IFAC

INTERNATIONAL FEDERATION
OF AUTOMATIC CONTROL

WARSZAWA 1969

Multivariable Systems Design

Fourth Congress of the International Federation of Automatic Control
Warszawa 16–21 June 1969

TECHNICAL SESSION
61

Organized by
Naczelna Organizacja Techniczna w Polsce

INTERNATIONAL FEDERATION OF AUTOMATIC CONTROL

Multivariable Systems Design

TECHNICAL SESSION No 61

FOURTH CONGRESS OF THE INTERNATIONAL FEDERATION OF AUTOMATIC CONTROL
WARSZAWA 16 – 21 JUNE 1969

Organized by
Naczelna Organizacja Techniczna w Polsce

Contents

Wydawnictwa Czasopism Technicznych NOT - Polska

Zakład Poligraficzny WCT NOT. Zam. 72/69.

A NONMINIMUM PHASE INDEX AND ITS APPLICATION TO INTERACTING MULTIVARIABLE CONTROL SYSTEMS

By

E. J. Davison

Department of Electrical Engineering

University of Toronto, Canada

SUMMARY

A nonminimum phase index is defined for a linear time invariant multivariable system. It is then used to give a measure of the degree of difficulty of stabilizing the system when two or more control systems, each of which controls one output variable of the system, are applied simultaneously to the system. The index is simple to compute and so should be useful in predicting when interaction will occur in large multivariable control systems such as which occur in process control. A numerical example of a distillation column with pressure control and temperature control is included.

1. Introduction

Often the control of large systems is accomplished by controlling output variables of the system as though they were independent of each other. That is, a control system is found which satisfactorily controls one output variable of the system and a control system is then found to satisfactorily control another output variable of the system. The control of both variables is then attempted by applying the two control schemes simultaneously. This is especially true in process control and chemical engineering where the system being controlled is very large. The main reasons for this have been that the plant being considered is generally far too complex to consider applying modern optimization techniques to it (often several hundred differential equations will describe the plant), and that the control schemes empirically found have often been satisfactory. Occasionally, however, (as would be expected) one control loop will react strongly with another control loop and strong interaction will exist in the system. It is obviously important to know the circumstances under which this may occur.

Previous methods of measuring interaction[1] have either proved to be unreliable (e.g. when applied to the examples given by Rosenbrock[2]) or difficult to calculate[3]. The purpose of this paper is to give a method by which it may be readily determined if interaction will occur in a specified control configuration.

2. Development

Consider the system given in Figure 1. The system is described by:

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \tag{1}$$

$$\underline{y} = \underline{D}\,\underline{x} \tag{2}$$

where

$$\underline{B} = (\underline{B}_1, \underline{B}_2, \ldots, \underline{B}_\Omega), \quad \underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_\Omega \end{pmatrix} \tag{3}$$

and

$$\underline{D} = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \tag{4}$$

where

$$\begin{aligned} d_i &= 1 \ , \quad i = r, s, t, \ldots \\ &= 0 \ , \quad i = 1, 2, \ldots, n \\ & \qquad\quad i \neq r \\ & \qquad\quad\ \ \neq s \\ & \qquad\quad\ \ \neq t \\ & \qquad\quad\ \ \ \cdot \\ & \qquad\quad\ \ \ \cdot \\ & \qquad\quad\ \ \ \cdot \end{aligned} \tag{5}$$

$\underline{x}$ is an n-dimensional vector describing the state of the plant, $\underline{A}$ is an n x n constant matrix, $\underline{B}_i$ $(1 \leq i \leq \Omega)$ is an n-dimensional vector and u_i $(1 \leq i \leq \Omega)$ are the input forcing functions to the plant. $\underline{y}$ is an n-dimensional vector consisting of the measured variables. In the case that the γ state variables $x_r, x_s, x_t, \ldots$ cannot be directly observed, a transformation of coordinates can always be made so that in the new coordinate system they can be directly measured.

Suppose that there are γ control configurations applied separately to the system so that controller r controls ouput variable x_r, controller s controls output variable x_s etc. in the following way:

Controller r: The control system is given by:

$$\underline{u}(t) = \underline{u}^o(t) + \begin{pmatrix} -k_1^r \\ -k_2^r \\ \vdots \\ -k_\Omega^r \end{pmatrix} c_r(t) \tag{6}$$

$$c_r(t) = \theta_r \, x_r(t) \tag{7}$$

where $\underline{u}^o(t)$ are the input disturbances to the plant and the controlling signals u_i $(1 \leq i \leq \Omega)$ have negative feedback from x_r with gain $-\theta_r k_i^r$ $(1 \leq i \leq \Omega)$. (The reference signal for $x_r(t)$ is assumed to be zero).

The equations of the closed loop system therefore are:

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \sum_{i=1}^{\Omega} (-k_i^r \underline{B}_i) c_r + \underline{B}\,\underline{u}^o \tag{8}$$

or
$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \theta_r \sum_{i=1}^{\Omega} (-k_i^r \underline{B}_i) x_r + \underline{B}\,\underline{u}^o \tag{9}$$

Similarly the equations of the closed loop system for the other controllers are given by:

Controller s:
$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \theta_s \sum_{i=1}^{\Omega} (-k_i^s \underline{B}_i) x_s + \underline{B}\,\underline{u}^o \tag{10}$$

Controller t:
$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \theta_t \sum_{i=1}^{\Omega} (-k_i^t \underline{B}_i) x_t + \underline{B}\,\underline{u}^o \tag{11}$$

Now often satisfactory performance is obtained when the above controllers are connected separately to the plant, i.e., output variable x_r is satisfactorily controlled when controller r is applied and output variable x_s is satisfactorily controlled when controller s is applied, etc., but when more than one controller are connected together (in an attempt to control two or more output variables satisfactorily), performance may be severely degraded and unsatisfactory control obtained. In this case interaction is said to take place.

Suppose that the controllers s, t, ... are applied simultaneously to the plant and it is desired to determine their effect on the controller r. (i.e., When controller r is connected to the plant, to what extent, if any, will the performance of controller r be degraded?) Interaction will be severe if, when controller r is not connected, the effect of the controllers s, t ... is to cause the time response of x_r for a step function input to $c_r(t)$ to change initially in a direction which is opposite to the final value it eventually reaches. This will happen if the transfer function relating output variable x_r to the forcing function input $c_r(t)$ is nonminimum phase. It will be very difficult to achieve satisfactory control with controller r in this case and if an attempt is made, performance of the control system will be downgraded. An attempt will now be made to measure the degree of nonminimum phase in such systems.

The transfer function relating $\mathcal{L}[x_r(t)]$ to $\mathcal{L}[c_r(t)]$ may be written for the case when controllers s, t ... are connected as follows:

$$\mathcal{L}[x_r(t)] = -\frac{\det\left\{\left(\underline{A}-s\underline{I}+\theta_s\underline{D}_s\left[\sum_{i=1}^{\Omega}-k_i^s\underline{B}_i\right]+\theta_t\underline{D}_t\left[\sum_{i=1}^{\Omega}-k_i^t\underline{B}_i\right]+\dots\right)(\underline{I}-\underline{I}_r)+\underline{D}_r\left[\sum_{i=1}^{\Omega}-k_i^r\underline{B}_i\right]\right\}}{\det\left\{\underline{A}-s\underline{I}+\theta_s\underline{D}_s\left[\sum_{i=1}^{\Omega}-k_i^s\underline{B}_i\right]+\theta_t\underline{D}_t\left[\sum_{i=1}^{\Omega}-k_i^t\underline{B}_i\right]+\dots\right\}}\mathcal{L}[c_r(t)] \quad (12)$$

where

$$\underline{D}_j(\underline{\xi}) = (\underline{d}_1\ \underline{d}_2\ \dots\ \underline{d}_n),\quad 1 \le j \le n \quad (13)$$

and

$$\begin{aligned} \underline{d}_k &= \underline{\xi} &&, k = j \\ &= \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} &&, k = 1,2,\dots,n \\ & && \quad k \ne j \end{aligned}$$

where

$$\underline{I}_r = \begin{pmatrix} \omega_1 & & & 0 \\ & \omega_2 & & \\ & & \ddots & \\ 0 & & & \omega_n \end{pmatrix},\quad 1 \le r \le n \quad (14)$$

and

$$\begin{aligned} \omega_i &= 1 &&, i = r \\ &= 0 &&, i = 1,2,\dots,n \\ & && \quad i \ne r \end{aligned}$$

Now the zeros of this transfer function are given by the finite eigenvalues of the matrix $\underline{Z}$[4]:

$$\underline{Z} = \left(\underline{A}+\theta_s\underline{D}_s\left[\sum_{i=1}^{\Omega} -k_i^s\underline{B}_i\right]+\theta_t\underline{D}_t\left[\sum_{i=1}^{\Omega} -k_i^t\underline{B}_i\right]+\ldots\right)(\underline{I}-\underline{I}_r)+\Gamma\underline{D}_r\left[\sum_{i=1}^{\Omega} -k_i^r\underline{B}_i\right] \tag{15}$$

in the limit as $\Gamma \to \infty$. This may be verified by expanding the determinant det $(\underline{Z} - s\,\underline{I}) = 0$ about its r column and then comparing coefficients of the characteristic equation obtained with the corresponding coefficients of the equation obtained by expanding out the numerator of the right hand side of (12) and equating it to zero. The poles of the transfer function are given by the eigenvalues of the matrix $\underline{P}$:

$$\underline{P} = \underline{A} + \theta_s\,\underline{D}_s\left[\sum_{i=1}^{\Omega} -k_i^s\,\underline{B}_i\right] + \theta_t\,\underline{D}_t\left[\sum_{i=1}^{\Omega} -k_i^t\,\underline{B}_i\right] + \ldots \tag{16}$$

Let the zeros be denoted by $-z_1, -z_2, \ldots -z_m$ $(0 \le m \le n-1)$ and the poles by $-p_1, -p_2, \ldots -p_n$ in order of increasing magnitude (in the absolute sense).

A nonminimum phase index $D^r(\theta_s, \theta_t \ldots)$ will now be defined for controller r, when controllers s, t ... are connected to the plant with gains $\theta_s, \theta_t \ldots$, as being

$$D^r(\theta_s, \theta_t \ldots) = \frac{\Delta}{\Delta + TC_{dom}} \tag{17}$$

where Δ is the "dead zone" time interval occuring in the time response of $x_r(t)$ for a step function input when the controller r is not connected (see Figure 2), and TC_{dom} is the dominant time constant of the system. It is shown in Appendix 1 that an approximate value for this index is given as follows:

<u>Nonminimum Phase Index</u>

For the case that the dominant pole p_1 is real and distinct:

$$D^r(\theta_s,\theta_t\ldots) = \frac{\log\left[\prod_{i=1}^{m}\left(1-\frac{p_1}{z_i}\right)\right]}{1+\log\left[\prod_{i=1}^{m}\left(1-\frac{p_1}{z_i}\right)\right]} \quad \text{if} \quad \prod_{i=1}^{m}\left(1-\frac{p_1}{z_i}\right) > 1 \tag{18}$$

$$= 0 \quad \text{if} \quad \prod_{i=1}^{m}\left(1-\frac{p_1}{z_i}\right) \le 1 \text{ or if } m = 0$$

For the case that the dominant pole is complex with poles p_1 and p_2:

$$D^r(\theta_s,\theta_t,\ldots) = \frac{\log\left[\mathrm{Re}\prod_{i=1}^{m}\left(1-\frac{p_1}{z_i}\right)\right]}{1+\log\left[\mathrm{Re}\prod_{i=1}^{m}\left(1-\frac{p_1}{z_i}\right)\right]} \quad \text{if } \mathrm{Re}\prod_{i=1}^{m}\left[1-\frac{p_1}{z_i}\right] > 1$$

$$= 0 \quad \text{if } \mathrm{Re}\prod_{i=1}^{m}\left[1-\frac{p_1}{z_i}\right] \leq 1 \text{ of if } m = 0 \tag{19}$$

It is assumed that the system is stable so that $\mathrm{Re}(p_i) > 0$, $i = 1,2,\ldots,n$. If the dominant pole has repeated roots, similar expressions may be obtained for the nonminimum phase index using the methods of Appendix 1.

This index has the following properties:

(1) If $D^r(\theta_s,\theta_t \ldots) = 0$, the transfer function relating $\mathcal{L}[x_r(t)]$ and $\mathcal{L}[c_r(t)]$ is minimum phase and satisfactory control should be possible to achieve by adjusting θ_r properly.

(2) If $D^r(\theta_s,\theta_t \ldots) > 0$, the transfer function relating $\mathcal{L}[x_r(t)]$ and $\mathcal{L}[c_r(t)]$ is non-minimum phase, so that control is difficult to achieve.

(3) As $D^r(\theta_s,\theta_t \ldots) \to 1$, control becomes more difficult to achieve. It will equal unity when a zero of the transfer function approaches the origin from the right hand side of the complex plane. The system becomes extremely difficult to control in this case.

An interaction index will now be defined for the system which characterizes the effect of controllers s, t ... with gains θ_s, θ_t ... on controller r when controller r is connected to the system. The index gives a measure of the relative difficulty of stabilizing the system in the presence of nonminimum phase interactions.

<u>Interaction Index</u> $$I^r(\theta_s,\theta_t \ldots) = D^r(\theta_s,\theta_t \ldots) - D^r(0,0 \ldots) \tag{20}$$

This index has the following interpretation:

(1) If $I^r(\theta_s,\theta_t \ldots) < 0$, favourable interaction will occur when controller r is connected to the system.

(2) If $I^r(\theta_s,\theta_t \ldots) = 0$, no interaction will occur when controller r is connected to the system.

(3) If $I^r(\theta_s, \theta_t \ldots) > 0$, unfavourable interaction will occur when controller r is connected to the system.

(4) As $I^r(\theta_s, \theta_t \ldots) \to 1$, the interaction will become more severe.

It is seen that the numerical calculation of this index is simple. It is essentially just a matter of finding eigenvalues of the matrices $\underline{Z}$ and $\underline{P}$. A typical value of Γ that can be used for computational purposes is $\Gamma = 10^{15}$.[4]

3. Relation of Interaction Index with Conventional Means of Measuring Interaction.

It will now be shown that as $I^r(\theta_s, \theta_t \ldots) \to 1$, the proposed index gives the same result as the conventional index of interaction[1] which is given as follows: Interaction will be severe in a 2 variable control system if for large control gains

$$\frac{G_{12}G_{21}}{G_{11}G_{22}} = 1 \tag{21}$$

when $s \to 0$ where the G's are the transfer functions of the process.

Suppose that in the system considered, controller r is connected with the gain θ_r and controller s is connected with gain θ_s to the system. Suppose that $I^r(\theta_s) \to 1$. If it is assumed that $D^r(0) = 0$, this means that $D^r(\theta_s) \to 1$ or that the determinant of matrix $\underline{Z}$ given by (15) is:

$$\det\left\{\left[\underline{A}+\theta_s \underline{D}_s\left(\sum_{i=1}^{\Omega} -k_i^s \underline{B}_i\right)\right](\underline{I}-\underline{I}_r)+\Gamma \underline{D}_r\left(\sum_{i=1}^{\Omega} -k_i^r \underline{B}_i\right)\right\} = 0 \tag{22}$$

in the limit. If it is assumed that the gain θ_s is large, (22) then becomes:

$$\det\left\{\underline{A}(\underline{I}-\underline{I}_r-\underline{I}_s)+\underline{D}_r\left(\sum_{i=1}^{\Omega} -k_i^r \underline{B}_i\right)+\underline{D}_s\left(\sum_{i=1}^{\Omega} -k_i^s \underline{B}_i\right)\right\} = 0 \tag{23}$$

Equation (23) now implies that:

$$\det\left\{\underline{A}(\underline{I}-\underline{I}_r)+\underline{D}_r\left(\sum_{i=1}^{\Omega} -k_i^r \underline{B}_i\right)\right\}\det\left\{\underline{A}(\underline{I}-\underline{I}_s)+\underline{D}_s\left(\sum_{i=1}^{\Omega} -k_i^s \underline{B}_i\right)\right\} = \det\left\{\underline{A}(\underline{I}-\underline{I}_r)+\underline{D}_r\left(\sum_{i=1}^{\Omega} -k_i^s \underline{B}_i\right)\right\}\det\left\{\underline{A}(\underline{I}-\underline{I}_s)+\underline{D}_s\left(\sum_{i=1}^{\Omega} -k_i^r \underline{B}_i\right)\right\} \tag{24}$$

and when the determinants in (24) are divided by $\det(\underline{A}-s\underline{I})$, the following relation is obtained (for the case that $s \to 0$):

$$G_{11}G_{22} = G_{12}G_{21} \tag{25}$$

where the G's are the transfer functions of the two variable control system. It is the conventional measure of interaction used in two variable control system study[1].

It should be noted, however, that the converse relationship is <u>not</u> true. Eq$\underline{^{n}}$ (25) does not at all imply (22) and so it is possible to have systems in which (25) is true, yet which have little interaction (e.g. when applied to the examples of Rosenbrock[2]).

The following conclusion is therefore made - the conventional index of performance is a valid measure of interaction only when it indicates that little interaction will occur in a system.

4. Numerical Examples

The first example is taken from Rosenbrock[5]. He considered the system shown in Figure 3 where -

$$G_{11} = \frac{1}{(1+s167)(1+s1)(1+s0.1)^4}$$

$$G_{22} = \frac{-1}{(1+s167)(1+s1)^2}$$

$$G_{12} = \frac{-0.85}{(1+s83)(1+s1)^2}$$

$$G_{21} = \frac{0.85}{(1+s167)(1+s0.5)^4(1+s1)}$$

$$C_{11} = k_1$$

$$C_{22} = -k_2$$

x_1 is satisfactorily controlled when $k_1 = 150$ and $k_2 = 0$ and x_2 is satisfactorily controlled when $k_1 = 0$ and $k_2 = 75$. It is desired to determine if, when the bottom controller is connected with $k_2 = 75$, interaction will occur when the top controller is connected.

The following interaction index was obtained in this case:

$$I^1(k_2) = \frac{\log\left[1 - \frac{p_1}{z_1}\right]}{1+\log\left[1 - \frac{p_1}{z_1}\right]}$$

where $z_1 = -0.00758, \quad p_1 = 0.00600$

or $I^1(k_2) = 0.37$

which implies that severe* interaction will occur. This was in fact verified by simulation (see Figure 4). It is interesting to note that when a small change is made in the transfer function G_{12} so that

$$G_{12} = \frac{-0.85}{(1+s167)(1+s1)^2}$$

with all other transfer functions and gains the same, the interaction index becomes zero, implying that no interaction will occur. This was verified by simulation (see Figure 5) and was observed by Rosenbrock[6]**. This means that rather small changes in the dynamics of a system can occasionally cause spectacular changes in the controlled system.

The second example consists of a binary distillation column with a temperature controller and a pressure controller. Figure 6 illustrates the control system used.

The equations of the distillation column are given below[7]:

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}_1 u_1 + \underline{B}_2 u_2 + \underline{B}_k u_k, \quad n = 11$$

$$T_i = -46x_i + 50x_{11}, \quad i = 1,2, \ldots, 10$$

$$k_p = -10^8$$

where $\underline{A}$ is given in Table 1 and $\underline{B}_1$, $\underline{B}_2$ are given below:

---

* As guide lines for a measure of the severity of the interaction, the interaction would be considered weak if $I^r(\theta_s,\theta_t, \ldots) < 0.01$ (say) and severe if $I^r(\theta_s,\theta_t,\ldots) > 0.1$ (say) by the "Ziegler-Nichols rule"[5].

** It should be noted that the conventional measure of interaction gives $\frac{G_{12}G_{21}}{G_{11}G_{22}} = 0.72$ for both G_{12} transfer functions, implying that strong interaction will occur in both cases!

$$\underline{B}_1 = \begin{pmatrix} 0.0 \\ -0.3 \times 10^{-5} \\ -0.5 \times 10^{-5} \\ -0.5 \times 10^{-5} \\ -0.5 \times 10^{-5} \\ -0.5 \times 10^{-5} \\ -0.5 \times 10^{-5} \\ -2.0 \times 10^{-5} \\ -4.0 \times 10^{-5} \\ -2.0 \times 10^{-5} \\ 46.0 \times 10^{-5} \end{pmatrix} \qquad \underline{B}_2 = \begin{pmatrix} 0.0 \\ 3.0 \times 10^{-5} \\ 5.0 \times 10^{-5} \\ 5.0 \times 10^{-5} \\ 5.0 \times 10^{-5} \\ 5.0 \times 10^{-5} \\ 5.0 \times 10^{-5} \\ 5.0 \times 10^{-5} \\ 4.0 \times 10^{-5} \\ 2.0 \times 10^{-5} \\ 46.0 \times 10^{-5} \end{pmatrix}$$

There are 8 plates in the column and x_3 is the composition on plate 2 and x_{11} is the pressure in the column. T_3 is the temperature on plate 2. $u_1(t)$ represents a change of heat input to the column and $u_2(t)$ represents a change of heat output from the column.

