1(0) = -3$ and $X_2(0) = 2$. Let the desired terminal state be the origin and the specified final time be 5 seconds. The solution for the approximate problem is

$$u = 0 \; ; \; 0 \leq t \leq .5 + 3\epsilon \, ,$$

$$u = -1 \; ; \; .5 + 3\epsilon \leq t < 2.5 + 6\epsilon \, ,$$

and

$$u = 0 \; ; \; 2.5 + 6\epsilon \leq t < 5 \, .$$

When this control sequence is applied to the actual system, the final state is not the origin but $X_1 = -9\epsilon$ and $X_2 = -6\epsilon$. The amount of fuel used is $2 + 3\epsilon$. One can readily see that the minimum amount of fuel required for the actual system is equal to or greater than the transfer specified along the $X_2$ axis which is 2 for this problem. Thus the degree of sub-optimality is at most $3\epsilon$ and the error in boundary conditions is seen to be small also.

## 4.3 Example III

This example is similar to the second one; however, here we consider control along two coordinates. This could describe two coordinates of a docking maneuver by two satellites again over a small arc such as during final stages. The equations of motion are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{45}$$

Normalized position and velocity along one coordinate are represented by $x_1$ and $x_2$ respectively. The normalized thrust along this axis is $u_1$. Along the other axis $x_3$, $x_4$ and $u_2$ represent position, velocity and thrust respectively. These equations also could describe attitude control of a non-spinning symmetrical spacecraft. Let the objective be to transfer the system from some initial state to some final state in minimum time. The problem is not normal as is revealed by the test in paragraph 2.3. The $\underline{A}$ matrix may be approximated by

$$\hat{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon & 0 & 0 & 0 \end{bmatrix} \tag{46}$$

and the resulting problem is normal. To obtain some numerical results, take as initial conditions

$$x_1(0) = -100,$$

$$x_2(0) = 0,$$

$$x_3(0) = -40,$$

and $$x_4(0) = 0.$$

Let the desired terminal state be the origin. The solution for the approximate problem is

$$u_1 = +1;\ 0 \leq t < 10 + 1.05 \times 10^4\ \epsilon^2,$$

$$u_1 = -1;\ 10 + 1.05 \times 10^4\ \epsilon^2 \leq t < 20 - 1.82 \times 10^4\ \epsilon^2,$$

$$u_2 = +1;\ 0 \leq t < 7 - 1.93 \times 10^4\ \epsilon^2,$$

$$u_2 = -1;\ 7 - 1.93 \times 10^4\ \epsilon^2 \leq t < 17 + 1.56 \times 10^3\ \epsilon^2,$$

and $$u_2 = +1;\ 17 + 1.56 \times 10^3\ \epsilon^2 \leq t < 20 - 1.82 \times 10^4\ \epsilon^2.$$

When this control sequence is applied to the original system, the final state is not the origin, but

$$\underline{X}(t) = \begin{bmatrix} 3.92 \times 10^{4}\ \epsilon^{2} \\ -5.99 \times 10^{4}\ \epsilon^{2} \\ 2.1 \times 10^{5}\ \epsilon^{2} \\ -5.1 \times 10^{5}\ \epsilon^{2} \end{bmatrix} \tag{57}$$

This says that the final state of the actual system lies within a hypercube of dimension

$$2|\underline{\delta}| = 1.02 \times 10^{6}\ \epsilon^{2} \tag{48}$$

centered about the desired terminal state.

Using the bound given by equation (30), we know that the final state is within a hypercube of dimension

$$2|\underline{\delta}| = 1.94 \times 10^{12}\epsilon \tag{49}$$

centered about the desired terminal state. This seems to say that the bound of equation (30) is a loose one. However, since $\epsilon$ can be made arbitrarily small, even the loose bound can be made small. For computer calculations one might take $\epsilon$ to be $10^{-16}$ or $10^{-32}$ depending on the word size of the computer and the powers to which $\epsilon$ is to be carried in the solution.

To examine the degree of sub-optimality, the minimum cost of the original system can be determined by realizing that the first two components of the state are uncoupled from the last two components. The minimum time for the composite system is equal to or greater than the minimum time of each sub-system. For the boundary conditions specified here, the minimum times for the two sub-systems are 20 seconds for $x_1$ and $x_2$ and 12.64 seconds for $x_3$ and $x_4$. Comparing the largest of these, 20 seconds, with the time required by the approximating technique, we see that the approximation technique actually yielded a lower cost. This is explainable by the fact that the system has not met the boundary conditions precisely although it is very close.

## V. Conclusions

A technique has been presented which allows one to find in a straight forward manner approximate solutions for certain control problems which are not normal. The bounds on errors in boundary conditions are very useful. The bounds on degree of sub-optimality are not as easy to evaluate, but may be estimated in certain instances. Since the approximation can be made arbitrarily accurate (by taking $\epsilon$ arbitrarily small) the errors can be made arbitrarily small. The only limitation is computer accuracy.

The most obvious area for extension of this technique is to make the control closed loop, i.e. determine the switching surfaces for the approximate system and evaluate the errors incurred when this control law is ap-

plied to the original system. Also, if tighter bounds can be obtained, one might approximate time varying linear systems or slightly non-linear systems by constant linear systems and apply the optimal control of the constant linear systems to the original system. In this case one could not make $\epsilon$ arbitrarily small, thus the need for tight bounds. These areas are presently being investigated.

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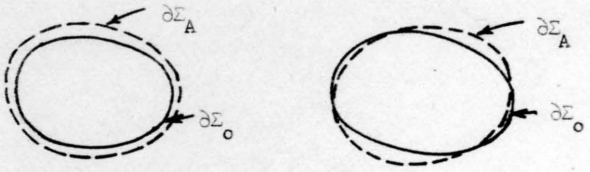


Figures 1.a and 1.b

Reachable Sets for the Original and Approximate Systems

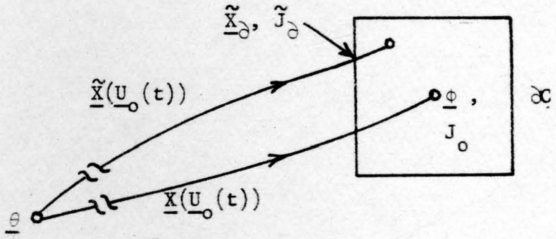


Figure 2.a

Optimal Trajectory for Original System and Associated Trajectory for Approximate System

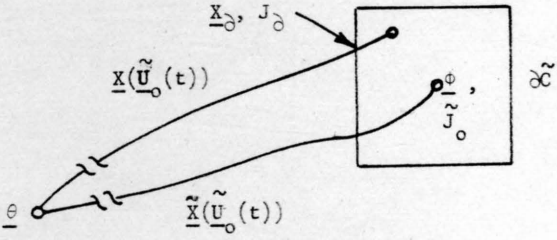


Figure 2.b

Optimal Trajectory for Approximate System and Associated Trajectory for Original System