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Identification

Correlation and Spectral Analysis Methods

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Organized by
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ON A METHOD OF CORRELATION ANALYSIS FOR MULTIVARIATE SYSTEMS

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INTRODUCTION

It is well-known, that correlation techniques in automatic control systems hinges on the result that, if white noise is applied to a linear system, the cross-correlation function of the input and the output gives the system impulse response function. In the case of multivariate systems, correlation techniques may be used as in the univariate system. But there are few papers treating the case generally when the input vector of multivariate systems has mutually correlated components or subvectors.

Here the author generalizes correlation techniques for univariate systems to the case of multivariate systems, and shows that the equation to estimate impulse responses of the system should be a multidimensional linear algebraic equation having a matrix of comparably high orders.

In the estimation of dynamic characteristics of systems by the use of correlation functions in the multivariate case, following two methods may be used as in the univariate case.

1) Mehtod by correlograms of inputs and outputs to find system impulse responses.
2) Method by spectral analysis through correlograms of inputs and outputs to find frequency response of the system.

Disscussions about the comparison of accuracies in the estimation by these two methods had been treated in some works.[1] In this paper, the first method in the multivariate case is mainly studied.

Up to now there are some discussions about the accuracy of impulse response functions estimated by methods of correlation techniques. Particularly about the accuracy of frequency response functions by spectral analysis methods, many statistical discussions were given.[1]

Consequently the concept of windows became familiar to engineers, however it is still difficult to choose adequate window-pairs for each practical problem. In contrast to discussions of frequency domain analysis, it seems

that statistical estimations of the accuracy of impulse response functions given by correlation techniques were not so successfull. Here some discussions about this problem are to be given.

THE CALCULATION OF IMPULSE RESPONSES BY USING CORRELATION FUNCTIONS IN THE MULTIVARIATE CASE

About correlation techniques of automatic control systems in multivariate cases, some results are obtained by Y.Sawaragi and N.Sugai[2]. In this paper a more general formula will be obtained.

Now consider the multivariate linear system as shown in Fig.1.

For the simplicity, we use vector notations of inputs, outputs and noises as follows.

$$\mathbf{X}(t)=\begin{pmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_n(t) \end{pmatrix} \qquad \mathbf{Y}(t)=\begin{pmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ Y_m(t) \end{pmatrix} \qquad \mathbf{N}(t)=\begin{pmatrix} N_1(t) \\ N_2(t) \\ \vdots \\ N_m(t) \end{pmatrix} \tag{1}$$

In general, capital letters stand for random variables, and small letters their realized values. But in some cases, let us use small letters to denote deterministic variables.

In Fig.1, without loss of generality we can assume that $\mathbf{X}(t)$ and $\mathbf{N}(t)$ are mutually uncorrelated. As in the case of univariate system, a component of the output vector $\mathbf{Y}(t)$, for example $Y_1(t)$, is represented by following an equation due to the convolution theorem.

$$y_1(t)=\sum_{j=1}^{n}\int_0^{\infty} h_{1j}(z)\, x_j(t-z)\,dz + n_1(t). \qquad j=1,2,\cdots n. \tag{2}$$

Where $h_{1j}(t)$ is the impulse response function of y_1 when x_j is unit impulse function, i.e. δ-function.

Using above formula, we can calculate the cross-correlation function of y_1 and x_j as

$$\phi_{y_1x_j}(\tau) = \frac{1}{T}\int_0^T y_1(t)\, x_j(t-\tau)\,dt = \frac{1}{T}\int_0^T x_j(t-\tau)\,dt \sum_{j=1}^{n}\int_0^{\infty} h_{1j}(z)x_j(t-z)\,dt + \frac{1}{T}\int_0^T n_1(t)\cdot x_j(t-\tau)\,dt \tag{3}$$

Where T is a time-interval in which correlation should be calculated.

In the above calculation, the ergodic property of the input process was used, because of the assumption that the input signal was derived from a stationary normal process. Further the correctionfactor $(1-\frac{|\tau|}{T})$ depending on the finiteness of the time-interval T was assumed nearly equal to 1 by the choice of T enough large in comparison with τ.

Owing to the existence of the integral (3), we can change the order of the integration as follows.

$$\phi_{y_1x_1}(\tau)=\sum_{j=1}^{n}\int_0^\infty h_{1j}(z)dz\frac{1}{T}\int_0^T x_1(t-\tau)x_j(t-z)dt+\frac{1}{T}\int_0^T n_1(t)x_1(t-\tau)dt \qquad (3')$$

Since we can write

$$\phi_{x_ix_j}(z-\tau)=\frac{1}{T}\int_0^T x_i(t-\tau)x_j(t-Z)dt$$

$$\phi_{n_ix_j}(\tau)=\frac{1}{T}\int_0^T n_i(t)x_j(t-\tau)dt$$

then

$$\phi_{y_1x_1}(\tau)=\sum_{j=1}^{n}\int_0^T h_{1j}(z)\phi_{x_1x_j}(z-\tau)\,dz+\phi_{n_ix_1}(\tau) \qquad (4)$$

As mentioned above, $\phi_{n_ix_j}(\tau)$ is the cross-correlation of the input signal and disturbances in the system, and without loss of generality we can assume the input signal is such that it approximates to a random input uncorrelated with any system noise. If in the case of the existence of correlation between them, $\mathbb{N}(t)$ should be regarded as a subvector of $\mathbb{X}(t)$.