The purpose of the control system is to keep the temperature on plate 2 and pressure of the column as constant as possible against disturbances from other sources represented by the terms $\underline{B}_k u_k$. The control scheme proposed is a very common way of achieving it. However, it is well known that severe interaction may often result with this control configuration[3].

This interaction may be easily predicted using the interaction index proposed which is as follows:

$$I^{T_3}(k_p) = \frac{\log\left[\prod_{i=1}^{9}\left(1 - \frac{p_1}{z_i}\right)\right]}{1+\log\left[\prod_{i=1}^{9}\left(1 - \frac{p_1}{z_i}\right)\right]}$$

where

$z_1 = -0.100 \times 10^{-2}$ $\qquad p_1 = 0.101 \times 10^{-2}$

$z_2 = 0.617 \times 10^{-2}$

$z_3 = 0.138 \times 10^{-1}$

$z_4 = 0.337 \times 10^{-1} + i\, 0.164 \times 10^{-2}$

$z_5 = 0.337 \times 10^{-1} - i\, 0.164 \times 10^{-2}$

$z_6 = 0.473 \times 10^{-1} + i\, 0.120 \times 10^{-1}$

$z_7 = 0.473 \times 10^{-1} - i\, 0.120 \times 10^{-1}$

$z_8 = 0.700 \times 10^{-1}$

$z_9 = 0.981 \times 10^{-1}$

or

$$I^{T_3}(k_p) = 0.25$$

which means that with the control scheme proposed, severe interaction will occur. This severe interaction is obtained over a large range of values of the pressure control gain.

It should be noted, however, that when composition on plate 2 (x_3) is controlled instead of temperature, the interaction index is zero. This means that interaction can be eliminated or at least reduced if compositions rather than temperatures are controlled in the column. This observation has been made previously[3,7].

Conclusions

An interaction index has been proposed to give a measure of the degree of difficulty of stabilizing a system when two or more control systems, each of which controls one output variable of the system, are simultaneously applied to the system. The index should be especially useful in process control.

References

1. Mitchell, D.S., Webb, C.R., "A Study of Interaction in a Multi-loop Control System". Proc. 1st IFAC Congress, Moscow. (Butterworth). (1960).
2. Rosenbrock, H.H., "On the Design of Linear Multivariable Control Systems". Proc. 3rd IFAC Congress, London, June (1966).
3. Rijnsdorp, J.E., "Interaction in Two-Variable Control Systems for Distillation Columns - II", Automatica, Vol.3, p.29 (1965).
4. Davison, E.J., "A Numerical Method for Finding the Poles and Zeros of a Control System". Proc. 3rd IFAC Congress, London, June (1966).
5. Perlmutter, D.D., Introduction to Chemical Process Control, Wiley & Sons p.143 (1965).
6. Rosenbrock, H.H., "The Control of Distillation Columns" Trans. Instn. Chem. Engrs., Vol. 40, p.35, (1962).
7. Davison, E.J., "Control of a Distillation Column with Pressure Variation". Trans. Instn. Chem. Engrs, Vol. 45, No.6, p.T229, (1967).
8. Davison, E.J., "A Method for Simplifying Linear Dynamic Systems". IEEE Trans. on Automatic Control, Vol. AC-11, No.1, p.93, (1966).

Appendix I

It is desired to obtain a value for $\frac{\Delta}{\Delta + TC_{dom}}$ where Δ is the "dead zone" time interval occurring in the time response of $x_r(t)$ for a step function input to $c_r(t)$ when the controller r is not connected and TC_{dom} is the dominant time constant of the system (see Figure 2).

The Laplace Transform of $x_r(t)$ for a unit step function input in $c_r(t)$ may be written as follows (assuming that the poles are distinct for algebraic simplicity):

$$\mathcal{L}[x_r(t)] = K\frac{(s+z_1)---(s+z_m)}{s(s+p_1)--(s+p_n)} \tag{1a}$$

where K is a constant and the zeros and poles are given by the eigenvalues of matrix $\underline{Z}$ and matrix $\underline{P}$ respectively. The following time respnnse is then obtained by partial fraction expansion (assuming that the poles and zeros do not cancel):

$$x_r(t) = A_0 + A_1 e^{-p_1 t} + --- + A_n e^{-p_n t} \tag{2a}$$

where

$$A_i = K\left.\frac{(s+z_1)(s+z_2)---(z+z_m)(s+p_i)}{s(s+p_1)--------(s+p_i)------(s+p_n)}\right|_{s=-p_i} \quad i=0,1,2,...,n \tag{3a}$$

and

$$p_0 = 0 .$$

It is desired now to solve the equation

$$A_0 + A_1 e^{-p_1\Delta} + A_2 e^{-p_2\Delta} + --- + A_n e^{-p_n\Delta} = 0 \tag{4a}$$

for Δ. It is difficult to obtain an analytical solution to this equation and so instead, the time response of (2a) will be approximated by a third order time response and Δ will then be obtained from this approximate response.

The approximate time response will be taken to be

$$x_r(t) \doteq A_0 + A_1 e^{-p_1 t} + (-A_0 - A_1)e^{-p_2^* t} \tag{5a}$$

where p_2^* is to be determined later. This fitted third order time response has the same initial and steady-state values as (2a) and the same dominant modal contribution. It is, therefore, a good approximation to (2a)[7]. It is now desired to solve

$$A_0 + A_1 e^{-p_1\Delta} - (A_0 + A_1)e^{-p_2^*\Delta} = 0 \tag{6a}$$

for Δ, $\Delta \neq 0$ or

$$x^\theta = \left(1 + \frac{A_0}{A_1}\right)x - \frac{A_0}{A_1} \tag{7a}$$

for x, $x \neq 1$ where

$$\theta = \frac{p_1}{p_2^*} , \; x = e^{-p_2^*\Delta} \quad ; 0 < \theta < 1 \quad , \quad 0 < x < 1 \tag{8a}$$

and this will have a solution $x \neq 1$ only if $0 \leq -\frac{A_0}{A_1} < 1$.

An approximate solution to this equation for $x \neq 1$ is:

$$x \doteq \left[-\frac{A_\theta}{A_1}\frac{1}{(1-\theta)}\right]^{\frac{1}{\theta}}, \text{ if } 0 \leq -\frac{A_0}{A_1}\frac{1}{(1-\theta)} \leq 1 \qquad (9a)$$

which will become exact as $x \to 1$. The following expression is now obtained on solving for Δ:

$$\begin{aligned}\Delta &\doteq \frac{1}{p_1}\log\left[-\frac{A_1}{A_0}(1-\theta)\right] && \text{if } -\frac{A_1}{A_0}(1-\theta) > 1 \\ &\doteq 0 && \text{if } -\frac{A_1}{A_0}(1-\theta) \leq 1\end{aligned} \qquad (10a)$$

which will become exact as $\Delta \to 0$. The following nonminimum phase index is then obtained on substituting for Δ:

$$\begin{aligned}D &= \frac{\log\left[-(1-\theta)\frac{A_1}{A_0}\right]}{1+\log\left[-(1-\theta)\frac{A_1}{A_0}\right]} && \text{if } -(1-\theta)\frac{A_1}{A_0} > 1 \\ &= 0 && \text{if } -(1-\theta)\frac{A_1}{A_0} \leq 1\end{aligned} \qquad (11a)$$

and on substituting for A_1 and A_0 the following expression is obtained:

$$\begin{aligned}D &= \frac{\log\left[\left(1-\frac{p_1}{p_2^*}\right)\frac{\prod_{i=1}^{m}\left(1-\frac{p_1}{z_i}\right)}{\prod_{i=2}^{n}\left(1-\frac{p_1}{p_i}\right)}\right]}{1+\log\left[\left(1-\frac{p_1}{p_2^*}\right)\frac{\prod_{i=1}^{m}\left(1-\frac{p_1}{z_i}\right)}{\prod_{i=2}^{n}\left(1-\frac{p_1}{p_i}\right)}\right]} && \text{if } \left(1-\frac{p_1}{p_2^*}\right)\frac{\prod_{i=1}^{m}\left(1-\frac{p_1}{z_i}\right)}{\prod_{i=2}^{n}\left(1-\frac{p_1}{p_i}\right)} > 1 \\ &= 0 && \text{if } \left(1-\frac{p_1}{p_2^*}\right)\frac{\prod_{i=1}^{m}\left(1-\frac{p_1}{z_i}\right)}{\prod_{i=2}^{n}\left(1-\frac{p_1}{p_i}\right)} \leq 1\end{aligned} \qquad (12a)$$

Now it is well know that if

$$\mathcal{L}[x_r(t)] = K\frac{1}{s(s+p_1)---(s+p_n)} \qquad (13a)$$

then the first non-zero derivative of x_r at $t = 0$ is K and a nonminimum phase response similar to Figure 2 cannot occur with this system. p_2^* will be chosen therefore so that when $m = 0$ (i.e., there are no finite zeros in $\mathcal{L}[x_r(t)]$), $\Delta = 0$ in the approximate time response. This will occur when

$$\left(1 - \frac{p_1}{p_2^*}\right) = \prod_{i=2}^{n} \left(1 - \frac{p_1}{p_i}\right) \tag{14a}$$

On substituting (15a) into (13a) the expression given by (14) is finally obtained for the nonminimum phase index.

For the case that the dominant pole is complex with poles p_1 and p_2 the time response of (2a) is now given by

$$x_r(t) = A_0 + 2\mathrm{Re}\,(A_1)e^{-\mathrm{Re}(p_1)t}\left(\cos[\mathrm{Im}(p_1)t] + \frac{\mathrm{Im}(A_1)}{\mathrm{Re}(A_1)}\sin[\mathrm{Im}(p_1)t]\right) + A_3e^{-p_3t} + \text{---} + A_ne^{-p_nt} \tag{15a}$$

and the "fitted" third order response will now be taken as:

$$x_r(t) \doteqdot A_0 + 2\mathrm{Re}(A_1)e^{-\mathrm{Re}(p_1)t} + [-A_0 - 2\mathrm{Re}(A_1)]e^{-p_3^*t} \tag{16a}$$

where p_3^* is determined in the same way as p_2^*. The same analysis as before is then carried out and the expression given by (15) is finally obtained.

Table 1 - Matrix A

-0.014	0.0043	0	0	0	0	0	0	0	0	0
0.0095	-0.0138	0.0046	0	0	0	0	0	0	0	-3.0_{10}^{-4}
0	0.0095	-0.0141	0.0063	0	0	0	0	0	0	-5.0_{10}^{-4}
0	0	0.0095	-0.0158	0.011	0	0	0	0	0	-8.0_{10}^{-4}
0	0	0	0.0095	-0.0312	0.015	0	0	0	0	-8.0_{10}^{-4}
0	0	0	0	0.0202	-0.0352	0.022	0	0	0	-8.0_{10}^{-4}
0	0	0	0	0	0.0202	-0.0422	0.028	0	0	-8.0_{10}^{-4}
0	0	0	0	0	0	0.0202	-0.0482	0.037	0	-6.0_{10}^{-4}
0	0	0	0	0	0	0	0.0202	-0.0572	0.042	-3.0_{10}^{-4}
0	0	0	0	0	0	0	0	0.0202	-0.0483	0
0.0255	0	0	0	0	0	0	0	0	0.0255	-0.0185

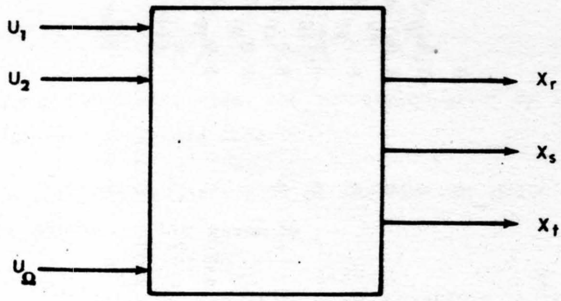


Figure 1 System under Consideration

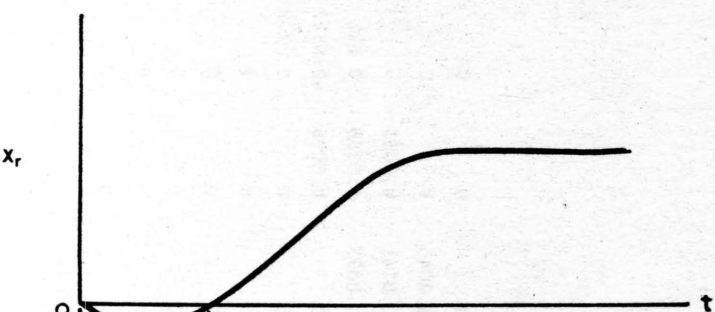


Figure 2 Response of Variable x_r for a Step Function Input

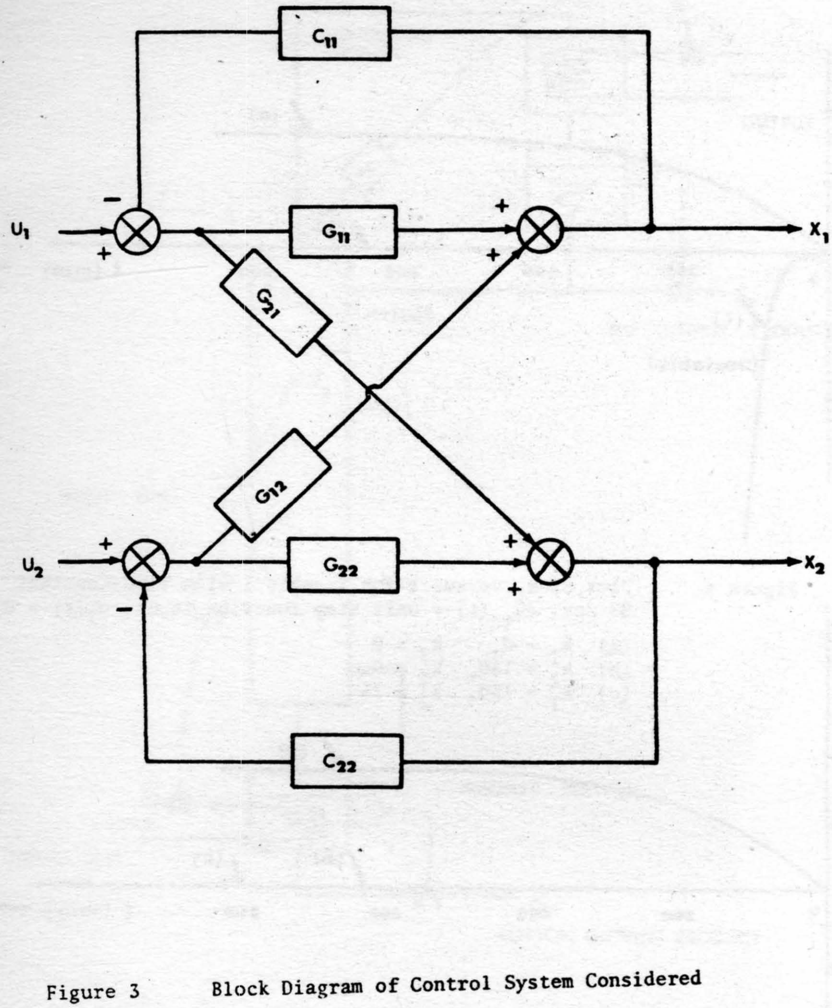


Figure 3 Block Diagram of Control System Considered

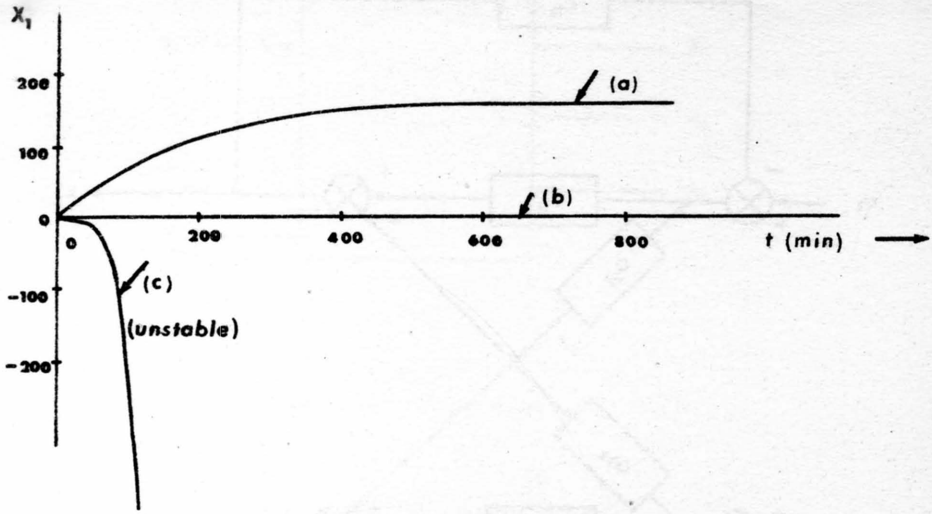


Figure 4 Plot of x_1 versus t for Example 1 with Time-Constant 83 sec. (u_1 (t) = unit step function input, $u_2(t) = 0$).

(a) $k_1 = 0, \quad k_2 = 0$
(b) $k_1 = 150, \quad k_2 = 0$
(c) $k_1 = 150, \quad k_2 = 75$

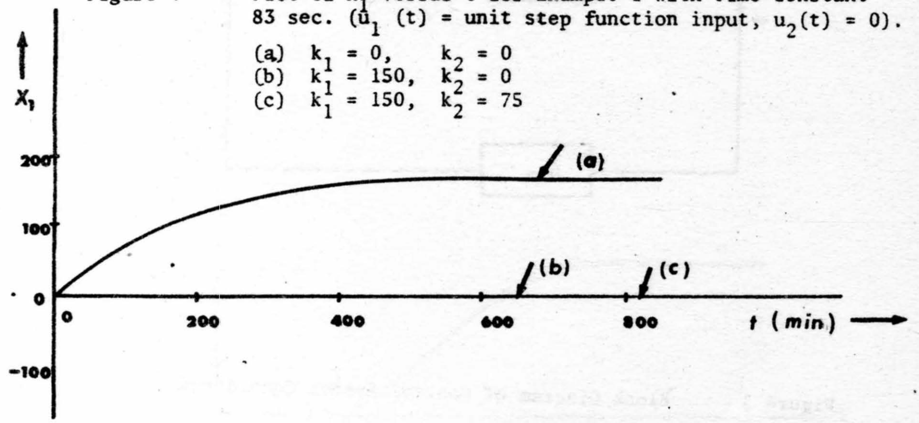


Figure 5 Plot of x_1 versus t for Example 1 with Time-Constant 167 sec. (u_1 (t) = unit step function input, u_2 (t) = 0).

(a) $k_1 = 0, \quad k_2 = 0$
(b) $k_1 = 150, \quad k_2 = 0$
(c) $k_1 = 150, \quad k_2 = 75$

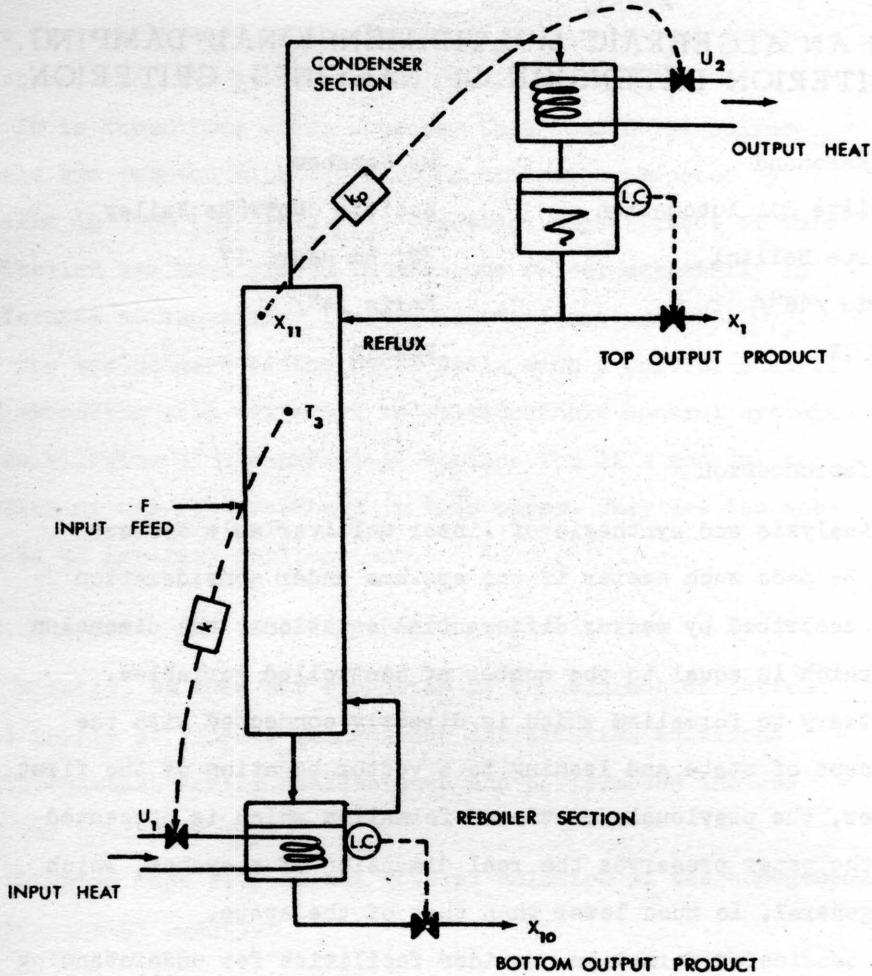


Figure 6 A Distillation Column under Control

ON AN ALGEBRAIC MULTIDIMENSIONAL DAMPING CRITERION EXTENSION OF NASLIN'S CRITERION

D. Marchand
Société ECA Automation
8, rue Bellini
Paris /16°/
FRANCE

M. Menahem
Société Contrôle Bailey
32, Bd Henri IV
Paris /4°/
FRANCE

I. Introduction

Analysis and synthesis of linear multivariable systems can be made much easier if the systems under consideration are described by vector differential equations, the dimension of which is equal to the number of controlled variables. Contrary to formalism which is directly connected with the concept of state and leading to a vector equation of the first order, the previously mentioned formalism which is presented in the paper preserves the real dimension of a system, which in general, is much lower than that of the state. The considered formalism provides facilities for understanding the algebraic structure of a system. Moreover, its application makes possible to use the concept of "the matrix transmittance" which seems to replace "the transfer matrix". This formalism has already been the subject of two works [1], [2].

In the first part of the paper standard matrix polynomials /representing differential operators/ are considered. They result from the generalization of a certain scalar polynomials which

-are frequently used in the domain of unidimensional control systems, viz. the standard Naslin polynomials [3].

It is known from other sources that considered polynomials are related with the damping criterion proposed by Naslin in 1960 [3], [4], [5]. Practical significance of this criterion has been proved in numerous cases, especially in reference to industrial feedback control systems [6], [7].

The second part of the paper deals with a partial analysis of extending this criterion to multivariable control systems. Possibilities of the practical application of a generalized criterion are not discussed in this paper. They are the subjects of separate publications.

II. The principle of the proposed generalization

Firstly, we turn our attention to the connection between the family of linear, homogeneous differential equations with constant scalar coefficients and performance indices of the quadratic type.

Suppose that $x(t)$ is the general solution of the homogeneous differential equation

$$\sum_{0}^{n} a_i \, x^{(i)}(t) = 0 \qquad a_o = a_n = 1 \qquad /1/$$

It can be proved that no matter what the initial conditions are, the solution $x(t)$ minimizes the n-th order performance index

$$J_n = \int_{0}^{\infty} \sum \varphi_{2i} \; x^{(i)^2}(t) \, dt \qquad /2/$$

provided that the following relations involving coefficients

φ_{2i} and a_i, $i = 0, 1, \ldots, n$, hold

$$\begin{aligned}
\varphi_0 &= 1 \\
\varphi_2 &= \left[a_1^2 - 2a_2\right] \\
\varphi_4 &= \left[a_2^{\,2} - 2a_1\, a_3\right] + 2\, a_4 \qquad /3/ \\
&\text{- - - - - - - - - - - - - - - -} \\
\varphi_{2i} &= \left[a_i^{\,2} - 2\, a_{i-1}\, a_{i+1}\right] + 2\, a_{i-2}\, a_{i+2} - \ldots \\
&\text{- - - - - - - - - - - - - - - -} \\
\varphi_{2\,n-1} &= \left[a_{n-1}^{\,2} - 2\, a_{n-2}\right] \\
\varphi_{2n} &= 1
\end{aligned}$$

One-to-one correspondence between coefficients φ_{2i} and a_i exists only under assumption that the corresponding differential equation (1) is stable.

Suppose the differential operator associated with equation (1) to be

$$a\,(p) = \sum_{0}^{n} a_i\, p^i \quad ; \quad a_1 = a_n = 1$$

where p denotes the operation of differentiation.

The square of the modulus of this operator can be written as

$$a(p)\; a(-p) = 1 - \varphi_2\, p^2 + \varphi_4\, p^4 - \varphi_6\, p^6 + \ldots + (-1)^n\, p^{2n} \qquad /5/$$

Properties of some standard differential operators /or polynomials in p/ can be easily described in terms of the coefficients φ_{2i}. Such a case is for Butterworth polynomials obtained under assumption that $\varphi_{2i} = 0$ for $i = 1, 2, \ldots, n - 1$ as well as for Naslin polynomials. The latter are derived assuming following recurrent relation

$$a_i^2 - \alpha a_{i-1} a_{i+1} = 0 \qquad /6/$$

/damping of these polynomials is a function of the parameter α/. In particular, damping corresponding to $\alpha = 2$ is satisfactory from the practical point of view. The choice of such a value of the parameter α relies on reducing to zero those terms of equations (3) which are in the brackets.

Given a differential matrix operator of the form

$$A(p) = \sum_{0}^{n} A_i \, p^i \qquad /7/$$

Coefficients A_i are constant matrices of order m. Let us consider the square of the modulus of this operator, represented by the Hermitian matrix

$$A'(-p)\, A(p) = \phi_0 + \phi_1 p - \phi_2 p^2 - \phi_3 p^3 + \phi_4 p^4 + \ldots\ldots$$

$$\ldots\ldots + (-1)^n \phi_{2n} \; p^{2n} \qquad /8/$$

with matrix coefficients satisfying the following conditions

$$\phi_i = \phi'_i \; , \quad \text{if i is even} \quad /\text{symmetry}/$$

$$\phi_i = -\phi'_i \; , \quad \text{if i is odd} \quad /\text{antisymmetry}/$$

These coefficients can be written in terms of A_i. There results

$$\phi_o = A_o' A_o$$

$$\phi_1 = (A_o' A_1 - A_1' A_o)$$

$$\phi_2 = A_1' A_1 - (A_o' A_2 + A_2' A_o)$$

$$\phi_3 = (A_1' A_2 - A_2' A_1) - (A_o' A_3 - A_3' A_o)$$

$$\phi_4 = A_2' A_2 - (A_1' A_3 + A_3' A_1) + (A_o' A_4 + A_4' A_o)$$

- -

$$\phi_{2i} = A_i' A_i - (A_{i-1}' A_{i+1} + A_{i+1}' A_{i-1}) + $$
$$+ (A_{i-2}' A_{i+2} + A_{i+2}' A_{i-2}) - \dots$$

$$\phi_{2i+1} = (A_i' A_{i+1} - A_{i+1}' A_i) - (A_{i-1}' A_{i+2} - A_{i+2}' A_{i-1}) + \dots$$

- -

$$\phi_{2n-1} = A_{n-1}' A_n - A_n' A_{n-1}$$

$$\phi_{2n} = A_n' A_n \qquad /9/$$

Note, in the capacity of an example, that if $x(t)$ is the solution of the homogeneous vector differential equation

$$x + A_1 \dot{x} + A_2 \ddot{x} = 0 \qquad /10/$$

then it minimizes the following quadratic performance index

$$J_n = \int_0^{\infty} (x'x + x'\phi_1 \dot{x} + \dot{x}'\phi_2 \dot{x} + \dot{x}'\phi_3 \ddot{x} + \ddot{x}'\phi_4 \ddot{x}) \, dt$$

where

$$\phi_1 = A_1 - A_1'$$

$$\phi_2 = A_1' A_1 - (A_2 + A_2')$$

$$\phi_3 = A_1' A_2 - A_2' A_1$$

$$\phi_4 = A_2' A_2 \qquad /11/$$

Analogically to the scalar relations (3) and equations (6) defining Naslin polynomials, we suggest to generalize these polynomials to the matrix domain assuming

$$A_i' A_i - \frac{\alpha}{2} \left(A_{i-1}' A_{i+1} + A_{i+1}' A_{i-1} \right) = 0$$

$$A_i' A_{i+1} - A_{i+1}' A_i = 0 \qquad /12/$$

where α is a positive scalar

Relations given above can be considered as a system of equations linear with respect to the variable A_{i+1}. In fact, the equation /12.1/ results from the generalization of equation /6/. Examining the former one, it can be seen that a_i^2 is replaced by the square of the modulus of A_i and that the product $a_{i-1} a_{i+1}$ converts to the symmetric component of the product $A_{i-1}' A_i$. The relation /12.2/ effects only upon antisymmetric coefficients of /8/.
In the next paragraph the procedure of generating such polynomials is discussed in more detail.

III. Standard matrix polynomials

1° Naslin scalar polynomials

Given

$$f(p) = a_o + a_1 p + \dots + a_n p^n$$

and

$$\alpha_i = \frac{a_i^2}{a_{i-1}\, a_{i+1}} \quad , \quad i = 1, 2, \dots, n-1$$

/ α_i - the characteristic ratio of the i-th order/

f(p) is called,by definition, a Naslin polynomial of α parameter, if

$$\alpha_i = \alpha$$

holds for all i = 1,2, . . . , n-1

Let the symbol $f(p \,;\, n\,, \alpha\,, a_o\,, a_1)$ denotes an n-th order Naslin polynomial of parameter α, with the first and second coefficients equal to a_o and a_1 respectively and the independent variable being p.

2° Naslin matrix polynomials

Let us consider the polynomial

$$A(p) = \sum_{o}^{n} A_i\, p^i$$

with the matrix coefficients of order m satisfying the relations /12/. Only such polynomials for which $A_o = I$ are taken into considerations.

The relation /12.2/ shows for i = 0 that A_1 is a symmetric matrix.

Hence

$$A_1 \triangleq S_1 = \Omega D_1 \Omega' \qquad /13/$$

where D_1 is a diagonal matrix $\{\lambda_1, \lambda_2, \ldots\ldots, \lambda_m\}$

Ω is an orthogonal matrix

Let us examine equations /12/ assuming that i = 1.

We have

$$S_1^2 = \frac{\alpha}{2}\left(A_2 + A_2'\right)$$

$$S_1 A_2 = A_2' S_1 \qquad /14/$$

If the matrix S_1 is supposed to be regular, then equation /14/ yields

$$I = \frac{\alpha}{2}\left(S_1^{-1} A_2 S_1^{-1} + S_1^{-1} A_2' S_1^{-1}\right) \qquad /15/$$

$$A_2 S_1^{-1} = S_1^{-1} A_2'$$

Assuming that

$$A_2 S_1^{-1} \triangleq S_2 \qquad /16/$$

is a symmetric matrix, equation /15.1/ can be written in the form

$$I = \frac{\alpha}{2}\left(S_1^{-1} S_2 + S_2 S_1^{-1}\right) \qquad /17/$$

This equation has a unique solution with respect to S if the matrix S_1^{-1} has no eigenvalues of the same modulus but opposite signs [8]. So under assumption that all the eigenvalues λ_i are positive, the condition for uniqueness is satisfied. The obvious solution is

$$S_2 = \frac{S_1}{\alpha} \qquad /18/$$

Then

$$A_2 = \frac{S_1^{\,2}}{\alpha} = \Omega \frac{D_1^{\,2}}{\alpha} \Omega' \qquad /19/$$

Now, let us suppose that

$$A_{i-1} = \Omega D_{i-1} \Omega' \triangleq S_{i-1} \quad \text{and} \quad A_i = \Omega D_i \Omega' \triangleq S_i \qquad /20/$$

Moreover, it is assumed that both matrices are positive definite. It will be shown that

$$A_{i+1} = \Omega D_{i+1} \Omega' \quad \text{where} \quad D_i^{\,2} = \alpha D_{i-1} D_{i+1}$$

For this purpose the following system is considered

$$S_1^{\,2} = \frac{\alpha}{2} \left(A'_{i+1} S_{i-1} + S_{i-1} A_{i+1} \right)$$

$$S_i A_{i+1} = A'_{i+1} S_i \qquad /21/$$

Suppose that

$$A_{i+1} S_i^{-1} \triangleq S \qquad /22/$$

then equation /21.1/ can be written in the form

$$I = \frac{\alpha}{2} \left(S S_{i-1} S_i^{-1} + S_i^{-1} S_{i-1} S \right) \qquad /23/$$

We know that $S_{i-1} S_i^{-1} = \Omega D_{i-1} D_i^{-1} \Omega'$. So, the condition for uniqueness is satisfied and equation /23/ has the solution

$$S = \frac{S_i S_{i-1}^{-1}}{\alpha} \qquad /24/$$

where

$$A_{i+1} = \frac{1}{\alpha} S_i S_{i-1}^{-1} S_i = \Omega \frac{D_i^2 D_{i-1}^{-1}}{\alpha} \Omega' \qquad /25/$$

It shows that all the coefficients are well determined.

Let us define a family of standard matrix polynomials

$$F(p\, ;\, n, \alpha, I, S_1) = \Omega D(p\, ;\, n, \alpha, I, D_1) \Omega' \qquad /26/$$

where

$$D(p\, ;\, n, \alpha, I, D_1) = \{f_1(p), f_2(p), \dots \dots f_m(p)\} \qquad /27/$$

and

$$f_k(p) = f(p\, ;\, n, \alpha, 1, \lambda_k) \qquad /28/$$

Note 1. The defined polynomials are symmetric. This suggests that they can be derived using relations of the form

$$A_i A_i' = \frac{\alpha}{2} \left(A_{i+1} A_{i-1}' + A_{i-1} A_{i+1}' \right)$$

$$A_{i+1} A_i' - A_i A_{i+1}' = 0 \qquad A_0 = I \qquad /29/$$

The obtained equations are the result of replacing the square of modulus by the matrix

$A(p)\,A'(-p)$..

Note 2. $\det\left[F(p)\right] = f_1(p)\, f_2(p) \,\ldots\, f_m(p)$ /30/

It implies that the damping factor of the zeros of the determinant of a standard matrix polynomial can be affected by choice of α. Conversely, the modulus of these zeros depend upon α and D_1.

Note 3. The zeros of the determinant of a standard matrix polynomial have negative real parts if all λ_k are positive /the S_1 matrix is positive definite/ and $\alpha > 1{,}463$ /$n \geqslant 3$/, as it results from the investigations of Naslin scalar polynomials.

Note 4. Let α be a symmetric, positive definite matrix which commutes with the S_1 matrix

Let us assume that

$$\alpha = \Omega \Delta \Omega'$$

where

$$\Delta = \left\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_m\right\} \qquad /31/.$$

It is easy to show that this assumption makes possible to define the generalized standard polynomials

$$F\left(p\ ;\ n, \alpha\ , I,\ S_1\right) = \Omega\, D\left(p\ ;\ n, \Delta\ , I,\ D_1\right)\Omega' \qquad /32/$$

where

$$D(p) = f_1(p),\ f_2(p), \ldots\ldots,\ f_m(p) \qquad /33/$$

and

$$f_k(p) = f\left(p\ ;\ n, \alpha_k\ ,\ 1, \lambda_k\right) \qquad /34/$$

In such a manner, damping of every element of $D(p)$ can be changed indepedently on othefs.

Note 5. For

$$A_o = I \; ; \; A_n = T^n \qquad /35/$$

where T is a symmetric,positive definite matrix.
The structure of standard polynomials becomes the canonical one, such as initially used by Naslin. Then,for n = 2, 3, 4 the following polynomials are obtained

$$I + \sqrt{\alpha}\; T\, p + T^2\, p^2$$

$$I + \alpha\, T\, p + \alpha\, T^2\, p^2 + T^3\, p^3$$

$$I + \alpha\sqrt{\alpha}\; T\, p + \alpha^2\, T^2\, p^2 + \alpha\sqrt{\alpha}\; T^3\, p^3 + T^4\, p^4 \qquad /36/$$

it is assumed here that α is a symmetric, positive definite matrix with the same directions of eigenvectors as those of the time constant matrix T.

Note 6. If the coefficients of the matrix polynomial

$$A(p) = \sum_{o}^{n} A_i\, p^i$$

satisfy the relation /12/, then this polynomial is equal to the product of the A_o matrix and the standard polynomial of the α matrix parameter/multiplication on the right/

$$A(p) = F(p \; ; n, \alpha, I, S_1)\, A_o \qquad /37/$$

where

$$S_1 = A_1\, A_o^{-1} \quad \text{is a positive definite matrix} \qquad /38/$$

If for these coefficients relation /12/ holds, then

$$A(p) = A_0 \cdot F(p\ ;\ n, \alpha\ , I, S_1) \qquad /39/$$

where

$$S_1 = A_0^{-1} A_1 \quad \text{is a positive definite matrix} \qquad /40/$$

IV. An extension of the Naslin algebraic damping criterion

Let us recall that in the scalar domain the Naslin criterion can be formulated as follows

$$a_i^2 - \alpha a_{i+1} \ a_{i-1} \geqslant 0 \qquad /41/$$

for i = 1, 2,....., n-1. If the considered operator a(p) depends upon a number of free parameters, then /n-1/ simultaneous inequalities define admissible regions in the parameter space. These inequalities can be systematically used in many cases and they make possible to obtain better dynamical properties /which can be characterized by the α parameter/. If the number of free parameters is less than /n-1/, then the discussed criterion can be satisfied only in part: to show it,one has to consider the appropiate number of relations of the form /41/, starting from i = 1.

The generalization of this criterion to matrix operators relies on setting up the following conditions

$$\Psi_{2i} \triangleq A_i' A_i - \frac{\alpha}{2}\left(A_{i-1}' A_{i+1} + A_{i+1}' A_{i-1}\right) \geqslant 0$$

$$\Psi_{2i+1} \triangleq A_i' A_{i+1} - A_{i+1}' A_i = 0 \qquad /42/$$

If we start from considering terms of high order, then it is convenient to replace equation /42.2/ by

$$\Psi_{2i-1} \triangleq A_{i-1}' A_i - A_i' A_{i-1} = 0 \qquad /43/$$

/having known A_{i+1} and A_i the expression for A_{i-1} can be determined/. In such a manner, the criterion imposes upon Ψ_{2i} the condition that it has to be a positive semidefinite matrix. On the other hand, it aims at the elimination /or reduction/ of the operator antisymmetry.

Note that the discussion of the problem of making modifications of the antisymmetry can be advantageous and as a consequence, the problem of defining polynomials of two controlled parameters α and β can arise. But it will not be considered here.

Further discussion has to show by means of an example a computing procedure for the generalized criterion.

Consider a stable linear system described by the matrix transmittance

$$\left(I + A\,p + B\,p^2 + C\,p^3\right)^{-1} \qquad /44/$$

The matrix transmittance is a differential operator /of an assumed dimension m/ which transforms the input vector a into the output vector x. Moreover, it expresses the fact that the system satisfies the linear vector differential equation with constant coefficients

$$C\,\dddot{x} + B\,\ddot{x} + A\,\dot{x} + x = a \qquad /45/$$

Let assume that the system is a closed-loop control system, fig.1. The system consists of a controller of the conventional PID structure. The control signal generated by the controller is a function of the error signal $x_o - x$, where x_o is the desired value of x.

The controller transmittance is

$$R(p) = R_{-1} \frac{1}{p} + R_o + R_1 p \qquad /46/$$

where R_{-1}, R_o, R_1 are square matrices of order m.