Therefore, assuming that the term $\phi_{n_ix_j}(\tau)$ is negligible, we get

$$\phi_{y_ix_1}(\tau)=\sum_{j=1}^{n}\int_0^\infty h_{1j}(z)\,\phi_{x_1x_j}(z-\tau)dz \qquad (5)$$

In the discrete case (5) becomes

$$\phi_{y_ix_1}(\tau)=\sum_{j=1}^{n}\sum_{\ell=0}^{s}h_{1j}(\ell)\,\phi_{x_ix_j}(\ell-k) \qquad (k=0,1,2,\cdots\cdots s) \qquad (6)$$

If we write as derived in (6), the crosscorrelation of $Y_1(t)$ and $X_1(t)$ in the discrete form, we obtain the following relation; (note $\phi_{x_i x_i}(-\tau) = \phi_{x_i x_i}(\tau)$)

$$\begin{pmatrix} \phi_{y_1x_1}(0) \\ \phi_{y_1x_1}(1) \\ \vdots \\ \phi_{y_1x_1}(s) \end{pmatrix} = \begin{pmatrix} \phi_{x_1x_1}(0) & \phi_{x_1x_1}(1) & \cdots & \phi_{x_1x_1}(s) \\ \phi_{x_1x_1}(1) & \phi_{x_1x_1}(0) & \cdots & \phi_{x_1x_1}(s-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{x_1x_1}(s) & \phi_{x_1x_1}(s-1) & \cdots & \phi_{x_1x_1}(0) \end{pmatrix} \begin{pmatrix} h_{11}(0) \\ h_{11}(1) \\ \vdots \\ h_{11}(s) \end{pmatrix} + \cdots$$

$$+ \begin{pmatrix} \phi_{x_1x_n}(0) & \phi_{x_1x_n}(1) & \cdots & \phi_{x_1x_n}(s) \\ \phi_{x_1x_n}(-1) & \phi_{x_1x_n}(0) & \cdots & \phi_{x_1x_n}(s-1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{x_1x_n}(-s) & \phi_{x_1x_n}(1-s) & \cdots & \phi_{x_1x_n}(0) \end{pmatrix} \begin{pmatrix} h_{1n}(0) \\ h_{1n}(1) \\ \vdots \\ h_{1n}(s) \end{pmatrix} \tag{7}$$

or simply,

$$\Phi_{y_1x_1} = \left[\Phi_{x_1x_1}\right] h_{11} + \cdots + \left[\Phi_{x_1x_n}\right] h_{1n} \tag{8}$$

In general, the correlation of $Y_u(t)$ and $X_v(t)$ can be written as

$$\Phi_{y_ux_v} = \left[\Phi_{x_vx_1}\right] h_{v1} + \cdots + \left[\Phi_{x_vx_n}\right] h_{vn} \tag{9}$$

$$(u=1, \ldots, m \quad v=1, \ldots, n)$$

Now consider the treatment of this system of equations (9). It contains m. n unknown vectors h_{uj}, and also contains m. n matrix equations. So it should be solved with respect to h_{uj} uniquely if (9) satisfy some conditions. First write this system of matrix equations including for example, only $h_{11}, \ldots, h_{1n}$, as follows:

$$\begin{pmatrix} \Phi_{y_1x_1} \\ \Phi_{y_1x_2} \\ \vdots \\ \Phi_{y_1x_n} \end{pmatrix} = \begin{bmatrix} \Phi_{11} \Phi_{12} \cdots \Phi_{1n} \\ \Phi_{21} \Phi_{22} \cdots \Phi_{2n} \\ \vdots \\ \Phi_{n1} \Phi_{n2} \cdots \Phi_{nn} \end{bmatrix} \begin{pmatrix} h_{11} \\ h_{12} \\ \vdots \\ h_{1n} \end{pmatrix} \tag{10}$$

Submatrices $\Phi_{11}, \Phi_{22} \ldots\ldots, \Phi_{nn}$ in main diagonal are positive definite, and hence nonsingular, because they are autocovariance matrices of $X_1(t), X_2(t), \ldots\ldots, X_n(t)$.

So it is natural, to assume that the matrix of (10) to be nonsigular, but it needs a proof. Remember that a autocovariance matrix of $X_i(s)$ can be expressed as follows: (by using the expectation operator $\mathcal{E}$)

$$\Phi_{ii} = \mathcal{E}\left[X_i(s) X_i'(s)\right] \tag{11}$$

where the primed vector $X'(s)$ denotes a transposed vector, namely

$$X_i'(s) = \left[X_i(s), X_i(s-1), \ldots, X_i(0)\right]$$

on the other hand the matrix of (10) may be expressed as follows.