The matrix differential equation of the closed - loop system has the form

$$\left[R_{-1} + \left(R_o + I\right) p + \left(R_1 + A\right)p^2 + B\,p^3 + C\,p^4\right] x =$$
$$= \left(R_{-1} + R_o\,p + R_1\,p^2\right) a \qquad /47/$$

The application of the Naslin criterion reduces to the consideration of the operator on the left-hand side of equation /47/ /it has effect upon the loop stability/ and to the requirement that the operator coefficients satisfy certain conditions. The inequality /42.1/, turned into equality, is used simultaneously with the condition /43/ and $\alpha = 2$ is chosen.

The discussed operator is written in the form

$$U + V\,p + W\,p^2 + B\,p^3 + C\,p^4 \qquad /48/$$

where

$$U = R_{-1}\,, \quad V = R_o + I\,, \quad W = R_1 + A$$

The generalized criterion imposes the following relations

$$\begin{cases} U'\,V - V'U = 0 \\ V'V - \left(U'W + W'U\right) = 0 \end{cases} \qquad /49.1/$$

$$\begin{cases} V'W - W'V = 0 \\ W'W - \left(V'B + B'V\right) = 0 \end{cases} \qquad /49.2/$$

$$\begin{cases} W'B - B'W = 0 \\ B'B - \left(W'C + C'W\right) = 0 \end{cases} \qquad /49.3/$$

Every one of the systems /49.i/ is linear with respect to one of the unknowns U, V, and W. It makes possible to determine W from equation /49.3/. If W exists, U can be computed using /49.1/.

Firstly, we shall consider the system /49.3/. If W_1 and W_2 are two distinct solutions of this system, then it is easy to show that the difference $W_1 - W_2$ is the solution of the homogeneous system

$$B'W - W'B = 0$$

$$C'W + W'C = 0 \qquad /50/$$

If the only solution of equations /50/ is X = 0, then the system /49.3/has a unique solution. But if non-zero solutions exist, two cases have to be taken into consideration: the system /49.3/ is inconsistent or it has an infinite number of solutions.

Suppose B to be a regular matrix. Then

$$W' = B'W B^{-1} \qquad /51/$$

It enables to write

$$\left(B^{-1} C\right)' Y + Y\left(B^{-1} C\right) = 0 \qquad /52/$$

where $Y \triangleq B'W$ is a symmetric matrix

The necessary and sufficient condition that Y = 0 be the only solution of equation /52/ is that none of the eigenvalues of the B^{-1} C matrix be equal to zero and that this matrix have no eigenvalues of the same modulus but opposite sign. In particular, it results in requirement that the C matrix be regular.

Note. In order to make clear the interpretation of such a constraint, we consider a simple system consisting of a second order process, the transmittance of which is

$$\left(I + A\,p + B\,p^2\right)^{-1} \qquad /53/$$

and a proportional controller

$$R(p) = R_o \qquad /54/$$

We should now consider the operator

$$U + A\,p + B\,p^2, \quad \text{where} \quad U = I + R_o \qquad /55/$$

The generalized criterion imposes the conditions

$$\begin{cases} U'A - A'U = 0 \\ A'A - \left(U'B + B'U\right) = 0 \end{cases} \qquad /56/$$

Suppose that A is a regular matrix. Then

$$U\,A^{-1} = A'^{-1}\,U' \qquad /57/$$

On the base of this equation, it is possible to assume that

$$U\,A^{-1} \triangleq S \quad \text{is a symmetric matrix} \qquad /58/$$

Using it, equation /56.2/ can be written as

$$I = S\,B\,A^{-1} + \left(B\,A^{-1}\right)'S \qquad /59/$$

If all the eigenvalues of the $B\,A^{-1}$ matrix have positive real parts /it is said that such a matrix is "stable"/, this equation has a unique solution. The solution S is a positive definite matrix [9]. If S is known, then

$$U = SA \qquad /60/$$

If the transmittance /53/ describes the system, whose block diagram is shown in fig.2, we have

$$A = T_1 \qquad B = T_2 T_1 \qquad /61/$$

So

$$B A^{-1} = T_2$$

Thus, if the time constant matrices T_1 and T_2 are "stable", equation /59/ has a unique solution of the form

$$U = S T_1 \qquad /62/$$

where S is a symmetric, positive definite matrix which is the unique solution of the equation

$$I = S T_2 + T_2' S \qquad /63/$$

Let us return to the system /49.3/. If the B matrix is regular, the matrix

$$W B^{-1} \triangleq S_o \qquad /64/$$

is symmetric. It satisfies the equation

$$I = S_o \left(C B^{-1}\right) + \left(C B^{-1}\right)' S_o \qquad /65/$$

If $C B^{-1}$ is a "stable" matrix, then the solution of equation /65/ is unique. Moreover, this solution is a positive definite matrix. It follows that

$$W = S_o B \qquad /66/$$

is a regular matrix.

Using equation /49.2/ we can show that

$$V\,W^{-1} \triangleq S_1 \qquad /67/$$

is a symmetric matrix.

The S_1 matrix satisfies the equation

$$I = S_1\left(B\,W^{-1}\right) + \left(B\,W^{-1}\right)' S_1 \qquad /68/$$

But $B\,W^{-1} = S_o^{-1}$. Hence, this matrix is also "stable". The performed analysis proves that the solution of equation /68/ is a positive definite matrix which is uniquely determined. Since an obvious solution of /68/ is

$$S_1 = \frac{S_o}{2} \qquad /69/$$

V can be expressed as

$$V = S_1\,W = \frac{1}{2}\,S_o^{\,2}\,B \qquad /70/$$

A similar procedure allows to establish that the only solution of equation /49.1/ is

$$U = \frac{1}{8}\,S_o^{\,3}\,B \qquad /71/$$

In conclusion, if it is assumed that B and C are regular matrices and on the other hand that the $C\,B^{-1}$ matrix turns out to be "stable", the system /49/ has a unique solution.

The operator /48/ can be written in the form

$$\left(\frac{1}{8}\,S_o^{\,3} + \frac{1}{2}\,S_o^{\,2}\,p + S_o\,p^2 + p^3\right) B + C\,p^4 \qquad /72/$$

where S_0 is the solution of equation /65/.

If the applied controller is of PI type $\left(R_1 = 0\right)$, the operator /48/ can be expressed by the formula

$$U + V p + A p^2 + B p^3 + C p^4 \qquad /73/$$

If it is assumed that A and B are regular matrices and moreover, the matrix $B A^{-1}$ turns out to be "stable", then the application of the generalized Naslin criterion transforms /73/ to the form

$$\left(\frac{1}{2} S_0^{\,2} + S_0 p + p^2\right) A + B p^3 + C p^4 \qquad /74/$$

where S_0 is the solution of the equation

$$I = S_0 \left(B A^{-1}\right) + \left(B A^{-1}\right)' S_0 \qquad /75/$$

Evidently, the practical interest paid to the generalized criterion is justified not only by the fact that it assures the uniquenes of solutions by means of some imposed conditions. A number of problems still merits discussion, for example:

1°. Stability of transformed operators of the type of /72/ or /74/.

2°. Properties of the settings R_i resulting from the information about the U, V, and W coefficients.

3°. The nature of transient states occurring in the closed - loop system.

These questions are currently under investigation and at the present time it is too early for detailed answers. However, it seems that it has been shown that concepts used to describe one-variable systems can be advantageously extended to cover a wide range of problems arising in the domain of multivariable systems. To anyone of new problems resulting from such a generalization a serious attention has to be paid, because it can provide facilities for better understanding of the properties of multivariable systems.

REFERENCES:

1. Menahem M., "Notes on the Functional Representation and the Analysis of Linear Multidimensional Processes". III IFAC Congress, London, 1966.

2. Menahem M., "Notes on the Theory of Multivariable Systems. The Factorization of Polynomial Matrices". IFAC Symposium on Multivariable Control Systems, Duesseldorf, 1968.

3. Naslin P., "Nouveau critère d'amortissement". Automatisme, June, 1960.
"Polynômes normaux et critère algébrique d'amortissement". Automatisme, June and July - August, 1963.
"Retor sur un critère algébrique d'amortissement". Automatisme, September, 1964.

4. Naslin P., "Les régimes variables dans les systèmes linéares et non linéaires". DUNOD, 1962.

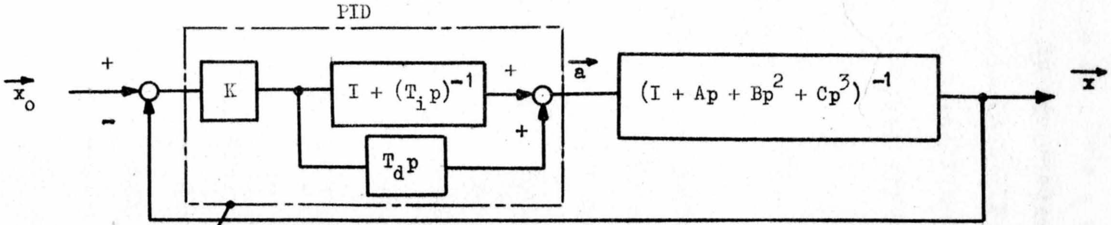
5. Naslin P., "Introduction à la commande optimale". DUNOD, 1966.

6. Menahem M., "Régulation des systèmes industriels complexes." Automatisme, January, 1963.

7. Chaussard R., Grauvogel J., Davoust G.,: "Rational Adjustment of the Controls of a Thermal Power Station". III IFAC Congress, London 1966, Paper 21.A.

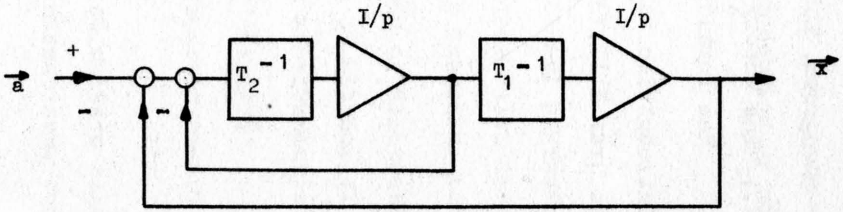
8. Gantmacher F.R., "The Theory of Matrices". Chelsea, New York, 1959.

9. Bellman R., "Introduction to Matrix Analysis." Mc Graw Hill, 1960.



$$R(p) = (I + (T_i p)^{-1} + T_d p)\, K \triangleq R_{-1} \cdot \frac{1}{p} + R_o + R_1\, p$$

Figure 1.



$$\vec{x} = (I + T_1\, p + T_2\, T_1\, p^2)^{-1}\, \vec{a}$$

Figure 2.

USE OF GENERALIZED MOHR CIRCLES FOR MULTIVARIABLE REGULATOR DESIGN

A.G.J. MacFarlane and N. Munro
Control Systems Centre,
University of Manchester Institute of Science and Technology,
England

1. INTRODUCTION

The state space analysis of a multivariable control system depends upon the concept of an operator on a linear vector space. This operator is usually characterised by its eigenvalues and eigenvectors, the eigenvectors defining modes of system behaviour[1], and the eigenvalues providing a link with the frequency domain concepts of classical control theory. In other engineering disciplines in which such matrix operators are used, such as stress analysis, a more complete representation is available by making use of constructions such as Otto Mohr's circle first introduced in 1882.[2] Mohr's original construction was devised to show the effects of a rotation of coordinate axes on the components of stress and strain tensors; its wide utility stems from the fact that it not only gives the principal axis values (i.e. the eigenvector values) but gives a complete representation of planar stress components. The object of this paper is to show that similar constructions can be used in the design of multivariable feedback control systems. In order to establish as close a link as possible with classical control concepts, the circle construction will be defined in terms of the familiar complex frequency plane.[4]

The construction is illustrated in terms of a linear proportional regulator design. Let a multivariable linear feedback controller be applied to a linear constant coefficient dynamical system whose state space equations are

$$\dot{x} = Ax + Bu \qquad (1)$$
$$y = Cx$$

where x is the system state vector (of order n), u is a vector of manipulated variables (of order ℓ), and y is a vector of measured outputs (of order m). Proportional feedback is defined by

$$u = K(r - y) \qquad (2)$$

where r is a vector of reference inputs (of order m). In these equations A,B,C, and K are constant coefficient matrices of orders $n \times n$, $n \times \ell$, $m \times n$ and $\ell \times m$ respectively. The corresponding closed-loop state space equations are

$$\dot{x} = (A - BKC)x + BKr$$
$$y = Cx \tag{3}$$

If r is a vector of constant inputs and the system is asymptotically stable, then the steady state response is given by

$$y(\infty) = - C (A - BKC)^{-1} BKr(\infty)$$
$$= Er(\infty) \tag{4}$$

where $y(\infty)$ and $r(\infty)$ denote the steady state values of $y(t)$ and $r(t)$, and E is a matrix of constants which will specify the accuracy of steady state regulation. The basic linear multivariable regulation problem may then be stated as : given A,B,C and an acceptable E , choose K so that all the eigenvalues of (A - BKC) lie in a specified region of the complex plane. If we put

$$F = -BKC \tag{5}$$

, then it is convenient to call F the system feedback matrix. The stability and general dynamical behaviour of the closed-loop system are determined by the properties of the matrix (A+F), and the steady state accuracy depends on setting acceptably high values for the elements of K. The circle construction gives a means of graphically exhibiting, in an additive manner, the effects of the elements of K on the dynamical properties of (A + F).

2. GENERALIZED MOHR CIRCLES

Let E_1 and E_2 be a pair of orthonormal vectors emanating from the origin of the state space. For any state vector x lying in the plane spanned by E_1 and E_2 such that

$$x = aE_1 + bE_2 \tag{6}$$

define a related vector y, orthogonal to x and in this plane, by

$$y = -bE_1 + aE_2 \tag{7}$$

For any linear state space flow defined by

$$\dot{x} = Ax \tag{8}$$

let a complex number ρ be defined by

$$\rho = \frac{\langle x, Ax \rangle}{\langle x, x \rangle} + j \frac{\langle y, Ax \rangle}{\langle y, y \rangle} \tag{9}$$

The real and imaginary parts of ρ correspond to the components of instantaneous velocity along x (which we may call the recession component) and along y (which we may call the spin component in this plane). This is directly analogous to the resolution of stress into direct stress and shear stress in the conventional Mohr construction[3].

Equation (9) defines a mapping from the plane spanned by E_1 and E_2 in the state space on to the complex plane. If we denote, for any given plane in the state space, the complex numbers determined by (A+F), A and F by $\rho($ A+F $)$, $\rho(A)$ and $\rho(F)$ respectively, we have

$$\begin{aligned}\rho(A+F) &= \frac{\langle x, (A+F)x\rangle}{\langle x, x\rangle} + j\,\frac{\langle y, (A+F)x\rangle}{\langle y, y\rangle} \\ &= \frac{\langle x, Ax\rangle}{\langle x, x\rangle} + j\frac{\langle y, Ax\rangle}{\langle y, y\rangle} + \frac{\langle x, Fx\rangle}{\langle x, x\rangle} + j\frac{\langle y, Fx\rangle}{\langle y, y\rangle} \\ &= \rho(A) + \rho(F) \qquad (10)\end{aligned}$$

It is this additive property of the mapping which is crucial to feedback controller synthesis, since it enables one to graphically display the effect of F on the closed-loop system matrix (A+F).

If any vector x in the selected plane is multiplied by a real positive constant k , then y by definition is also multiplied by the same constant, and we have the corresponding new value of ρ for a matrix A as

$$\begin{aligned}\rho &= \frac{\langle kx, Akx\rangle}{\langle kx, kx\rangle} + j\,\frac{\langle ky, Akx\rangle}{\langle ky, ky\rangle} \\ &= \frac{k^2\langle x, Ax\rangle}{k^2\langle x, x\rangle} + j\,\frac{k^2\langle y, Ax\rangle}{k^2\langle y, y\rangle}.\end{aligned}$$

This shows that the value of ρ is constant along any line emanating from the origin of the state space. Since the whole plane can be swept out by rotating such a ray about the origin, and since each ray maps into a point in the complex plane, it follows that the whole plane in the state space maps into a closed locus in the complex plane.

To find the form of this locus, let the state plane be swept out by taking

$$x = \cos\theta\, E_1 + \sin\theta\, E_2$$
$$y = -\sin\theta\, E_1 + \cos\theta\, E_2$$

where θ is varied continuously. Since in this case $\langle x,x\rangle$ and $\langle y,y\rangle$ are both equal to unity we have

$$\begin{aligned}\rho &= \langle x, Ax\rangle + j\,\langle y, Ax\rangle \\ &= \rho_R + j\,\rho_I\end{aligned} \tag{11}$$

Simple calculations then give

$$\begin{aligned}\rho_R - \alpha_1 &= \beta_1 \cos 2\theta + \beta_2 \sin 2\theta \\ \rho_I - \alpha_2 &= -\beta_1 \sin 2\theta + \beta_2 \cos 2\theta\end{aligned} \tag{12}$$

where

$$\alpha_1 = \tfrac{1}{2}\left(E_1^t A E_1 + E_2^t A E_2\right)$$
$$\alpha_2 = \tfrac{1}{2}\left(E_2^t A E_1 - E_1^t A E_2\right)$$
$$\beta_1 = \tfrac{1}{2}\left(E_1^t A E_1 - E_2^t A E_2\right)$$
$$\beta_2 = \tfrac{1}{2}\left(E_2^t A E_1 + E_1^t A E_2\right)$$

Squaring and adding equations (12) then gives

$$(\rho_R - \alpha_1)^2 + (\rho_I - \alpha_2)^2 = \beta_1^2 + \beta_2^2 \tag{13}$$

This shows that the ρ-locus in the complex plane is a circle with the following characteristics:

(i) Centre at (α_1, α_2)

(ii) Radius of $\sqrt{\beta_1^2 + \beta_2^2}$

(iii) Lines through the origin of the state space at an angle of θ to each other give points subtending an angle 2θ at the centre of the circle. Thus orthogonal lines in the state space determine diametrically opposite points on the circle.

3. PROPERTIES OF GENERALIZED MOHR CIRCLES

Denote the symmetric and skew-symmetric parts of the matrix A by A_+ and A_- respectively where

$$A_+ = \tfrac{1}{2}\left(A + A^t\right) \qquad A_- = \tfrac{1}{2}\left(A - A^t\right)$$

From the additive property of the mapping we immediately have that

$$\rho(A) = \rho(A_+) + \rho(A_-)$$

For the symmetric part we have

$$\alpha_2 = \frac{1}{2}\left(E_2^t A_+ E_1 - E_1^t A_+ E_2\right) = 0$$

since $\quad E_2^t A_+ E_1 = (E_2^t A_+ E_1)^t = E_1^t A_+ E_2$

It follows from this that the symmetric matrices map into circles centred on the real axis of the complex plane. Such circles are precisely analogous to the Mohr circles obtained for plane stress from Cartesian tensors[3].

For the skew-symmetric part we have

$\alpha_1 = 0$ since $E_1^t A_- E_1 = 0$ and $E_2^t A_- E_2 = 0$

We also have that

$$E_2^t A_- E_1 = (E_2^t A_- E_1)^t = E_1^t A_-^t E_2 = -E_1^t A E_2$$

so that $\beta_1 = 0$ and $\beta_2 = 0$.

It follows from this that skew-symmetric matrices map into points on the imaginary axis in the complex plane.

3.1 Real Eigenvalues

Let w_i be a real eigenvector of A associated with the real eigenvalue λ_i. We then have

$$\rho_R = \frac{\langle w_i, \lambda_i w_i \rangle}{\langle w_i, w_i \rangle} = \lambda_i \qquad \rho_I = 0$$

It follows from this that :

(i) If the plane being mapped contains such an eigenvector, the eigenvector is mapped into the associated real eigenvalue.

(ii) A plane spanned by two real eigenvectors w_i and w_j, associated with two real eigenvalues λ_i and λ_j, is mapped into a circle which cuts the real axis in the complex plane at the real eigenvalues λ_i and λ_j. This is directly analogous to the conventional Mohr circle cutting the direct stress axis at the principal values of stress.

3.2 Complex Eigenvalues

Let $u_i \pm jv_i$ be a pair of complex conjugate eigenvectors associated with the pair of complex conjugate eigenvalues $\sigma_i \pm j\omega_i$. We then have

$$A(u_i + jv_i) = (\sigma_i + j\omega_i)(u_i + jv_i) \qquad (14)$$

from which we obtain the pair of relations

$$\begin{aligned} A u_i &= \sigma_i u_i - \omega_i v_i \\ A v_i &= \omega_i u_i + \sigma_i v_i \end{aligned} \qquad (15)$$

It is an immediate consequence of equations (15) that the plane spanned by vectors u_i and v_i is invariant under the action of the operator A. Consider the mapping of this invariant plane into the complex plane. Let the real vectors u_i and v_i be of unit length and let y_u and y_v be a pair of related orthonormal vectors as shown in Figure 1.

When u_i is mapped into the complex plane we have

$$\rho_R = \langle u_i, (\sigma_i u_i - \omega_i v_i)\rangle = \sigma_i - \omega_i \langle u_i, v_i\rangle$$
$$\rho_I = \langle y_u, (\sigma_i u_i - \omega_i v_i)\rangle = -\omega_i \langle y_u, v_i\rangle$$

When v_i is mapped into the complex plane we have

$$\rho_R = \langle v_i, (\omega_i u_i + \sigma_i v_i)\rangle = \sigma_i + \omega_i \langle v_i, u_i\rangle$$
$$\rho_I = \langle y_v, (\omega_i u_i + \sigma_i v_i)\rangle = \omega_i \langle y_v, u_i\rangle$$

An inspection of the relationships in Figure 1 shows that

$$\langle y_v, u_i\rangle = \cos\phi = \sin\theta$$

$$\langle y_u, v_i\rangle = \cos(2\theta+\phi) = \cos(90^\circ+\theta) = -\sin\theta$$

This shows that the map of both u_i and v_i have the same imaginary part $\omega_i \sin\theta$ and have real parts $\sigma_i \pm \omega_i \cos\theta$, so that the invariant plane will map into a circle as shown in Figure 2.

3.3 Eigenvalue Bounds

Let λ_i, μ_i and ν_i (i=1,2,......,n) be the eigenvalues of A, A_+ and A_- respectively, and suppose that all the respective eigenvalues have been ordered such that subscripts 1 and n denote the largest and smallest eigenvalues respectively. Bromwich's inequalities[5] then give that

$$\begin{aligned} \mu_n &\leq Re(\lambda_i) \leq \mu_1 \\ \nu_n &\leq Im(\lambda_i) \leq \nu_1 \end{aligned} \qquad i = 1,2,\dots,n \tag{16}$$

where $Re(\lambda_i)$ and $Im(\lambda_i)$ denote the real and imaginary parts of λ_i respectively. The Courant-Fischer[6] min-max relationships for symmetric matrices give

$$\begin{aligned} \mu_1 &= \max_x \langle x, A_+ x\rangle = \max_x \rho_R(A_+) \\ \mu_n &= \min_x \langle x, A_+ x\rangle = \min_x \rho_R(A_+) \end{aligned} \tag{17}$$

As shown above, the skew-symmetric matrix A_- always maps into a point on the imaginary axis in the complex plane. It also follows from the general discussion of the mapping of the invariant plane associated with complex eigenvectors that the totality of points obtained from the imaginary axis map of A_- will include γ_1 and γ_n. Combining all these results we obtain the inequalities

$$\begin{aligned} \min_x \rho_R(A_+) \leqslant Re(\lambda_i) \leqslant \max_x \rho_R(A_+) \\ \min_{x,y} \rho_I(A_-) \leqslant Im(\lambda_i) \leqslant \max_{x,y} \rho_I(A_-) \end{aligned} \tag{18}$$

4. EXAMPLES OF THE USE OF CIRCLE MAPPINGS IN DESIGN

As a first example of the circle mapping, consider the current-regulating system shown in Figure 3. This maintains the inductor currents constant at a pair of specified values. Suppose we take as a provisional specification:

(i) No interaction between output variables in the steady state.

(ii) Transient settling time of both outputs less than 0·5 seconds.

(iii) Critical damping, for step change in references, on both outputs.

(iv) Steady state error on both outputs less than 5%.

The steady state accuracy matrix E in then given by

$$E = \begin{pmatrix} 0{\cdot}95 & 0{\cdot}0 \\ 0{\cdot}0 & 0{\cdot}95 \end{pmatrix}$$

and the state space equations, for the component values shown, are

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

As the system B and C matrices are both non-singular in this case, the matrix K is readily obtained from

$$K = B^{-1} A\, C^{-1} (I - E^{-1})^{-1}$$

giving the feedback matrix as

$$F = - A\, C^{-1} E^{*} C$$

where $$E^{*} = (I - E^{-1})^{-1} \tag{19}$$

Inserting the appropriate numerical values gives

$$E^{*} = \begin{pmatrix} -19 & 0 \\ 0 & -19 \end{pmatrix};\ F = \begin{pmatrix} -38 & -19 \\ -19 & -38 \end{pmatrix};\ A + F = \begin{pmatrix} -40 & -20 \\ -20 & -40 \end{pmatrix}$$

The open-loop system matrix A maps into the circle shown in Figure 4(a). The closed-loop matrix (A+F) maps into a circle with centre at (−40,0) and radius $\sqrt{(0^2 + 20^2)}$ as shown in Figure 4(b). The closed-loop system eigenvalues are given by the intersection of this latter circle with the real axis and are therefore −60 and −20. Thus the steady state accuracy and settling time requirements have been achieved, but the system response will not be critically damped. To determine whether this latter requirement can be satisfied, we can use the circle mapping to determine the required form of matrix F. For a specific choice of E, say

$$E = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

we get

$$E^{*} = \begin{pmatrix} \gamma & 0 \\ 0 & \mu \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 2\gamma & \mu \\ \gamma & 2\mu \end{pmatrix}$$

where $$\gamma = \left(\frac{\alpha}{\alpha - 1}\right) \quad \text{and} \quad \mu = \left(\frac{\beta}{\beta - 1}\right).$$

The matrix A is symmetric and the matrix F decomposes into

$$F_{+} = \frac{1}{2}\begin{pmatrix} 4\gamma & \mu+\gamma \\ \mu+\gamma & 4\mu \end{pmatrix} \quad ; \quad F_{-} = \frac{1}{2}\begin{pmatrix} 0 & \mu-\gamma \\ \gamma-\mu & 0 \end{pmatrix}$$

The matrix ($A+F_{+}$) maps into a circle with

$$\text{Centre} = (-2 + \gamma + \mu ,\ 0)$$

and Radius = Modulus $\left(\gamma-\mu, \frac{\mu+\gamma-2}{2}\right)$.

The matrix F_- maps into a point on the imaginary axis at $\left[0, \frac{1}{2}(\gamma-\mu)\right]$. Consideration of the additive combination of $(A + F_+ + F_-)$ then shows that, for critical damping, the circle must be tangent to the real axis in the complex plane, and so we must have that

$$\frac{\gamma-\mu}{2} = \text{Modulus}\left(\gamma-\mu, \frac{\mu+\gamma-2}{2}\right).$$

If we take both steady state accuracies to be the same, then we have that

$$\gamma = \mu = 1$$

Now these values of γ and μ will not give an acceptable steady state accuracy. The circle mapping has therefore shown that the original specification is unattainable since good regulation, for this particular system, is incompatible with a critically damped response. If the critical damping requirment is removed then a non-interacting (in the steady state) design is readily obtained for the required accuracy with a gain matrix

$$K = \begin{pmatrix} 38 & 19 \\ -19 & 19 \end{pmatrix}.$$

4.1 Cascaded Blending Tank Example

For a third order system representative of process control work, consider the regulation of liquid levels in a set of three cascaded blending tanks shown in Figure 5. For the component values shown the system state space equations are

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -0{\cdot}02 & 0{\cdot}02 & 0 \\ 0{\cdot}01 & -0{\cdot}02 & 0{\cdot}01 \\ 0 & 0{\cdot}005 & -0{\cdot}01 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0{\cdot}2 & 0 & 0 \\ 0 & 0{\cdot}1 & 0 \\ 0 & 0 & 0{\cdot}05 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Suppose the liquid level regulator has to be designed to the following specification:

(i) No steady state interaction between output variables.

(ii) Steady state error on all outputs less than 5%.

(iii) Decay time of all output transients less than 100 seconds.

For the form of steady state error matrix E dictated by the system specification, the feedback matrix F will have the form

$$F = \begin{pmatrix} -2\alpha & 2\alpha & 0 \\ \alpha & -2\alpha & \alpha \\ 0 & 0.5\alpha & -\alpha \end{pmatrix} \qquad (20)$$

We may use the inequalities given by equation (18) to construct a boundary rectangle for the eigenvalues of the matrix (A+F). Doing this for the form of F given by equation (20) shows that acceptable damping factors may be obtained, and so the whole bounding rectangle can be moved to the left in the complex plane simply by increasing α . The result for the open-loop matrix A and the closed-loop matrix (A+F) where

$$F = \begin{pmatrix} -0{\cdot}58 & 0{\cdot}58 & 0 \\ 0{\cdot}29 & -0{\cdot}58 & 0{\cdot}29 \\ 0 & 0{\cdot}145 & -0{\cdot}29 \end{pmatrix}$$

are shown in Figures 6(a) and 6(b) respectively. A better indication of the corresponding closed-loop dynamical behaviour is given by the generalized Mohr circle through the largest and smallest eigenvalues of the symmetric part of (A+F) shown in Figure 7. The gain matrix corresponding to this value of F is

$$K = \begin{pmatrix} 2{\cdot}9 & -2{\cdot}9 & 0 \\ -2{\cdot}9 & 5{\cdot}8 & -2{\cdot}9 \\ 0 & -2{\cdot}9 & 5{\cdot}8 \end{pmatrix}$$

and the steady state error matrix is

$$E = \begin{pmatrix} 0{\cdot}97 & 0 & 0 \\ 0 & 0{\cdot}97 & 0 \\ 0 & 0 & 0{\cdot}97 \end{pmatrix}$$

The corresponding closed-loop system eigenvalues are

$$\lambda_1 = -0{\cdot}0679 \qquad \lambda_2 = -0{\cdot}377 \qquad \lambda_3 = -1{\cdot}055$$

so that the overall design specification has been achieved.

5. CONCLUSIONS

Most of the current state-space techniques used in multivariable system theory are related to concepts used in classical field theory. For example, the Lyapunov stability theory uses concepts analogous to potential and Lagrangian derivative, and the Pontryagin optimal control theory uses techniques analogous to Hamilton's wavefront techniques in optics. This paper has shown how the idea of circular representations of an operator acting on a plane, first introduced in a simple form by Mohr in stress analysis, can form the basis of a design technique for linear multivariable regulators. Two simple examples of the use of this technique have been given which show that multiple feedback loops are easily handled, at least for systems of low order. It is of very great interest that such close links should exist between control theory and general field theory; further links should be sought which could cast fresh light on the design of multivariable systems.

6. ACKNOWLEDGEMENT

The authors are grateful to Dr. R. Sabouni of the Control Systems Centre, UMIST, for assistance in these investigations.

7. REFERENCES

(1) Rosenbrock,H.H.R.:"Distinctive problems of process control", Chemical Engineering Progress, Vol.58,1962,p.43.

(2) Mohr,O.:"Uber die Darstellung des Spannungszustandes und des Deformationszustandes eines Korperelementes und uber die Anwendung derselben in der Festigkeitslehre", Civilingenieure, Vol. 28, 1882, p.112.

(3) Jaeger,L.G.:"Cartesian tensors in engineering science", Pergamon Press, 1966.

(4) MacFarlane,A.G.J. and Munro, N.:"Mappings of the state space into the complex plane and their use in multivariable systems analysis",International Journal of Control, Vol.7, 1968,p.507.

(5) Mirsky, L.:"Introduction to linear algebra",Oxford,1955.

(6) Beckenbach, E. and Bellman, R.: " Inequalities ", Ergebnisse der Mathematik und ihrer Grengebiete, Vol. 30, Springer-Verlag, Berlin, 1961.

8. PRINCIPAL SYMBOLS USED

x = vector of order n , components $x_1, x_2, \ldots, x_n$

$\dot{x}$ = vector of order n , components $\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}$

x^t = transpose of vector x

A = matrix of order n^2, components a_{ij}

A^t = transpose of matrix A

$\langle x, y \rangle$ = scalar product of vectors x and y

$= \sum_i x_i y_i$

$\langle x, (y+z) \rangle = \langle x, y \rangle + \langle x, z \rangle$

$\langle x, Ax \rangle = x^t Ax$

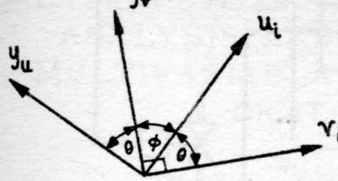
I = unit matrix of order n

ρ_R = real part of complex number ρ

ρ_I = imaginary part of complex number ρ

(ρ_R, ρ_I) = explicit component form of writing complex number ρ

modulus $(\rho) = \sqrt{(\rho_R^2 + \rho_I^2)}$.



$\phi = \cos^{-1} \langle y_v , u_i \rangle$
$= \sin^{-1} \langle u_i , v_i \rangle$
$\theta = \cos^{-1} \langle u_i , v_i \rangle$

FIGURE 1

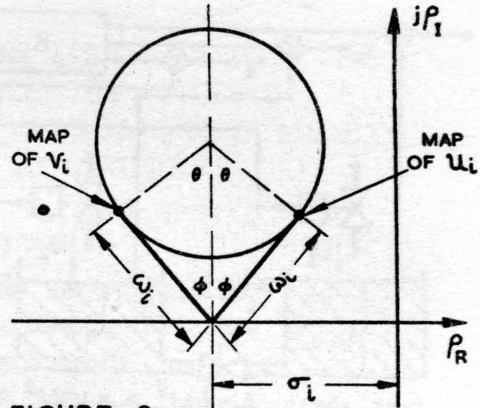


FIGURE 2

r_1, r_2 SPECIFY REFERENCE VALUES OF CURRENTS x_1 AND x_2.

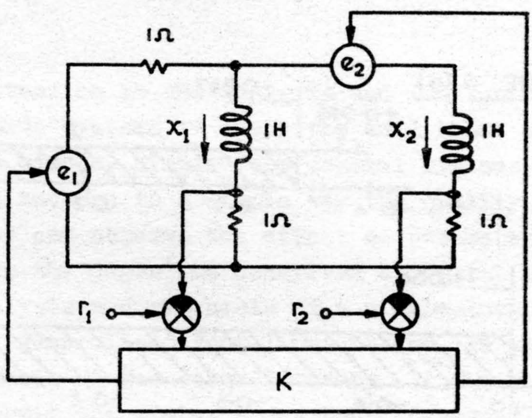


FIGURE 3

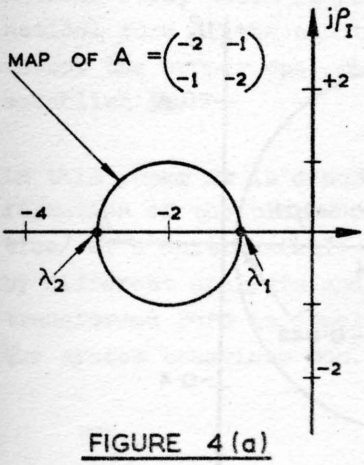


FIGURE 4 (a)

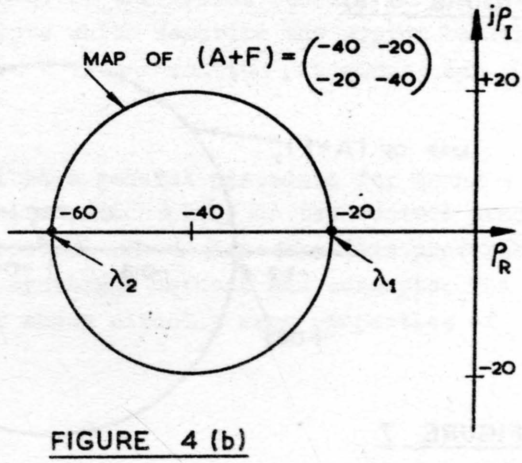
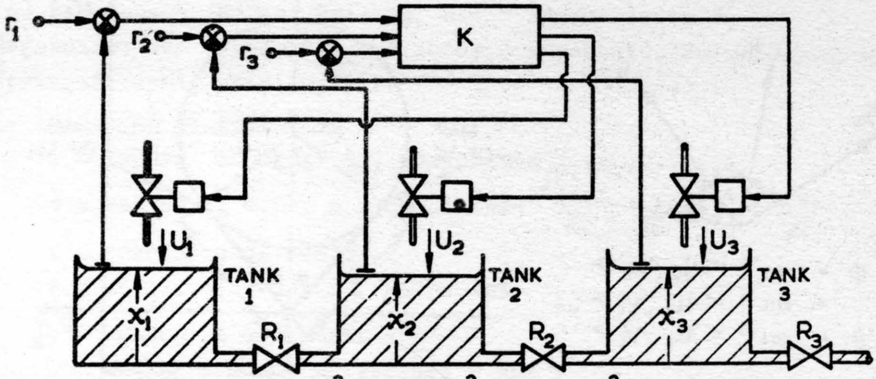


FIGURE 4 (b)

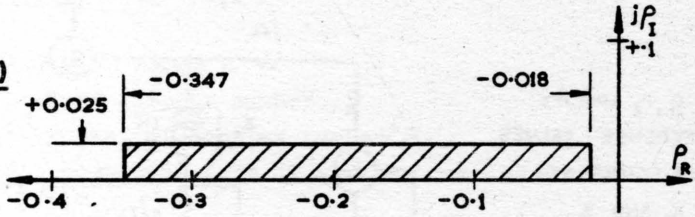


TANK AREAS: $A_1 = 5M^2$, $A_2 = 10M^2$, $A_3 = 20M^2$.

HYDRAULIC RESISTANCES: $R_1 = R_2 = R_3 = 10 \text{ SEC}/M^2$.

FIGURE 5

FIGURE 6 (a)



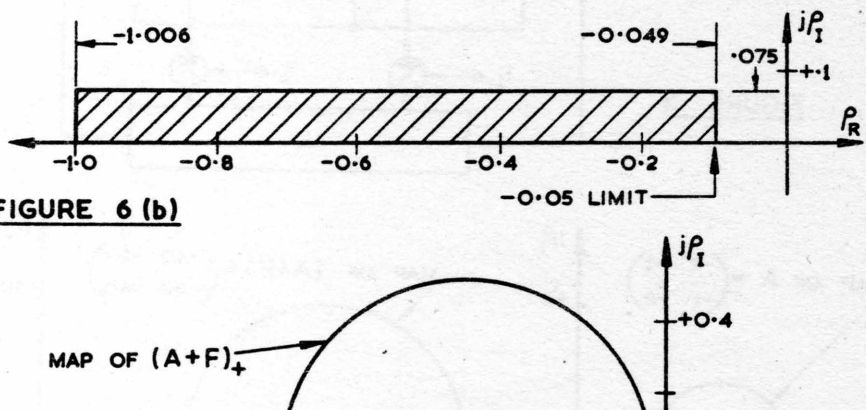


FIGURE 6 (b)

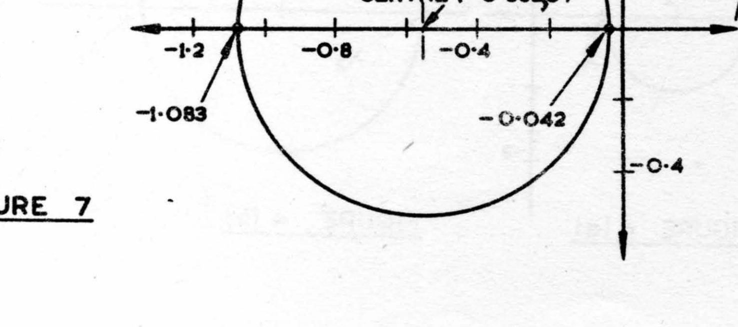


FIGURE 7

A TRANSFORMATION METHOD FOR THE ANALYSIS AND THE SYNTHESIS OF MULTIVARIABLE CONTROL SYSTEMS BY DIGITAL COMPUTER

J. Gyürki

Research Institute for Automation
of the Hungarian Academy of Sciences
Budapest, Hungary

Introduction

There is a natural intention in the analysis and the synthesis of multivariable control systems to apply the well known methods and procedures of the simple, single-loop control systems.
In this manner we can measure in a simple way the quality of the system performance and can observe the effect of parameter and structural changes. In the paper[1] is described a digital computer procedure for the analysis and synthesis of a single-loop control system on the base of generalised root-locus method. To apply it and many other methods of the control theory in the practice it is necessary to submit the system under test in proper mathematical form, which is needed by the method itself. If the mathematical form of the equations which describe the system behaviour is not the proper one, then a transformation procedure can establish it.