$$\Phi = \begin{bmatrix} \phi_{11}(0) \ldots \phi_{11}(s) & \phi_{12}(0) \phi_{12}(1) \ldots \phi_{12}(s) \ldots \\ \vdots & \vdots \\ \phi_{11}(s) \ldots \phi_{11}(0) & \phi_{12}(-s) \phi_{12}(1-s) \ldots \phi_{12}(0) \ldots \\ \phi_{21}(0) \ldots \phi_{21}(s) & \phi_{22}(0) \ldots\ldots \phi_{22}(s) \ldots \\ \phi_{21}(-1) \ldots \phi_{21}(s-1) & \vdots \\ \vdots & \vdots \\ \phi_{21}(-s) \ldots \phi_{21}(0) & \phi_{22}(s) \ldots\ldots \phi_{22}(0) \ldots \\ \vdots & \vdots \end{bmatrix}$$

$$= \mathcal{E}\left\{\begin{pmatrix} x_1(s) \\ x_1(s-1) \\ \vdots \\ x_1(0) \\ x_2(s) \\ \vdots \\ x_2(0) \\ \vdots \\ x_n(s) \\ \vdots \\ x_n(0) \end{pmatrix} \left[x_1(s), x_1(s-1), \ldots, x_1(0), x_2(s), \ldots, x_2(0), \ldots, x_n(s), \ldots, x_n(0)\right]\right\} \tag{12}$$

In vector notation, (12) becomes

$$\Phi = \mathcal{E}[XX'] \tag{13}$$

where

$$X' = \left[x_1(s), x_1(s-1), \ldots, x_1(0), x_2(s), \ldots, x_2(0), \ldots, x_n(s), \ldots, x_n(0)\right] \tag{14}$$

In comparson with (11), it is clear that the matrix should be positive definite. This can be proved directly also by noticing that $\Phi_{x_i x_j}(s) = \Phi_{x_j x_i}(-s)$, and hence the matrix in (12) is real valued, symmetric and positive definite.

Accordingly, the matrix equation (10) can be solved algebraically and uniquely, because the matrix is nonsingular.

The solution of the equation (10) becomes

$$H_1 = \Phi^{-1}\Phi_{yx} \tag{15}$$

where

$$\mathbf{h}_1 = \begin{bmatrix} h_{11} \\ \cdot \\ \cdot \\ \cdot \\ h_{1n} \end{bmatrix} \qquad \Phi_{yx} = \begin{bmatrix} \Phi_{y_1x_1} \\ \cdot \\ \cdot \\ \cdot \\ \Phi_{y_1x_n} \end{bmatrix}$$

Thus, simultaneous solutions of impulse responses $h_{11}, h_{12}, \ldots,$ h_{1n} are obtained.

As shown above, the equation (10) has a matrix of comparably high order. A convenient technique used so far is to construct components of a random vector $\mathbf{X}(t)$ so as to be mutually uncorrelated white noises. By using this experimental technique the equation (10) is extremely simplified. In this case, the evaluation of the accuracy of estimations of impulse responses is not so difficult. But at present, statistical evaluations of the accuracy of impulse responses $\mathbf{h}$ in (10) by the direct method is still important from theoretical points of view. Here one method is proposed, by an application of some techniques in the field of multivariate statistical analysis.

A STATISTICAL EVALUATION OF THE ACCURACY OF IMPULSE RESPONSES -CONFIDENCE REGIONS OF IMPULSE RESPONSES:

In above described ways, a discrete type Wiener-Hopf equation for multivariate systems was obtained, and was shown that the equation is quite similar to the equation of the univariate case. Now let us consider the accuracy of impulse responses given by (15). Here the accuracy of simultaneous solution of (15) is to be evaluated. A conventional way of discussions about the accuracy of (15) seems such that, from sample distribution of covariance matrix Φ and correlation vector Φ_{yx} to calculate the sample distribution of $\mathbf{h}$ precisely. But it requires tedious and difficult calculations to obtain sample distribution of $\mathbf{h}$ by this orthodox way.

Here the author proposes another approach of discussing the accuracy of impulse responses $\mathbf{h}$.

The solution of eq. (10) is of the form (15). By transposing (15) it becomes

$$\mathbf{h}'_1 = \Phi'_{yx}\Phi^{-1} \tag{16}$$

because Φ is a symmetric matrix.

Similarly, in the univariate case we obtain

$$h'_{11} = \Phi'_{yx_1} \Phi^{-1} \tag{17}$$

Observing (16), (17), we notice that they are quite analogous to so-called regression matrices. Here some explanations about regression matrices should be shown. By putting a covariance matrix in the form (18), then the regression matrix of $\mathfrak{X}_1$ on $\mathfrak{X}_2$ is given by β in (19). Where $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are random vectors.[3]

$$\mathcal{E}\left\{\begin{pmatrix}\mathfrak{X}_1\\ \mathfrak{X}_2\end{pmatrix}(\mathfrak{X}'_1 \;\; \mathfrak{X}'_2)\right\} = \begin{pmatrix}\Sigma_{11}, \Sigma_{12}\\ \Sigma_{21}, \Sigma_{22}\end{pmatrix} = \Sigma \tag{18}$$

$$\beta = \Sigma_{12}\Sigma_{22}^{-1} \tag{19}$$

In our problem, Σ_{12} appears as a row vector Φ_{yx} and the original covariance matrix of our case can be written as

$$\mathcal{E}(\mathbb{R}\mathbb{R}') = \begin{pmatrix}\phi_{yy}(0), \phi_{yx}(0), \phi_{yx}(1) \cdots\cdots \phi_{yx}(s)\\ \phi_{yx}(0)\\ \phi_{yx}(1)\\ \vdots \qquad\qquad \Phi_{xx}\\ \phi_{yx}(s)\end{pmatrix} = \begin{pmatrix}\Phi_{yx}(0), \Phi'_{yx}\\ \Phi_{yx} \quad \Phi_{xx}\end{pmatrix} \tag{20}$$

where $\mathbb{R}'$ is a row vector such that

$$\mathbb{R}' = \left[Y(s), X(s), X(s-1), \cdots\cdots, X(1), X(0)\right] \tag{21}$$

It is now clear, that our problem is reduced to the multiple-regression problem of finding a linear combination of the form $y = \sum_{j=0}^{s} h_j x(s-j)$ by the method of least squares from given sets of realized values of a random process $\left[y(s_\alpha), x(s_\alpha), x(s_\alpha - 1), \cdots\cdots\cdots, x(1_\alpha), x(0_\alpha)\right]$ considering $x(s_\alpha - 1)$ fixed.