In this paper it is described a general procedure for trans - formation of the block-diagram /widespread in the control practice/ of a multivariable control system into the form preferable by different analysis and synthesis methods and concepts. The transformed form in itself shows directly many properties of the system behaviour too.

Our investigations are restricted to the linear, time invariant control systems.

The generation of the state matrix

Let us suppose the system under test consist of M linear blocks, which are connected each other in complicated manner. The output signal of any component part of the entire system may be significant according to the system performance, so it is named as output variable, but it is not necessary that each block output signal is output variable. Let us suppose that the noises and disturbances and the external, useful input signals appear only at the input points of the component parts of the system. If the situation is not a such one, on the base of the original block-diagram we can generate an equivalent block in addition to the blocks existing in the system which holds that signal on its input.

Let the transfer function of the i-th component part of the system $W_i(s)$. In general case this block is characterized by a numerator of m_i degree, and a denominator of n_i degree. The most complicated case exists, when the output signal of this block is connected to each element's input point, and there are connections from each element's output to its input point. The input-output connections appear in form of weight factors which are real numbers /negative or positive/. The i-th block as part of the entire system is plotted on the Fig. 1. The mathematical form of the transfer function is:

$$W_i(s) = \frac{\sum_{j=0}^{m_i} a_j s^j}{\sum_{j=0}^{n_i-1} b_j s^j + s^{n_i}} \tag{1}$$

We restrict now ourselves to the case when $m_i < n_i$. /The case of $m_i = n_i$ is discussed in the Appendix I., but $m_i > n_i$ is not an existing real element./

It is well known from the literature[2] that the rational fractional transfer function equivalent to a set of first order differential equations. One of the possible set is the canonical form, which in matrix form:

$$\begin{bmatrix} \dot{x}_i \\ \dot{x}_{i2} \\ \dot{x}_{i3} \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_{in_i} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -b_0 & -b_1 & -b_2 & -b_3 & \cdots & -b_{n_i-1} \end{bmatrix} \cdot \begin{bmatrix} x_i \\ x_{i2} \\ x_{i3} \\ \vdots \\ \vdots \\ \vdots \\ x_{in_i} \end{bmatrix} + \begin{bmatrix} z_{i1} \\ z_{i2} \\ z_{i3} \\ \vdots \\ \vdots \\ \vdots \\ z_{in_i} \end{bmatrix} [u_i] \qquad (2)$$

or

$$\dot{X}_i = A_i X_i + Z_i u_i \qquad (3)$$

where

$$\begin{aligned} z_{i1} &= a_{n_i-1} \\ z_{ik} &= a_{n_i-k} - \sum_{l=n_i-1}^{n_i-k+1} b_l z_{i(l-n_i+k)} \end{aligned} \qquad (4)$$

Another possible form without modification of the genuine parameters, also in matrix form:

$$\begin{bmatrix} \dot{x}_i \\ \dot{\underline{x}}_{i2} \\ \dot{\underline{x}}_{i3} \\ \vdots \\ \vdots \\ \vdots \\ \dot{\underline{x}}_{in_i} \end{bmatrix} = \begin{bmatrix} -b_{n_i-1} & 1 & 0 & 0 & \cdots & 0 \\ -b_{n_i-2} & 0 & 1 & 0 & \cdots & 0 \\ -b_{n_i-3} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ -b_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} x_i \\ \underline{x}_{i2} \\ \underline{x}_{i3} \\ \vdots \\ \vdots \\ \vdots \\ \underline{x}_{in_i} \end{bmatrix} + \begin{bmatrix} a_{n_i-1} \\ a_{n_i-2} \\ a_{n_i-3} \\ \vdots \\ \vdots \\ \vdots \\ a_0 \end{bmatrix} [u_i] \qquad (5)$$

or

$$\dot{\underline{X}}_i = \underline{A}_i X_i + \underline{Z}_i u_i \qquad (6)$$

The equations (2), (4) and (5) can be derived from the transfer function (1) by simple arithmetic. In the equations (2), (3), (5), (6) U_i represents a symbolic input variable, which according to the Fig. 1. consist of the external input signal of the i-th element and the combination of the output signals of each other block. Therefore:

$$U_i = U_{ii} + \sum_{j=1}^{M} k_{ij} x_j \qquad (7)$$

The preceding matrix representations are only two possibilities among the possible forms, which are dependent on the choosing of the state vector. The main advantage of the equations (2) and (5) are that the input signal appears in them as a scalar variable. The derivates of the input signals do not appear, which reject many problems of the input functions containing discontinuities.

On the base of matrix descriptions (2) and (5) of the component parts can construct the state matrix of the entire system by the combination of the elementary sub-matrices. The different sub-system build the state matrix of the entire system through the Z_i and $\underline{Z}_i$ column vectors, which in connection with the equation (7) represent the real connections between the different blocks. The construction of the state matrix in this way is a formal procedure. The equation (8) gives the form of this state matrix. The "input" matrix B has a simple generating procedure too, owing to the assumptions mentioned above about the input signals.

$$\begin{bmatrix}\dot x_1\\ \dot x_{12}\\ \vdots\\ \dot x_{1n_1}\\ \vdots\\ \dot x_i\\ \dot x_{i2}\\ \vdots\\ \dot x_{in_i}\\ \vdots\\ \dot x_M\\ \dot x_{M2}\\ \vdots\\ \dot x_{Mn_M}\end{bmatrix} = \left[\begin{array}{c|c|c|c|c} k_{11}[Z_1]+[A_1] & \cdots & k_{1i}[Z_1]\ \ 0 & \cdots & k_{1M}[Z_1]\ \ 0 \\ \hline \vdots & & \vdots & & \vdots \\ \hline k_{i1}[Z_i]\ \ 0 & \cdots & k_{ii}[Z_i]+[A_i] & \cdots & k_{iM}[Z_i]\ \ 0 \\ \hline \vdots & & \vdots & & \vdots \\ \hline k_{M1}[Z_M]\ \ 0 & \cdots & k_{Mi}[Z_M]\ \ 0 & \cdots & k_{MM}[Z_M]+[A_M] \end{array}\right] \cdot \begin{bmatrix}x_1\\ x_{12}\\ \vdots\\ x_{1n_1}\\ \vdots\\ x_i\\ x_{i2}\\ \vdots\\ x_{in_i}\\ \vdots\\ x_M\\ x_{M2}\\ \vdots\\ x_{Mn_M}\end{bmatrix} +$$

$$+ \left[\begin{array}{c|c|c|c|c} [Z_1] & \cdots & [0] & \cdots & [0] \\ \hline \vdots & & \vdots & & \vdots \\ \hline [0] & \cdots & [Z_i] & \cdots & [0] \\ \hline \vdots & & \vdots & & \vdots \\ \hline [0] & \cdots & [0] & \cdots & [Z_M] \end{array}\right] \cdot \begin{bmatrix}u_{11}\\ \vdots\\ u_{ii}\\ \vdots\\ u_{MM}\end{bmatrix} \qquad (8)$$

or:

$$\dot X = A \cdot X + B \cdot U$$

The transformation of the state matrix into canonical form

The state matrix of the entire system carries both the quality and the quantity characteristics of the system, but not in proper form. The stability of the system, the possibilities of the control among others are dependent on the eigenvalues of this matrix. The connections existing between the input signals and the output variables and the "weight" of these connections are very important characteristics, too. The purpose of the transformation method is to explain these connections in explicit form, and to give the possibility for determination of the quality characteristics and other properties of the input-output signal connections. Both the quality and the quantity characteristics could be comprised in a single transfer element which appears as a block between one output variable and each input signals. The zeros and the poles of this rational transfer function in itself, and their dependence on the system parameters give the possibility to investigate the system behaviour, insure optimal work and so on.

If we can transform the state matrix, constructed on the base of the block diagram, into canonical form relating to one of the original output variable, the transformed form visualises all the connections existing between the chosen output variable and each input signals in proper form. On the base of the canonical form we can compute the different connections in transfer function form too by inversion of the relations (4). The information which are comprised in a transfer element are now available /poles and zeros, the whole course of the frequency response, etc./.

The simplification of the block-diagram, also its reduction according to one output variable is performed really by this transformation. After this transformation the analysis and the synthesis of multivariable system is lead back to the analysis and synthesis of a single control system which has more input signals. The transformed form also represents a block diagram which is plotted on the Figure 2.

The transformation of the state equations (8) into canonical form related to one of the original block output signal formally is a mathematical operation to choose such intermediate new variables which are linear combinations of the original ones that the new variables give canonical form instead of the equation (8). These mathematical operations, the substitution and generation of the new variables conclude to a matrix transformation.

Let us choose the connection between the original state variables and the new coordinates which are the phase coordinates belonging to one of the output variables in a matrix form:

$$X_i = F \cdot X \qquad (9)$$

or in detailed form, in accordance with the equation (8):

$$\begin{bmatrix} X_i \\ X_i' \\ X_i'' \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_i^N \end{bmatrix} = \begin{bmatrix} \\ \\ \\ & F & \\ \\ \\ \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_{12} \\ \vdots \\ X_2 \\ X_{22} \\ \vdots \\ X_i \\ X_{i2} \\ \vdots \\ X_M \\ X_{M2} \\ \vdots \end{bmatrix} \qquad (10)$$

By substitution of equation (10) into equation (8), on condition that F is an invertible matrix, we get:

$$\dot{X}_i = F \cdot A \cdot F^{-1} X_i + F \cdot B \cdot U \qquad (11)$$

The problem of the transformation is to choose the matrix F in such a form that the $F \cdot A \cdot F^{-1}$ matrix represents a cano - nical matrix. In this case one of the original output variables

is unchangeable and the elements of the new state variable appear as its derivates, i.e. phase coordinates belonging to the chosen output variable.
In the Appendix II. it is proved that if we choose the F matrix in the following form:

$$\left[F \right] = \begin{bmatrix} c_i \\ c_i A \\ c_i A^2 \\ \vdots \\ c_i A^{N-1} \end{bmatrix} \tag{12}$$

where c_i represents a special row vector, which given the connection between the chosen output variable and the original state vector:

$$x_i = c_i X \tag{13}$$

Namely if the block diagram consist of such blocks that $m_i < n_i$ is always true /the state equation agrees with equation (8)/ and the chosen output variable appears as the output signal of the i-th block of the genuine block diagram, then in the vector c_i the i-th element equal 1, but the others are equal 0. If the genuine block diagram contains of blocks characterized by $m_i = n_i$, which situation is discussed in the Appendix I., the vector c_i depends on the matrix C .

The general theory of the preceding matrix transformation, which is essentially a similarity transformation, is known. The mathematical questions of the "dynamic" and "static" similarity of matrices and the rigorous proof of the pro - perties of this similarity was described by Markus[3]. Some problems which are connected to this problem was discussed by Coppel[4]. The preceding papers deal with the situation, when the matrix elements are function of the time too.

In this paper we were restricted to the time invariant case and on the base of the general theory a procedure for transformation a constant matrix into canonical form was developed.

The idea was to visualise the signal flow simplification as a matrix transformation and to make a digital computer programme to perform the generation of the state matrix from the block-diagram and to transform it into canonical form by digital computer. The procedure give us the possibility to analyse a multivariable control system like to the simple single-loop systems.

The digital computer programme

The digital computer programme performs partly the operations connected to the construction of the state equations on the base of the block diagram, and partly the transformation of the state matrix into canonical form. These two parts are separated, in some cases only the generation of the state equations is needed, but in others only the transformation of a state matrix, which was generated by the first programme part or any other procedure. In addition to the preceding two programme parts the entire programme contain some analysis and synthesis procedures of the classical control engineering. These procedures visualise certain quality and quantity characteristics of the entire system and the properties of the input-output signal connections, which are very useful data with regard to the control of the system.

In detail the main parts of the entire programme contain the following computations:

1st part: On the base of the sufficient data of the block diagram /the degrees of the individual blocks and its parameters/ converts each block into canonical form by help of equation 4 . From the canonical sub-matrices with the connection pattern generates the matrices A, B, C and D on the base which was described in the first part of this paper and in the Appendix I.

__2nd part:__ In possession of the matrices A, B, C and D, which were generated by the 1-st part or by other procedure, the programme generates the c_i vectors and then the F matrix belonging to the different output variables. After this in every case computes the F . B matrix, and by help of the equation (II - 13) the k' row vector. The final computation is the reduction of the transformed equations to the transfer function form by help of inversion of the equations (4). The results are the degrees and the parameters of the transfer functions existing between the output variable and the genuine input signals.

__3rd part:__ The additional procedures compute:

a./ The frequency response /Bode plot or the Nyquist diagram/ belonging to the different input-output connections.

b./ The eigenvalues of the state matrix /or the poles of the system/ and the zeros of the individual input-output transfer functions.
This procedure give us the possibility to compute the root-locus /pole-locus and the zero locus/ belonging to the change of whichever realistic parameter of the genuine block diagram.

__Example__

In many cases appears in the control system such an element, which mathematically is described by partial differential equation. For example the heat transfer in a bar /Figure 4./ isolated at the one end, and fed on the other by a heat source of infinite heat capacity is carried out according to the equation:

$$\frac{\partial^2 T}{\partial x^2} = a \cdot \frac{\partial T}{\partial t} \tag{14}$$

with

$$T(0,t) - T_k \quad ; \quad \frac{\partial T(L,t)}{\partial x} = 0$$

The transient response of the bar /unit step in the temperature T_k and the response in the T_L / and many other characteristics of the heat transfer along it are interesting for the control engineer, if the bar appears as part of a control system.
The usual technique in the simulation of partial equations is the finite difference method. The substitution of the exact partial differential quotient by finite differences /see Fig.4./:

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x=x_i} = \frac{T_{i-1} - 2T_i + T_{i+1}}{(\Delta x)^2} \tag{15}$$

concludes to a set of first order ordinary differential equations. The number of equations depends on the number of cells. The set of equations according to the boundary conditions are:

$$\frac{dT_1}{dt} = a\,\frac{T_k - 2T_1 + T_3}{(\Delta x)^2}$$

$$\frac{dT_i}{dt} = a\,\frac{T_{i-1} - 2T_i + T_{i+1}}{(\Delta x)^2} \qquad i = 2, 3, \dots (n-1) \tag{16}$$

$$\frac{dT_n}{dt} = a\,\frac{T_{n-1} - T_n}{(\Delta x)^2}$$

This set of equations could be visualised as the block diagram plotted on the Figure 5. The properties of this system depend on the number of cells and the boundary conditions too. This block diagram was analised by the digital computer programme. The important signals in this case were T_k and T_L .
The transfer function between them /on condition that $a/(\Delta x)^2 = 1$, at n = 10/ in the cases n = 5 and 10, are:

$$W_5(s) = \frac{1}{1 + 60s + 560\, s^2 + 1792\, s^3 + 2304\, s^4 + 1024\, s^5}$$

$$W_{10}(s) = \frac{1}{1+55s+495s^2+1716s^3+3003s^4+3003s^5+1820s^6+680s^7+153s^8+19s^9+s^{10}}$$

The pole configurations belonging to the preceding transfer functions, are:

n = 5		n = 10	
-0.2025435	-01	-0.2233824	-01
-0.1725699	+00	-0.1980622	+00
-0.4288377	+00	-0.5338963	+00
-0.7077138	+00	-0.1000000	+01
-0.9206240	+00	-0.1554969	+01
		-0.2149351	+01
		-0.2731123	+01
		-0.3246012	+01
		-0.3653588	+01
		-0.3910657	+01

The frequency responses at $n = 5$ and 10 are plotted on the Figure 6.

APPENDIX I.

Let us suppose the system given by the block diagram contains such blocks, or all the blocks belong to the system are characterized by a transfer function in which the numerator has the same degree as the denominator. In this case the construction of the state matrix differs from the procedure was described in the first part of this paper. Let the transfer function of the i-th component part of the systems be:

$$W_i(s) = \frac{\sum_{j=0}^{n_i} a_j s^j}{\sum_{j=0}^{n_i-1} b_j s^j + s^{n_i}} \qquad (I.-1)$$

The equation (I. - 1) could be modified by division of the numerator by the denominator:

$$W_i(s) = a_{n_i} + W_i'(s) \qquad (I.-2)$$

where $W_i'(s)$ is characterized by a numerator of less degree than the denominator, and the parameters of its numerator are in simple connection with the genuine parameters of the $W_i(s)$. The output signal of the i-th element now is:

$$X_i = W_i'(s) \cdot U_i + a_{n_i} U_i = X_i' + a_{n_i} U_i \qquad (I.-3)$$

On the base of the equation (I. - 3) the genuine block diagram can be splitted into two different block diagrams. The first part consist of linear dynamic blocks, derived from the genuine blocks by division of the numerator with the denominator. In this system all blocks have a less degree numerator than the denominator and there is applicable the procedure of generating of the state matrix was described in the preceding section. The second part contains only static elements, only constants. In the static system instead of the original blocks appear 0 if $m_i < n_i$ or a_{n_i} if $m_i = n_i$. All the connections agree with the genuine ones in the two subsystems, and the input signals too. In this way the genuine block diagram is characterized by the two subsystems, but mathematically the state equations of the entire system also is the sum of two parts. The dynamic system could be described by a system of differential equations between the modified X_i' variables and the input signals, but the static one by a system of linear equations between the modified and the genuine variables and the input signals. This later one gives the connection between the new variables and the genuine output variables.

We restrict here ourselves to the generation of the linear set of equations, which describe the behaviour of the static block diagram. From the modified dynamic block diagram we can construct the system of differential equations by the procedure was described in the first part of this paper.

The schematic plot of the i-th element of the static system is plotted on the Figure 3. The equation of its operation is:

$$x_i' + a_{n_i}(k_{i1}x_1 + k_{i2}x_2 + \cdots + k_{ii}x_i + \cdots + k_{iM}x_M) = x_i \qquad (I.-4)$$

By extending the preceding generation procedure to all elements, we get the connection between the genuine and the transformed variables. The equations in matrix-form became:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_i' \\ \vdots \\ x_M' \end{bmatrix} = \begin{bmatrix} 1-k_{11}a_{n_1} & -k_{12}a_{n_1} & -k_{13}a_{n_1} & \cdots & -k_{1i}a_{n_1} & \cdots & -k_{1M}a_{n_1} \\ -k_{21}a_{n_2} & 1-k_{22}a_{n_2} & -k_{23}a_{n_2} & \cdots & -k_{2i}a_{n_2} & \cdots & -k_{2M}a_{n_2} \\ -k_{31}a_{n_3} & -k_{32}a_{n_3} & 1-k_{33}a_{n_3} & \cdots & -k_{3i}a_{n_3} & \cdots & -k_{3M}a_{n_3} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ -k_{i1}a_{n_i} & -k_{i2}a_{n_i} & -k_{i3}a_{n_i} & \cdots & 1-k_{ii}a_{n_i} & \cdots & -k_{iM}a_{n_i} \\ \vdots & & \vdots & & & & \vdots \\ -k_{M1}a_{n_M} & -k_{M2}a_{n_M} & -k_{M3}a_{n_M} & \cdots & -k_{Mi}a_{n_i} & \cdots & 1-k_{MM}a_{n_M} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_M \end{bmatrix} \qquad (I.-5)$$

By inversion of the matrix equation (I. - 5) we can generate the connections between the new variables $x_1', x_2', \ldots, x_M'$ and the original variables $x_1, x_2, \ldots, x_M$.

$$X = C \cdot X' \qquad (I.-6)$$

There is also a direct static connection between the state variables and the external input signals. The matrix-form of these connections agree with the equation (I. - 5), but instead of the X_i' variables appear the $a_{n_i} U_{ii}$ variables. The inverse matrix, which was generated once in connection with the equation (I. - 5) could be modified so that the connection between the genuine output variables, the modified ones and the external input signals appear in the following matrix form

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_i \\ \vdots \\ X_M \end{bmatrix} = \begin{bmatrix} & & \\ & C & \\ & & \end{bmatrix} \cdot \begin{bmatrix} X_1' \\ X_2' \\ X_3' \\ \vdots \\ X_i' \\ \vdots \\ X_M' \end{bmatrix} + \begin{bmatrix} & & \\ & D & \\ & & \end{bmatrix} \cdot \begin{bmatrix} U_{11} \\ U_{22} \\ U_{33} \\ \vdots \\ U_{ii} \\ \vdots \\ U_{MM} \end{bmatrix} \qquad (I.-7)$$

or in a simpler form:

$$Y = C \cdot \underline{X}' + D \cdot U$$

The state equations of the entire system in the most general case became:

$$\begin{aligned} \dot{X}' &= A \cdot X' + B \cdot U \\ Y &= C \underline{X}' + D \cdot U \end{aligned} \qquad (I.-8)$$

<u>APPENDIX II.</u>

Let us given the state equations of a multivariable system in general form, which originates from a block diagram or was constructed by other procedure:

$$\dot{X} = A \cdot X + B \cdot U \qquad (II.-1/a)$$
$$Y = C \cdot X + DU \qquad (II.-1/b)$$

where:

A : $N \times N$ matrix ;

B : $N \times M$ matrix ;

C : $Q \times N$ matrix ;

D : $Q \times M$ matrix ;

X : state coordinates;

U : input variables ;

Y : output variables

N : number of state coordinates

M : number of the input signals, or blocks

Q : number of the output variables

We try to construct a transformation which can transform the matrix equation (II. - 1) into a kinematical similar system that have a canonical matrix related to one of the output variables.

$$\dot{Y}_i' = K \cdot Y_i + B_i U \qquad (II.-2)$$
$$i = 1, 2, \ldots, Q$$

with

$$y_i' = c_i X \qquad Y' = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_Q \end{bmatrix} = C \cdot X$$
$$Y = I Y' + D \cdot U$$

Andthe form of the K matrix is:

$$[K] = \left[\begin{array}{c:c} 0 & I \\ \hdashline \multicolumn{2}{c}{k'} \end{array}\right] \qquad (II.-3)$$

Let

$$y_i' = c_i X \qquad (II.-4)$$

where c_i is a vector which generates the i-th output variable from the genuine state coordinates in accordance with the equation (II. - 1/b.).