In the multiple regression theory, each realized value $X(s_\alpha - j)$ is considered as fixed. Therefore from this point of view the independence or the dependence of each vector $\mathbb{P}'(s) = \left[x(s_\alpha), x(s_\alpha - 1), \cdots\cdots, X(1_\alpha), x(0_\alpha)\right]$ has essentially no influences to the theory.

From above discussions it is clear that correlation techniques have sig-

nificant relations to the method of multiple regression analysis.

In particular, in the case where artificially formed nonrandom deterministic (but having δ-function typed autocorrelation function) signals are used, then the problem satisfies essentially conditions of multiple regression theory.

As the conclusion, it is clear that the confidence interval estimation of impulse responses h can be obtained directly by applying well-known results about simultaneous confidence regions of regression coefficients. In the case where we use artificial random signals with a known covariance matrix Φ_1 and if by random read outs we form input signals, then the Wishart-distribution may be applied to make the estimation precisely. But the calculation of Wishart-distribution seems too tedious from practical points of view. And even in the case of using a random process data with unknown covariance matrix, the application of the multiple regression theory to this case can be justified, by taking on account that we use sets of realized values of the process V_α' such that

$$V_\alpha' = \left[y(s_\alpha), \mathbf{P}'(s_\alpha)\right] = \left[y(s_\alpha),\ x(s_\alpha),\ x(s_\alpha 1),\ \cdots\cdots,x(l_\alpha), X(o_\alpha)\right] \tag{21}$$

In other words, without precise estimations of auto-correlation and cross-correlation, i.e. without a precise estimation of the covariance matrix $\mathcal{E}(RR')=\Phi$, still can we estimate confidence regions of the regression vector h (which in our terminology appears as impulse responses.).

In general multivariable cases, a similar conclusion can be obtained. In this case the regression vector h' becomes

$$h' = \Phi_{yx}' \Phi^{-1} = \left[\Phi_{yx_1}', \Phi_{yx_2}', \cdots\cdots, \Phi_{yx_n}\right] \Phi^{-1} \tag{22}$$

The original covariance matrix to get this regression vector can be constituted from a random vector T' such that

$$T' = \left[Y_1(s), X_1(s), \cdots, X_1(o),\ X_2(s), \cdots, X_2(o), \cdots\cdots, X_n(s), \cdots, X_n(o)\right] \tag{23}$$

Let us assume that we get q sets of realized vector t_α' from the

random vector T' at different time s_α ($\alpha=1,\ldots,q$).

$$t'_\alpha = \left[y(s_\alpha),\ P'_1(s_\alpha),\ P'_2(s_\alpha),\ \ldots,\ P'_n(s_\alpha) \right] = \left[y(s_\alpha),\ x'(s_\alpha) \right] \tag{24}$$

Now we consider the subvector $x'(s_\alpha)$ fixed, and constitute a following regression model.

$$y(s_\alpha) = h'x(s_\alpha) + \varepsilon_\alpha \tag{25}$$

In this model $y(s_\alpha)$ is a sample from normal population with mean $h'x(s_\alpha)$ and ε_α is distributed independently according to a normal distribution $N(0, \sigma^2)$.

The normal equation to estimate h becomes

$$\sum_{\alpha=1}^{q}(x_\alpha-\bar{x})(x_\alpha-\bar{x})'\cdot\hat{h} = \sum_{\alpha=1}^{q}(y_\alpha-\bar{y})(x_\alpha-\bar{x}) \tag{26}$$

where for simplicity let us denote $Y(s_\alpha) = Y_\alpha$, $X(s_\alpha) = X_\alpha$.

It is clear that the normal equation (26) coincides with the equation (10) expressed by sample values. Namely consider x_α as a random sample and write formally as following

$$\frac{1}{q-1}\Sigma(x_\alpha-\bar{x})(x_\alpha-\bar{x})' = \hat{\Phi}$$

$$\frac{1}{q-1}\Sigma(y_\alpha-\bar{y})(x_\alpha-\bar{x}) = \hat{\Phi}_{yx}$$

Then (10) becomes

$$\hat{\Phi}\cdot\hat{h} = \hat{\Phi}_{yx} \tag{27}$$

By solving and transposing (27) we get $\hat{h}'$ in the following form.