Let

$$Y_i' = F \cdot X \qquad (II.-5)$$

Let us suppose that the inverse matrix F^{-1} exists, then:

$$X = F^{-1} \cdot Y_i' \qquad (II.-6)$$

Let us choose the matrix F in special form:

$$F = \begin{bmatrix} c_i \\ c_i A \\ \vdots \\ c_i A^{N-2} \\ c_i A^{N-1} \end{bmatrix} \qquad (II.-7)$$

According to the equation (II. - 1/a.) we get:

$$F \dot{X} = F \cdot A \cdot X + F \cdot B \cdot U$$

or

$$\dot{Y}_i' = F \cdot A \cdot F^{-1} \cdot Y_i' + F \cdot B \cdot U = K \cdot Y_i' + B_i U \qquad (II.-8)$$

In the following part we shall proof, that $F A F^{-1}$ matrix always is a canonical matrix if F is connected with the matrix according the equation (II. - 7).

Proof:

$$F \cdot A \cdot F^{-1} = \begin{bmatrix} c_i \\ c_i A \\ c_i A^2 \\ \vdots \\ c_i A^{N-2} \\ c_i A^{N-1} \end{bmatrix} \cdot \begin{bmatrix} A \end{bmatrix} \cdot \begin{bmatrix} c_i \\ c_i A \\ c_i A^2 \\ \vdots \\ c_i A^{N-2} \\ c_i A^{N-1} \end{bmatrix}^{-1} = \begin{bmatrix} c_i \\ c_i A \\ c_i A^2 \\ \vdots \\ c_i A^{N-2} \\ \hline c_i A^{N-1} \end{bmatrix} \cdot \begin{bmatrix} c_i A^{-1} \\ \hline c_i \\ c_i A \\ \vdots \\ c_i A^{N-3} \\ c_i A^{N-2} \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} A' \\ \hline c_i A^{N-1} \end{bmatrix} \cdot \begin{bmatrix} c_i A^{-1} \\ A' \end{bmatrix}^{-1} = \begin{bmatrix} A' \\ \hline c_i A^{N-1} \end{bmatrix} \cdot \left[\begin{array}{c|c} \underline{c}_i \underline{A}_{-1} & \underline{A}_{-1} \end{array} \right] \qquad (II.-9)$$

The connection between the separated rows and columns and sub-matrices of equation (II. - 9) are:

$$\begin{bmatrix} c_i A^{-1} \end{bmatrix} \cdot \begin{bmatrix} \underline{c}_i \underline{A}_{-1} \end{bmatrix} = 1 \ ; \quad \begin{bmatrix} c_i A^{-1} \end{bmatrix} \cdot \begin{bmatrix} \underline{A}_{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}$$

$$\begin{bmatrix} A' \end{bmatrix} \cdot \begin{bmatrix} \underline{A}_{-1} \end{bmatrix} = \begin{bmatrix} I \end{bmatrix} \ ; \quad \begin{bmatrix} A' \end{bmatrix} \cdot \begin{bmatrix} \underline{c}_i \underline{A}_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (II.-10)$$

Therefore

$$\begin{bmatrix} A' \\ \hline c_i A^{N-1} \end{bmatrix} \cdot \left[\begin{array}{c|c} \underline{c}_i \underline{A}_{-1} & \underline{A}_{-1} \end{array} \right] = \left[\begin{array}{c|c} A' \underline{c}_i \underline{A}_{-1} & A' \underline{A}_{-1} \\ \hline c_i A^{N-1} \underline{c}_i \underline{A}_{-1} & c_i A^{N-1} \underline{A}_{-1} \end{array} \right] = \left[\begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{k'} \end{array} \right] \qquad (II.-11)$$

From the equation (II. - 11) the row vector of the canonical matrix can derive in simple way:

$$\left[\quad k' \quad\right] = \left[c_i A^{N-1} \underline{c}_i \underline{A}_{-1} \,\middle|\, c_i A^{N-1} \underline{A}_{-1} \right] = c_i A^{N-1} \left[\underline{c}_i \underline{A}_{-1} \quad \underline{A}_{-1} \right] =$$

$$= c_i A^{N-1} \begin{bmatrix} c_i A^{-1} \\ ---- \\ A' \end{bmatrix}^{-1} = c_i A^{N} \begin{bmatrix} c_i \\ c_i A \\ \vdots \\ c_i A^{N-1} \end{bmatrix}^{-1} = c_i A^{N} F^{-1} \qquad (II.-12)$$

And so:

$$\left[\begin{array}{c|c|c|c|c} c_i & c_i A & c_i A^2 & \cdots\cdots & c_i A^{N-1} \end{array} \right] \cdot \left[k' \right] = \left[c_i A^{N} \right] \qquad (II.-13)$$

L i t e r a t u r e :

1. Gyürki J.: Szabályozási rendszerek analizise és szintézise digitális számológépen gyök-helygörbe módszerrel
/Control system synthesis and analysis on digital computer by help of root-locus method/
Mérés és Automatika, XVI.évf. 1968. 9.szám

2. Davison, E.J.: A numerical method of finding the poles and zeros of a control system
III. IFAC Conference, London 1966, Paper 1.B.

3. Markus, L.: Continuous matrices and the stability of differential systems
Math.Zeitschrift, Bd.62. S.310-319, 1955.

4. Coppel, W.A.: Dichotomies and Reducibility
Journal of Differential Equations, 3, pp.500-521, 1967.

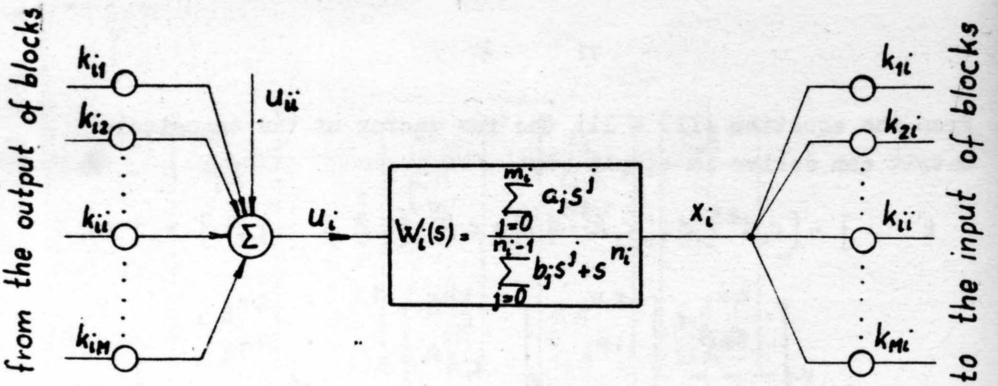


Fig. 1.

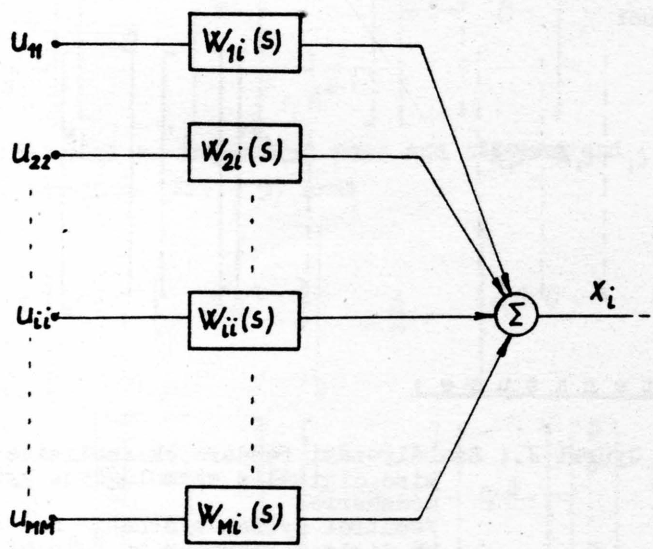


Fig. 2.

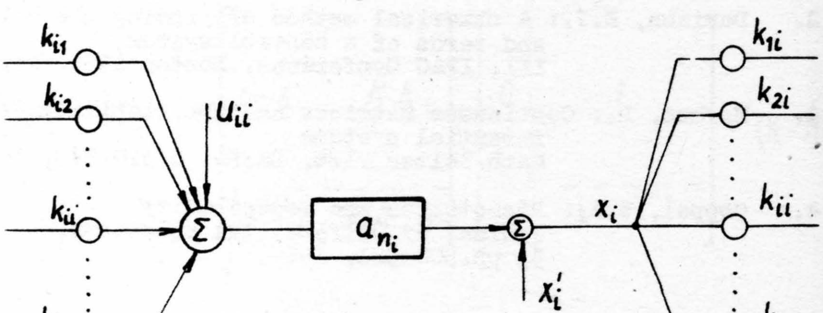
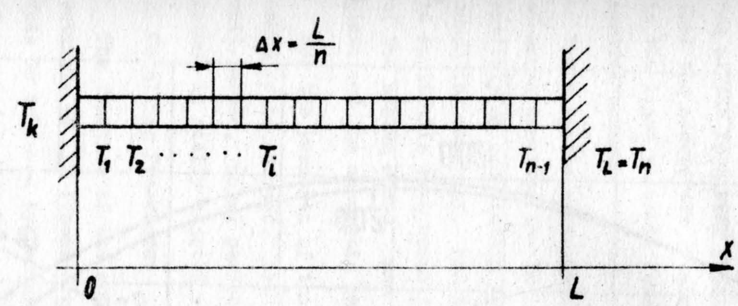
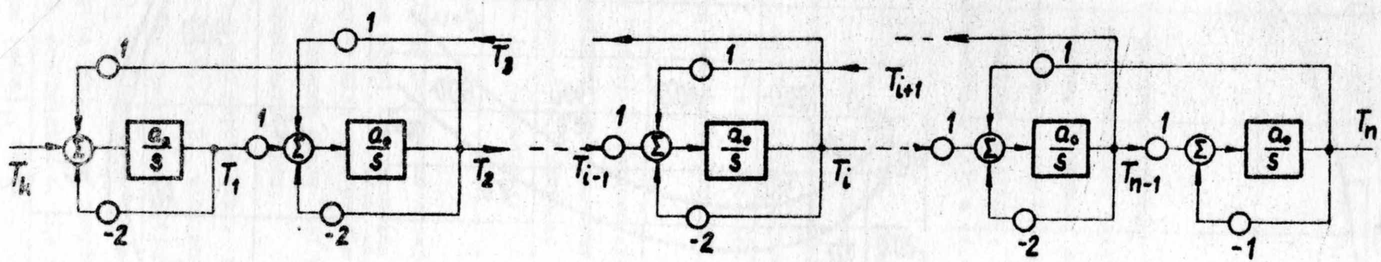


Fig. 3.



$\Delta x = \frac{L}{n}$
T_k
T_1 T_2 ...... T_i
T_{n-1}
$T_L = T_n$
x
0
L

Fig. 4.



T_k
$\frac{a_0}{s}$
T_1
T_2
T_3
T_{i-1}
T_i
T_{i+1}
T_{n-1}
$T_n = T_L$
1
-2
-1

Fig. 5.

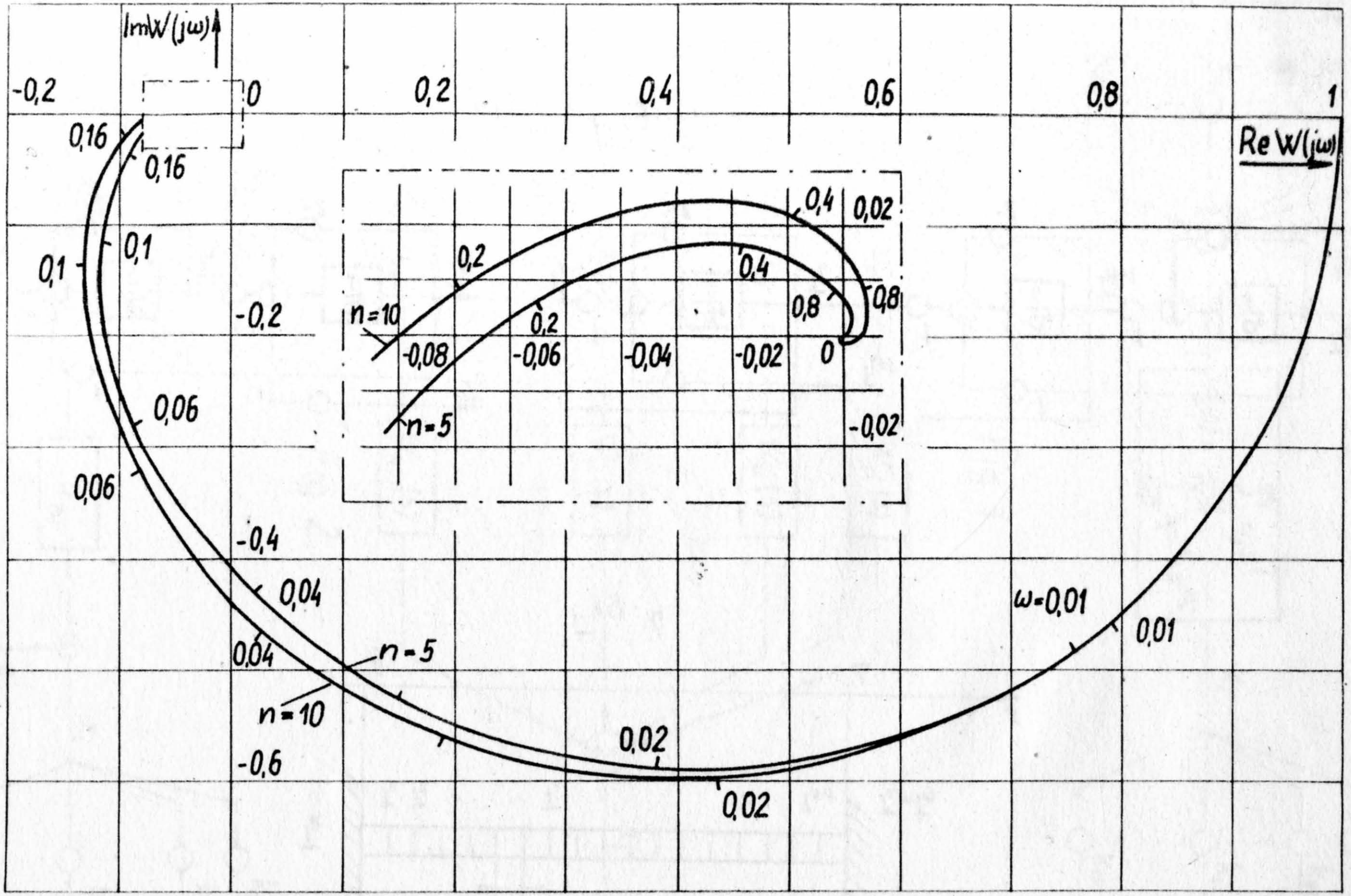

ImW(jω)
Re W(jω)
-0,2
0
0,2
0,4
0,6
0,8
1
-0,2
-0,4
-0,6
0,16
0,16
0,1
0,1
0,06
0,06
0,04
0,04
0,02
0,02
ω=0,01
0,01
n=5
n=10
n=10
n=5
0,2
0,2
0,4
0,4
0,02
0,8
0,8
-0,08
-0,06
-0,04
-0,02
0
-0,02

Fig. 6.

ON-LINE COMPUTER CONTROL USING WEIGHTING FUNCTION MODELS

H.A. Barker and A. Hepburn
University of Glasgow
Glasgow, Scotland.

1. Introduction:

The application of pseudorandom signals to the identification of systems by cross correlation techniques has been extensively described in the literature.[1] Relatively little has been written, however, about the use of the resulting models in control schemes. The reason for this is that the models are obtained in the form of weighting functions, and this is the least suitable form for immediate incorporation into a conventional control scheme.

For a system controlled by an on-line computer, however, this form of model provides the basis for a method of control. The method is based entirely on convolution type procedures, and is therefore economic in computing capacity, and ideally suited to a small digital computer. It is applicable when computing times are small relative to the time constants of the system modes, and has been considered for the control of chemical plant and the automatic landing of aircraft.

For the purposes of this paper, an ideal interface between system and computer is assumed. All conversions are performed synchronously with unit period, using perfect samples and zero-order hold devices. The method is easily adapted when the conversions are sequential.

2. Modelling Procedure:

The function of the modelling procedure is to provide an accurate estimate of the linear dynamics of the system about its operating point. This is achieved by perturbing the system inputs with appropriate phases of a pseudorandom binary signal and using a reference phase[2] of the pseudorandom binary sequence in subsequent crosscorrelation.

In the absence of disturbances, the system has inputs (xa), where a = 1,2,...,A, and outputs (zsb), where b = 1,2,...,B, with corresponding sampled sequences $(xa)_i$ and $(zsb)_i$. If the staircase functions which may be derived from the input sequences $(xa)_i$ by zero-order holds provide an adequate representation of the actual inputs (xa), then components $(ysba)_i$ of the system output sequences $(zsb)_i$ are related to the input sequences $(xa)_i$ by system weighting sequences $(wsba)_j$, where

$$(ysba)_i = \sum_{j=0}^{\infty} (wsba)_i (xa)_{i-j} \quad \text{for all a,b} \tag{1}$$

and

$$(zsb)_i = \sum_{a=1}^{A} (ysba)_i \quad \text{for all b} \tag{2}$$

The inputs (xa) are perturbed through zero-order holds by phases $(pa)_i$ of a pseudorandom binary sequence which is based on an m-sequence with period N and has values $(N-1)/N$ and $-(N+1)/N$. A reference phase $(p)_i$ of this sequence has the properties:

$$\sum_{i=0}^{N-1} (p)_i = \sum_{i=0}^{N-1} i(p)_i = 0 \tag{3}$$

and the phases $(pa)_i$ are chosen so that

$$(pa)_i = (p)_{i - \sum_{n=1}^{a-1} K_n} \quad \text{for all a} \tag{4}$$

where the integers K_n are such that

$$(wsba)_j = 0 \quad \text{for } j \geq K_a, \text{ all a,b} \tag{5}$$

and

$$\sum_{a=1}^{A} K_a = N \tag{6}$$

The perturbations give components $(qsba)_i$ of system output sequences $(rsb)_i$, where

$$(qsba)_i = \sum_{j=0}^{\infty} (wsba)_j (pa)_{i-j} \quad \text{for all a,b} \tag{7}$$

and

$$(rsb)_i = \sum_{a=1}^{A} (qsba)_i \quad \text{for all b} \tag{8}$$

In the presence of noise components $(nsb)_i$, the observed system output sequences $(usb)_i$ are therefore:

$$(usb)_i = (rsb)_i + (zsb)_i + (nsb)_i \quad \text{for all b} \tag{9}$$

During the period $0 \leq i \leq N-1$, a model estimate $(wmba)_j$ of each system weighting sequence $(wsba)_j$ is obtained by the crosscorrelation

$$(wmba)_j = \frac{1}{N+1} \left[\sum_{i=-N}^{-1} (p)_i (usb)_{i+j+\sum_{n=1}^{a-1} K_n} - \underset{j \to K_a - 1}{\text{Limit}} \sum_{i=-N}^{-1} (p)_i (usb)_{i+j+\sum_{n=1}^{a-1} K_n} \right]$$

$$\text{for all } 0 \leq j \leq K_a - 1, \text{ all a,b} \tag{10}$$

This estimate is complete when

$$i = \sum_{n=1}^{a} K_n - 1$$

and then becomes part of an established model of the system, replacing the estimate obtained during the previous period. The established model is thus sequentially updated, each part being updated once every period.