$$\hat{h}' = \hat{\Phi}'_{yx}\hat{\Phi}^{-1}$$

In the theory of regression analysis, following procedures are used for calculations

$$A = \sum_{\alpha=1}^{q}(x_\alpha-\bar{x})(x_\alpha-\bar{x})' \tag{29}$$

where x_α are regarded fixed. Let

$$x_\alpha - \bar{x} = \mathring{x}_\alpha \tag{30}$$

then $\hat{h}'$ can be expressed in the formula

$$\hat{h}' = \sum_{\alpha=1}^{q} (\overset{o}{y}_\alpha \overset{o}{x}_\alpha') \cdot \left(\sum_{\alpha=1}^{q} \overset{o}{x}_\alpha \overset{o}{x}_\alpha'\right)^{-1} = \sum_{\alpha=1}^{q} \overset{o}{y}_\alpha \overset{o}{x}_\alpha' \cdot A^{-1} \tag{31}$$

The mean vector and the covariance matrix can be calculated in the following way.

$$\mathcal{E}(\hat{h}') = \mathcal{E} \sum_{\alpha=1}^{q} (\overset{o}{y}_\alpha \overset{o}{x}_\alpha') \cdot A^{-1} = \sum_{\alpha=1}^{q} h' \overset{o}{x}_\alpha \overset{o}{x}_\alpha' \cdot A^{-1} = h' \tag{32}$$

$$\begin{aligned}
\mathcal{E}(\hat{h}-h)(\hat{h}-h)' &= A^{-1} \mathcal{E} \sum_{\alpha=1}^{q} (\overset{o}{y}_\alpha - \mathcal{E}\overset{o}{y}_\alpha) \overset{o}{x}_\alpha \sum_{\gamma=1}^{q} (\overset{o}{y}_\gamma - \mathcal{E}\overset{o}{y}_\gamma) \overset{o}{x}_\gamma' A^{-1} \\
&= A^{-1} \sum_{\alpha\gamma=1}^{q} \mathcal{E}(\overset{o}{y}_\alpha - \mathcal{E}\overset{o}{y}_\alpha)(\overset{o}{y}_\gamma - \mathcal{E}\overset{o}{y}_\gamma) \overset{o}{x}_\alpha \overset{o}{x}_\gamma' A^{-1} \\
&= A^{-1} \sum_{\alpha\gamma=1}^{q} \delta_{\alpha\gamma} \sigma^2 \overset{o}{x}_\alpha \overset{o}{x}_\gamma' A^{-1} \\
&= \sigma^2 A^{-1} A A^{-1} = \sigma^2 A^{-1}
\end{aligned} \tag{33}$$

Above formulas are well-known results in the field of mathematical statistics. In this case the statistic $\hat{h}$ follows a normal-distribution with the mean h, and the covariance matrix $\sigma^2 A^{-1}$. About this result some explanatory remarks should be given here. If we assume that the random vector $X(t)$ is subject to a stationary normal process, then the output vector $Y(t)$ as a linear transform of $X(t)$ is also subject to a stationary normal process. In this case the regression model (25) takes place exactly and $\mathcal{E}_\alpha$ in (25) corresponds to $N_\mu(t)$ in (1). In another case when the random $X(t)$ is not subject to a stationary normal process but to only a stationary process, then $Y(t)$ as a linear transform of $X(t)$ (i.e. $\sum_j h(Zj)X(t-Zj)\Delta Zj$) tends to approach to a stationary normal process. (Although the central limit theorem can not be applied strictly in this case.[4])

So in the case when the random vector $X(t)$ is subject to any stationary process, the regression model (25) takes place practically, and following confidence regions of regression coefficients can be used.

In the normal regression theory, (31) is expressed as in (34), but results are same.

$$\hat{h} = (X X')^{-1} XY \tag{34}$$

where $X = \{x_{\beta v}\}$ $(\beta = 1, 2, \ldots, (s+1)n, \quad v = 1, 2, \ldots, q)$

$Y' = (y_1, y_2, \ldots, y_q)$

An estimate of σ^2 is obtained by ESS (error square sum) in a following way.

$$\text{ESS} = \Sigma(y_\alpha - \bar{y})^2 - \hat{h}'\Sigma(x_\alpha - \bar{x})(y_\alpha - \bar{y}) \tag{35}$$

$$\sigma^2 = \frac{\text{ESS}}{q-(s+1)n-1} \tag{36}$$

The multiple correlation coefficient R, defining the correlation between $h'x$ and Y, is estimated by

$$\hat{R} = \sqrt{1 - \frac{\text{ESS}}{\Sigma(Y_\alpha - \bar{Y})^2}} \tag{37}$$

By using multiple correlation coefficient, we can verify the model of Fig. 1, but further discussions will be omitted here.

Now we are going to discuss confidence regions of regression coefficients $\hat{h}_\mu$. First to be shown a confidence region of $\hat{h}_\mu$, a component of regression vector $\hat{h}$, and secondly the simultaneous confidence regions of coefficients are to be introduced.

It is well-known that the confidence interval of $\hat{h}_\mu$ (μth component of $\hat{h}$) with a confidence coefficient 1-r is expressed by (38).

$$h_\mu - t_{q-(s+1)n-1}(r)\hat{\sigma}\sqrt{a^{\mu\mu}} \leqq h_\mu \leqq \hat{h}_\mu + t_{q-(s+1)n-1}(r)\hat{\sigma}\sqrt{a^{\mu\mu}} \tag{38}$$

where $a^{\mu\mu}$ stands for the (μ,μ) element of the matrix A^{-1}, and $t_{q-(s+1)n-1}$ is the 100 r o/o point of the t-distribution with q-(s+1)n-1 degrees of freedom. (38) derives from the fact that each statistic

$$t_\mu = \frac{\hat{h}_\mu - h_\mu}{\sigma\sqrt{a^{\mu\mu}}} \Big/ \frac{\hat{\sigma}}{\sigma} = \frac{\hat{h}_\mu - h_\mu}{\hat{\sigma}\sqrt{a^{\mu\mu}}} \tag{39}$$

follows a t-distribution with q-(s+1)n-1 degrees of freedom.