With this procedure, estimation errors are due solely to spurious correlations between the reference phase $(p)_i$ and the components $(zsb)_i$ and $(nsb)_i$ of the observed output sequences $(usb)_i$. From the reference phase properties and the form of the crosscorrelation, estimation errors due to constant, linear, and quadratic functions of time, which in practice form a considerable proportion of these components, are automatically eliminated.

Estimation errors are further reduced by a refinement of the basic procedure. In the refined procedure, estimates $(ymba)_i$ of the components $(ysba)_i$ are computed from the input sequences $(xa)_i$ and the established model weighting sequences $(wmba)_j$ to give

$$(ymba)_i = \sum_{j=0}^{K_a-1} (wmba)_j (xa)_{i-j} \quad \text{for all } a,b \qquad (11)$$

and estimates $(zmb)_i$ of the components $(zsb)_i$ are computed to give

$$(zmb)_i = \sum_{a=1}^{A} (ymba)_i \qquad \text{for all } b \qquad (12)$$

Subtraction of the estimated components $(zmb)_i$ from the observed output sequences $(usb)_i$ defines the sequences $(vmb)_i$, where

$$(vmb)_i = (usb)_i - (zmb)_i \qquad \text{for all } b \qquad (13)$$

and the sequences $(vmb)_i$ then replace the observed output sequences $(usb)_i$ in the crosscorrelation defined by equation (10), so that

$$(wmba)_j = \frac{1}{N+1}\left[\sum_{i=-N}^{-1} (vmb)_{i+j+\sum_{n=1}^{a-1} K_n} - \operatorname*{Limit}_{j \to K_a - 1} \sum_{i=-N}^{-1} (p)_i (vmb)_{i+j+\sum_{n=1}^{a-1} K_n}\right]$$

$$\text{for all } 0 \leq j \leq K_a - 1, \text{ all } a,b \qquad (14)$$

Estimation errors in the refined procedure are due to spurious correlations between the reference phase $(p)_i$ and the components $(zsb)_i - (zmb)_i + (nsb)_i$ of the sequences $(vmb)_i$, which are substantially less than in the basic procedure.

3. Feedforward Controller:

For the purposes of this paper, the system has a number of controlled inputs and an equal number of controlled outputs, the remaining input and outputs being uncontrolled. The function of the feedforward controller is to remove the effects of the uncontrolled inputs from the controlled outputs. This is achieved by computing appropriate control input sequences from estimates of these effects.

The principles are developed for the common case in which the system has one control input (x1) and one controlled output (us1). In this case it is required that $(zs1)_i$ is a null sequence, so the controlled component $(ys11)_i$ must be such that

$$(ys11)_i = -\sum_{a=2}^{A}(ys1a)_i \tag{15}$$

for which an estimate is

$$(ym11)_i = -\sum_{a=2}^{A}(ym1a)_i \tag{16}$$

From equation (11) this gives

$$\sum_{j=0}^{K_1-1}(wm11)_j(x1)_{i-j} = -\sum_{a=2}^{A}\sum_{j=0}^{K_a-1}(wm1a)_j(xa)_{i-j} \tag{17}$$

The required control input sequence $(x1)_i$ is contained implicitly in this equation, but to compute it explicitly, the equation must be implemented as the recursive relationship

$$(x1)_i = \frac{-1}{(wm11)_0}\left[\sum_{j=1}^{K_1-1}(wm11)_j(x1)_{i-j} + \sum_{a=2}^{A}\sum_{j=0}^{K_a-1}(wm1a)_j(xa)_{i-j}\right] \tag{18}$$

This relationship defines the ideal control input sequence $(x1)_i$ which is applied through a zero-order hold to give the required feedforward control input (x1). In practice, some modification is necessary to insure that it is realisable.

If $(wm11)_0$ is zero, then the relationship is evidently unrealisable. This is because it attempts to implement a prediction to cancel a time delay. The presence of a time delay in the control dynamics, so that the first few values of $(ws11)_j$ are zero, is quite common, the more serious aspect of this being that, due to estimation errors, the first few values of $(wm11)_j$ will differ slightly from zero and mask the true condition. The first few values of the model weighting sequences $(wmba)_j$ are therefore always subjected to simple processing before establishment to determine the first realistic non-zero value $(wmba)_{gba}$ and to set to zero all $(wmba)_i$ for $i < gba$. The control input sequence $(x1)_i$ is then computed from the relationship

$$(x1)_i = \frac{-1}{(wm11)_{g11}} \left[\sum_{j=1}^{K_1-g11-1} (wm11)_{j+g11} (x1)_{i-j} + \sum_{a=2}^{A} \sum_{j=0}^{K_a-h1a-1} (wm1a)_{j+h1a} (xa)_{i-j} \right] \qquad (19)$$

where

$$h1a = g11 \text{ or } g1a, \text{ whichever is least} \qquad (20)$$

This remains ideal if $g11 < g1a$ for all a, otherwise the errors are accepted.

If the pulse transfer function

$$\sum_{j=0}^{K_1-g11-1} (wm11)_{j+g11} \, z^{-j}$$

has a zero on or outside the unit circle, then the relationship is unstable; this is because it attempts to implement the inverse pulse transfer function which has a pole on or outside the unit circle and is consequently unstable. Fortunately this condition occurs only rarely in practice, and its treatment falls outside the scope of this paper.

Finally, there may be a physical constraint, such as saturation, which prevents the ideal control input sequence $(x1)_i$ from being implemented. In this case, the closest approach to the ideal $(x1)_i$ is implemented and the errors are accepted.

When the system has two control inputs (x1) and (x2), and two controlled outputs (us1) and (us2), it is required that both $(zs1)_i$ and $(zs2)_i$ are null sequences. Proceeding as before gives the conditions:

$$\sum_{j=0}^{K_1-1} (wm11)_j (x1)_{i-j} + \sum_{j=0}^{K_2-1} (wm12)_j (x2)_{i-j} = - \sum_{a=3}^{A} \sum_{j=0}^{K_a-1} (wm1a)_j (xa)_{i-j} \qquad (21)$$

and

$$\sum_{j=0}^{K_1-1} (wm21)_j (x1)_{i-j} + \sum_{j=0}^{K_2-1} (wm22)_j (x2)_{i-j} = - \sum_{a=3}^{A} \sum_{j=0}^{K_a-1} (wm2a)_j (xa)_{i-j} \qquad (22)$$

from which the equations containing implicitly the required control input sequences $(x1)_i$ and $(x2)_i$ are obtained as

$$\sum_{j=0}^{K_1+K_2-2} \sum_{f=0}^{j} \left[(wm11)_f (wm22)_{j-f} - (wm21)_f (wm12)_{j-f} \right] (x1)_{i-j}$$

$$= - \sum_{a=3}^{A} \sum_{j=0}^{K_a+K_2-2} \sum_{f=0}^{j} \left[(wm1a)_f (wm22)_{j-f} - (wm2a)_f (wm12)_{j-f} \right] (xa)_{i-j} \quad (23)$$

and

$$\sum_{j=0}^{K_1+K_2-2} \sum_{f=0}^{j} \left[(wm11)_f (wm22)_{j-f} - (wm21)_f (wm12)_{j-f} \right] (x2)_{i-j}$$

$$= - \sum_{a=3}^{A} \sum_{j=0}^{K_a+K_1-2} \sum_{f=0}^{j} \left[(wm2a)_f (wm11)_{j-f} - (wm1a)_f (wm21)_{j-f} \right] (xa)_{i-j} \quad (24)$$

Equations (23) and (24) are basically similar to equation (17), with new weighting sequences computed by a convolution procedure. Recursive relationships similar to equation (18) may therefore be obtained and modified to give relationships similar to equation (19) from which the required feedforward control input sequences $(x1)_i$ and $(x2)_i$ may be computed. It may be noted that, in this case, the condition for an unstable relationship, though still uncommon, is not so rare as in the case of a single control input and controlled output, because of the subtraction involved in computing the new weighting sequences. The equivalent problem in continuous systems has been considered by Rosenbrock.[3]

The procedure for systems with more than two control inputs and controlled outputs follows logically from the above development.

4. Feedback Controller:

By the implementation of the feedforward controller the controlled outputs are made effectively independent of the uncontrolled inputs. The function of the feeback controller is to remove all remaining components, with the exception of components resulting from the modelling procedure, from the controlled outputs, and to implement all changes in the required values of the controlled outputs. This is achieved by computing appropriate control input sequences from estimates of the remaining components and errors between the controlled output values and their required values.

The principles are again developed for the common case in which the system has one control input (x1), and one controlled output (us1). The control input (x1) now contains two components, a feedforward control

control input component $(x1)^{FF}$, derived as shown in the previous section, and a feedback control component $(x1)^{FB}$ so that

$$(x1) = (x1)^{FF} + (x1)^{FB} \tag{25}$$

It is convenient to consider the feedforward control input $(x1)^{FF}$ as acting independently and ideally, so that by taking the observed output sequence $(us1)_i$ from equation (9) as

$$(us1)_i = (ys11)_i^{FB} + (rs1)_i + (ns1)_i \tag{26}$$

where

$$(ys11)_i^{FB} = \sum_{j=0}^{\infty}(ws11)_j(x1)_{i-j}^{FB} \tag{27}$$

the effects of imperfect feedforward control are included in the noise component $(ns1)_i$. An estimate $(nm1)_i$ of the noise component may be computed through

$$(nm1)_i = (us1)_i - \sum_{j=0}^{K_1-1}(wm11)_j(x1)_{i-j}^{FB} - \sum_{a=1}^{A}\sum_{j=0}^{K_a-1}(wm1a)_j(pa)_{i-j} \tag{28}$$

If the required control output sequence is $(c1)_i$, then since the actual controlled output sequence must also contain the component $(rs1)_i$ resulting from the modelling procedure, the ideal controlled output sequence is $(c1)_i + (rs1)_i$. Hence an estimate of the ideal component $(ys11)_i^{FBI}$ in equation (26) is

$$(ys11)_i^{FBI} = (c1)_i - (nm1)_i \tag{29}$$

so the estimate of the ideal feedback control sequence $(x1)_i^{FBI}$ is given by

$$\sum_{j=0}^{K_1-1}(wm11)_j(x1)_{i-j}^{FBI} = -\left[(us1)_i - (c1)_i - \sum_{j=0}^{K_1-1}(wm11)_j(x1)_{i-j}^{FB} - \sum_{a=1}^{A}\sum_{j=0}^{K_a-1}(wm1a)_j(pa)_{i-j}\right] \tag{30}$$

The distinction between the sequences $(x1)_i^{FB}$ and $(x1)_i^{FBI}$ should be noted. The sequence $(x1)_i^{FB}$ is the feedback control sequence actually applied to the system, and in the computation of the right hand side of equation (30) all the values $(x1)_{i-j}^{FB}$ are known with the exception of $(x1)_i^{FB}$, which is taken as equal to $(x1)_{i-1}^{FB}$. The sequence $(x1)_i^{FBI}$ is the ideal feeback control sequence, which may not be realisable. Equation (30) is developed in the same way as equation (17) to give a relationship for the realisable feedback control sequence $(x1)_i^{FBR}$, which is

$$(x1)_i^{FBR} = \frac{-1}{(wm11)_{g11}} \Bigg[\sum_{j=1}^{K_1-g11-1} (wm11)_{j+g11}(x1)_{i-j}^{FBR} + (us1)_i - (c1)_i - \sum_{j=0}^{K_1-1} (wm11)_j (x1)_{i-j}^{FB} - \sum_{a=1}^{A} \sum_{j=0}^{K_a-1} (wm1a)_j (pa)_{i-j} \Bigg] \quad (31)$$

and $(x1)_i^{FBR}$ is the control sequence which is implemented, unless a physical constraint, such as saturation, prevents this in which case the closest approach to $(x1)_i^{FBR}$ is implemented and the errors are accepted.

The development of this procedure for systems with more than one control input and controlled output corresponds to the development for feedforward control.

5. Example:

To illustrate an application of the method, an example is given of a system with one uncontrolled input, one controlled input, and one controlled output, with feedforward control applied as discussed in section (3). The block diagram of the resulting system is given in Fig. 1, and the results for a step disturbance at the uncontrolled input appear in Fig. 2. In the latter, amplitudes are normalised relative to the amplitude of the test signals (pa). The system error curve represents the deviation of the system output $(us1)_i$ from the ideal null sequence; by far its largest component is due to the pseudorandom binary test signals themselves, and is, of course, unavoidable when using this method.

6. Conclusions:

A method of on-line digital computer control of a multivariable system which fully exploits the possibilities of on-line modelling by pseudorandom binary signal perturbation has been established. The method involves the implementation of three distinct procedures which are designed to be non-interacting. A modelling procedure is developed to provide estimates of the system weighting sequences which are virtually free from errors due to spurious correlations. The model weighting sequences are used in feedforward and feedback controllers which operate independently and are based on the implementation of a recursive procedure, the limitations of which are discussed, for the computation of control input sequences. The feedforward controller is shown to make the controlled outputs independent of all observed input disturbances, and the feedback controller is shown to remove from the controlled outputs the effects of unobserved disturbances and deviations from the required values with the exception of the small perturbations resulting from the modelling procedure. Results obtained for a simulated system demonstrate the feasability of the approach.

7. References:

1) Briggs, P.A.N., Hammond, P.H., Hughes, M.T.G., and Plumb, G.O.: "Correlation analysis of process dynamics using pseudo-random binary test perturbations." Proc. Instn. Mech. Engrs., 1964-65, 179, (3H), pp. 37-51.

2) Barker, H.A.: "Choice of pseudorandom binary signals for system identification.", Electronics Letters, 1967, 3, pp. 524-526.

3) Rosenbrock, H.H.: "On the design of linear multivariable control systems." Proceedings of the Third IFAC Congress, London, 1966.

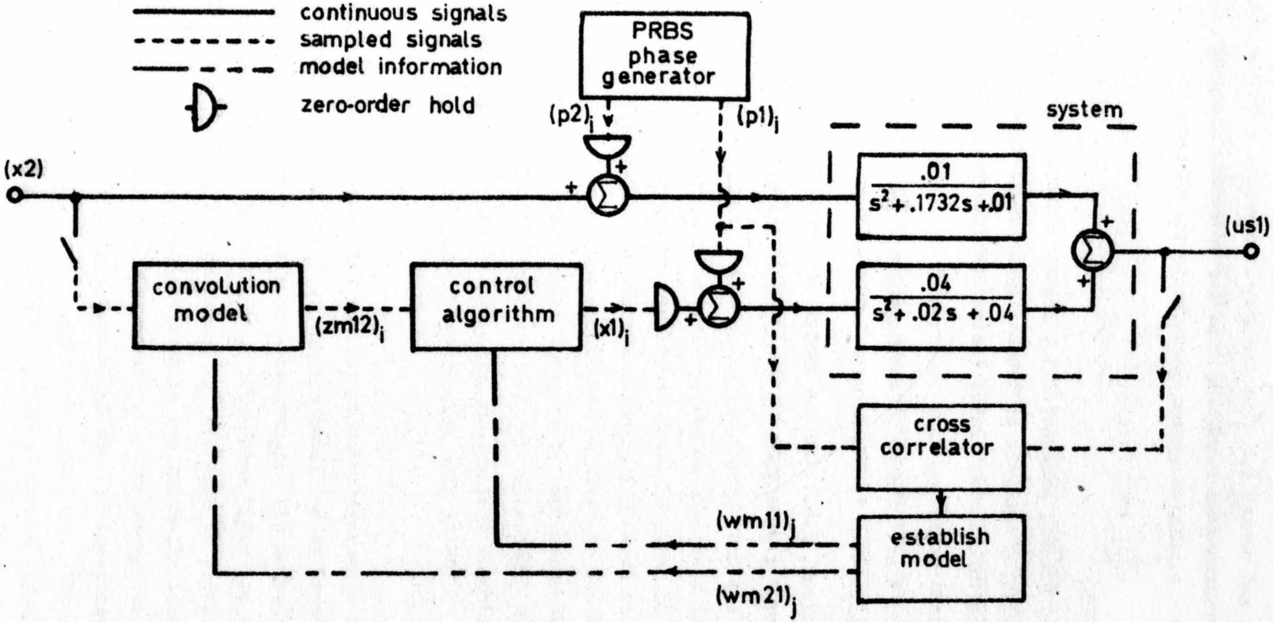


Fig.1: System used in example.

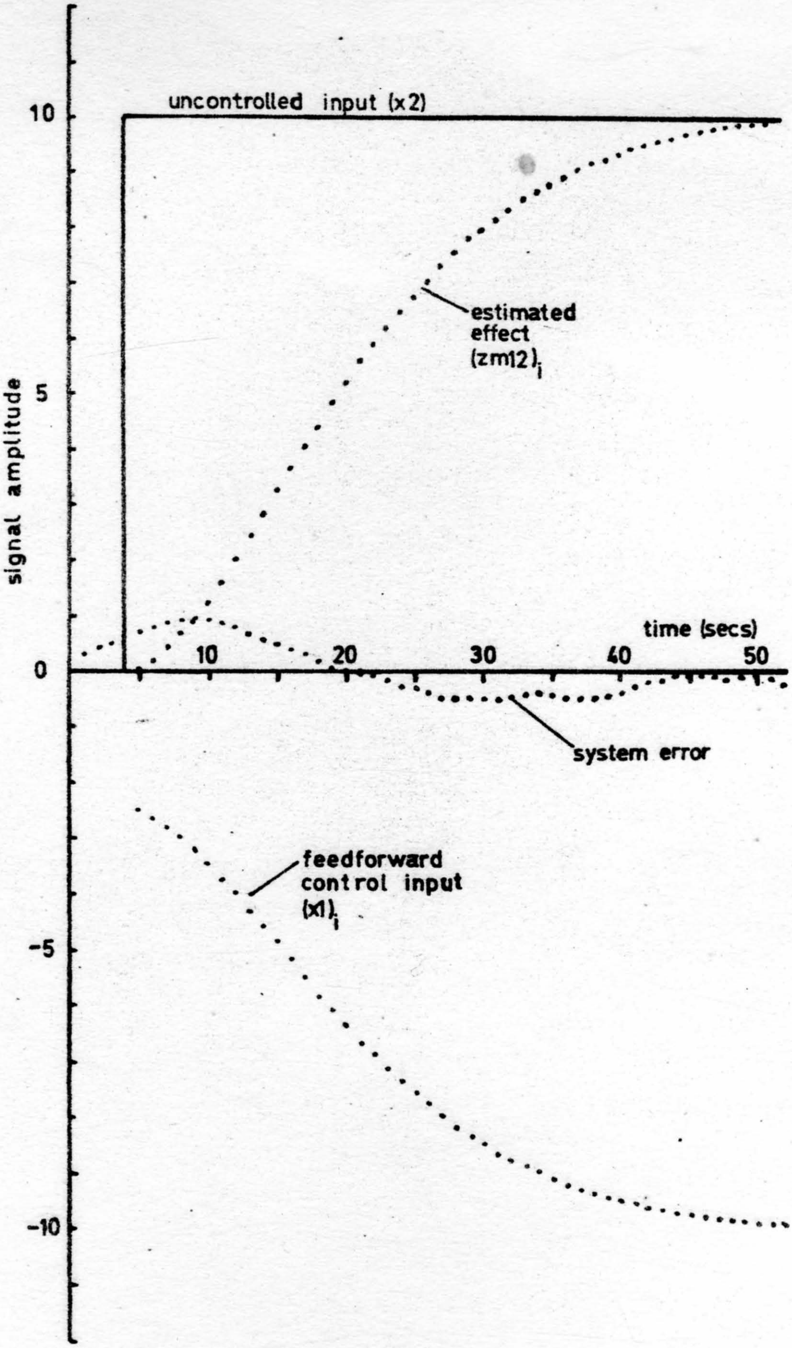


Fig. 2: Response of system to step disturbance.