In the case of using artificial signals, the matrix A often has a form such that

$$A = q\,E \tag{40}$$

where E is an unit matrix.

In this case each h_μ component of the vector h is estimated independently, so it is adequate to use confidence regions (38). But in more general cases, each component h_μ is estimated simultaneously, and $\hat{h}_\mu$ is not always independent each other. About simultaneous confidence interval estimations on regression coefficients we may refer to references[5,6].

Here well-known simultaneous confidence regions on regression coefficients h_μ with a confidence coefficient 1-r will be introduced.

This is

$$(\hat{\hbar}-\hbar)'A(\hat{\hbar}-\hbar) \leqq \{(s+1)n+1\}\,\hat{\sigma}^2 F_{(s+1)n+1,\,(\gamma)_{q-(s+1)n+1}} \tag{41}$$

where $F_{(s+1)n+1,(\gamma)_{q-(s+1)n+1}}$ is the upper 100 γ o/o point of the F-distribution with (s+1)n+1, and q-(s+1)n-1 degrees of freedom. Namely

$$\Pr\left\{ F > F_{(s+1)n+1,\,(\gamma)_{q-(s+1)n-1}} \right\} = \gamma \tag{42}$$

For example, let us consider the case when n=7, s=12, and q=1000. Then we get $F_{92,908.}(\gamma)$. In F-distribution tables, usually it is not shown values of F with large degrees of freedom. Let us consider an approximation method for F-distribution in such a case. By taking on account that $F^{n_1}_{n_2}$ is defined as $\frac{\chi_1^2/n_1}{\chi_2^2/n_2}$ and assuming $n_2 \gg n_1$, we can consider that χ_2^2/n_2 converges in probability to 1 and χ_1^2/n_1 converges in law to $N(1,1/n_1)$ as n_1 and n_2 tends to infinity. Then we can use the fact that the statistic $F^{n_1}_{n_2}$ converges to $N(1,1/n_1)$ in law. The error estimation is given by

$$\operatorname*{Sup}_{E<B} \left| P^{F^{n_1}_{n_2}}(E) - P^{N}(E) \right| \leqq 2\sqrt{I_{n_1,n_2}} \tag{43}$$

for any set belonging to B, where B is a Borel-type probability space. I_{n_1,n_2} is defined as follows[7].

$$I_{n_1,n_2} = \mathcal{E}_{f_{n_1,n_2}} \log \frac{f_{n_1,n_2}(x)}{\phi(1,1/n_1)} \tag{44}$$

$\mathcal{E}_{f_{n_1,n_2}}$ designates the expectation operator with respect to F-distribution with n_1,n_2 degrees of freedom.

APPLICATION OF PARTIAL CORRELATION TECHNIQUES

As an application of partial correlation techniques, let us consider the case, where some $\hbar_{1j}$, for example j=3,4,...,n, are known. In this case partial correlation techniques can be used. As mentioned above, $\hat{\hbar}$ is distributed according to a normal distribution with mean $\hbar$ and covariance matrix $\sigma^2 A^{-1}$. Let us partition the vector $\hat{\hbar}$ and the covariance matrix as follows.

$$\hat{h} = \begin{bmatrix} \hat{h}_{11} \\ \hat{h}_{12} \\ \hat{h}_{1j} \end{bmatrix} \qquad \sigma^2 A^{-1} = \begin{pmatrix} \Sigma_{11}, \Sigma_{12}, \Sigma_{13} \\ \Sigma_{21}, \Sigma_{22}, \Sigma_{23} \\ \Sigma_{31}, \Sigma_{32}, \Sigma_{33} \end{pmatrix} \tag{45}$$

It is known that h_{1j} are h^*_{1j} in reality. Let us use the concept of conditional distribution. Then the conditional mean and covariance of $\begin{pmatrix} \hat{h}_{11} \\ \hat{h}_{12} \end{pmatrix}$ are

$$\mathcal{E}\left[\begin{pmatrix} \hat{h}_{11} \\ \hat{h}_{12} \end{pmatrix} \middle| h^*_{1j}\right] = \begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix} + \begin{pmatrix} \Sigma_{13}\Sigma_{33}^{-1} \\ \Sigma_{23}\Sigma_{33}^{-1} \end{pmatrix} (h^*_{1j} - h_{1j}) \tag{46}$$

and

$$\mathcal{C}\left[\begin{pmatrix} \hat{h}_{11} \\ \hat{h}_{12} \end{pmatrix} \cdot \begin{pmatrix} \hat{h}_{11} \\ \hat{h}_{12} \end{pmatrix}' \middle| h^*_{1j}\right] = \begin{pmatrix} \Sigma_{11}, \Sigma_{12} \\ \Sigma_{21}, \Sigma_{22} \end{pmatrix} - \begin{pmatrix} \Sigma_{13} \\ \Sigma_{23} \end{pmatrix} \Sigma_{33}^{-1} \left(\Sigma_{31}, \Sigma_{32}\right) \tag{47}$$

Above formulas can be used for the correction of $\hat{h}_{11}$ and $\hat{h}_{12}$ and their confidence intervals in the case when h_{1j} $(j=3,\cdots\cdots n.)$ are known.

Partial correlations are considered also useful for the smoothing of impulse responses. For example, if $\hat{h}_{11}(5)$ and $\hat{h}_{11}(6)$, are partially positive correlated significantlly, then we can verify the smoothing of these two points of impulse response h_{11}. But further discussions will be omitted here.

CONCLUSION

In above discussions, the author shows that correlation techniques to estimate impulse responses have close relations to linear regression theories. He suggests the applicability of simultaneous confidence interval estimation of regression coefficients to correlation techniques. He also suggests the application of partial correlation techniques to identification problems. But the utility of these confidence regions must be tested by experiments.

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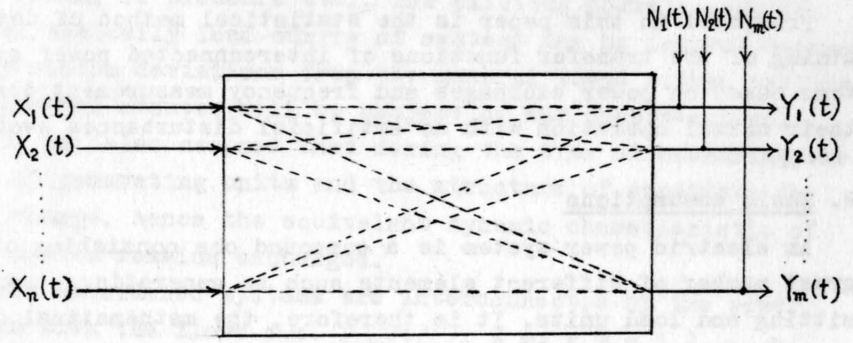

Figure 1: General linear system with multiple inputs and outputs

A METHOD TO DETERMINE THE TRANSFER FUNCTIONS OF POWER SYSTEMS

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1. Introduction

The problems connected with designing and setting of power and frequency automatic control system in the electric power systems require,primarily, a thorough investigation into their operational behaviour under various conditions,both in a transient and steady state. The control systems adjustment usually begins with the identification of the object and with construction of its mathematical model.To begin with the dynamic properties of the investigated object are to be determined. The disturbance in balance of power generated and load that initiate the transient phenomena, follows mainly the fluctuation in power demand. At some fixed moments power demand may be regarded as a random variable and to its analysis statistical methods may be applied. Much in the same way power exchange and frequency will be treated as stochastic processes.

Presented in this paper is the statistical method of determining of the transfer functions of interconnected power systems based on power exchanges and frequency measurement during their normal operation with no artificial disturbances evoked.

2. Basic assumptions

An electric power system is a compound one consisting of a great number of different elements such as generating, transmitting and load units. It is therefore, the mathematical description of system is very difficult or practically impossible if it is based only on the characteristics of the different elements, without carrying out measurements on the actual system. This concerns particularly the analysis of transient processes.More over, the characteristics of the different elements of the system are not exactly known and many of them vary during operation. The bests results can be ob-

tained on the base of measurements in the actual system during its normal operation but even in this case, certain simplifications will have to be adopted when the data are processed.

There will no consideration upon the characteristics of particular elements of the system, but only the characteristics of the system as a whole. Each system will be treated as a substitute generating unit, characterized by its system primary automatic control and inertia of the rotating masses and as one substitute load unit. The last one is, however, characterized by the natural relation between frequency and power demand. In the further considerations the generating and the load units will be treated as one control system. At the input of each of such systems the process i.e. $Z_r/t/$ /r= 1,2,...,n/ of power demand is observed. It is assumed that the power demand processes of different systems are statistically independent. This assumption is founded on the fact that the switching on and off of the particular receivers in one system does not cause similar changes in other systems; these changes occur in each system independently. The same may be said about the changes of the generated power caused by the disturbances in boilers /changes of the temperture of the steam, of pressure etc/. The existing correlations between the daily load curves of systems may be ignored, because only random deviations from the general trend within periods of several minutes will be subject of our consideration.

It is also assumed that during the time of measuring, the set of generating units and the structure of receivers do not change, hence the equivalent dynamic characteristic of the system remains unchanged.

The considered systems are interconnected by the power lines with the flows i.e. $P_{rm}/t/$ /$r \neq m$; r,m = 1,2,...,n/. These lines connect particular systems at the input /fig.1/ while at their outputs frequency processes i.e. $f_k/t/$ are observed. As a result a multi-input and multi-output system is obtained. Thus, there are n inputs /n-number of systems/, while the outputs may be as numerous as the points of frequency measurement. For such a system, therefore, exist as many transfer functions as there are combinations between

all the inputs and outputs, i.e. /n.N/. The most informative for each system will be the transfer function found in relation with the frequency measured in the given system, or determined with regard to the frequency measured in a central point chosen in a group of systems. Particularly, calculations may be carried out with reference to the frequency f/t/ measured at one point. This implies $f_1/t/ = f_2/t/, \ldots, = f_N/t/ = f/t/$ as it has been assumed in this paper. The determination of the transfer function with regard to the frequency measured at other points /it being known for one point/, does not present difficult problem.

In general, the processes of power load, power exchange and frequency are not stationary. It has been assumed, however, that after subtraction of their mean functions the residual processes are stationary and ergodic. It is, only, necessary to estimate properly the mean function of each of the considered processes.

Due to the relatively small deviations of the considered power and frequency, linearity of the system may be assumed.

It also must be noticed that loads of systems $Z_r/t/$, /r = 1,2,...,n/ can not be directly measured. Therefore a direct determination of their statistical characteristics is impossible. The problem, the solution of which has been given in this paper, consists in finding the transfer functions of each power system, when only exchange power and frequency processes are known. Then the determination of the statistical characteristics of load in an indirect way is possible.

A diagram of interconnected power systems is presented in fig. 1.

3. The basic relations

An arbitrary number n > 2 of the interconnected power systems is considered /fig. 1/. The centered processes $Z_r/t/$, $P_r/t/$ and f/t/ satisfy the relation

$$f(t) = -\int_0^\infty w_r(\tau)\left[Z_r(t-\tau) - P_r(t-\tau)\right]d\tau \qquad (r = 1,2,\ldots,n) \tag{3.1}$$

which can be rewritten in the form

$$f(t) - \int_0^\infty w_r(\tau) P_r(t-\tau)\,d\tau = -\int_0^\infty w_r(\tau) Z_r(t-\tau)\,d\tau \qquad (r = 1,2,\ldots,n). \tag{3.2}$$

Setting t + s instead of t yields:

$$(3.3)\quad f(t+s)-\int_0^\infty w_m(\tau)\,P_m(t+s-\tau)\,d\tau=-\int_0^\infty w_m(\tau)\,Z_m(t+s-\tau)\,d\tau \qquad (m=1,2,\ldots,n).$$

Multiplying both sides of equations /3.2/ and /3.3/ and taking the expected value of this expression, we obtain

$$E\big(f(t)\cdot f(t+s)\big)-E\Big(\int_0^\infty w_m(\tau)\cdot f(t)\cdot P_m(t+s-\tau)\,d\tau\Big)-E\Big(\int_0^\infty w_r(\tau)f(t+s)P_r(t-\tau)\,d\tau\Big)+$$

$$(3.4)\quad +E\Big(\int_0^\infty\int_0^\infty w_r(\tau)\,w_m(\eta)\,P_r(t-\tau)\,P_m(t+s-\eta)\,d\tau\,d\eta\Big)=$$

$$=E\Big(\int_0^\infty\int_0^\infty w_r(\tau)\,w_m(\eta)\,Z_r(t-\tau)\,Z_m(t+s-\eta)\,d\tau\,d\eta\Big).$$

Assuming the processes $Z_r/t/$ /r = 1,2,...,n/ to be uncorrelated, equation /3.4/ can be written in the form

$$E\big(f(t)f(t+s)\big)-\int_0^\infty w_m(\tau)E\big(f(t)\cdot P_m(t+s-\tau)\big)d\tau-\int_0^\infty w_r(\tau)\cdot E\big(f(t+s)P_r(t-\tau)\big)d\tau+$$

$$(3.5)\quad +\int_0^\infty\int_0^\infty w_r(\tau)\,w_m(\eta)\,E\big(P_r(t-\tau)P_m(t+s-\eta)\big)d\tau d\eta=0 \qquad (m\neq r;\ m,r=1,2,\ldots,n)$$

making use of the theorem that the expected value of product of statistically independent random variables with mean value zero equals zero. Further

$$E\big(f(t)f(t+s)\big)-\int_0^\infty w_r(\tau)E\big(f(t)P_r(t+s-\tau)\big)d\tau-\int_0^\infty w_r(\tau)E\big(f(t+s)P_r(t+s)\big)d\tau+$$

$$(3.6)\quad +\int_0^\infty\int_0^\infty w_r(\tau)\,w_r(\eta)\,E\big(P_r(t-\tau)\,P_r(t+s-\eta)\big)\,d\tau\,d\eta=$$

$$=\int_0^\infty\int_0^\infty w_r(\tau)\,w_r(\eta)\,E\big(Z_r(t-\tau)\,Z_r(t+s-\eta)\big)\,d\tau\,d\eta \qquad (r=1,2,\ldots,n).$$

From the known definitions of correlation functions, the equations /3.5/ and /3.6/ take the form

$$R_f(s)-\int_0^\infty w_m(\tau)\,R_{P_m f}(\tau-s)\,d\tau-\int_0^\infty w_r(\tau)\,R_{P_r f}(\tau+s)\,d\tau+$$

$$(3.7)\quad +\int_0^\infty\int_0^\infty w_r(\tau)\,w_m(\eta)\,R_{P_r P_m}(s+\tau-\eta)\,d\tau\,d\eta=0 \qquad (r\neq m;\ r,m=1,2,\ldots,n)$$

and

$$R_f(s) - \int_0^\infty w_r(\tau)\, R_{P_r f}(\tau - s)\, d\tau - \int_0^\infty w_r(\tau)\, R_{P_r f}(\tau + s)\, d\tau +$$

$$(3.8) \quad + \int_0^\infty \int_0^\infty w_r(\tau)\, w_r(\eta)\, R_{P_r P_r}(s + \tau - \eta)\, d\tau\, d\eta =$$

$$= \int_0^\infty \int_0^\infty w_r(\tau)\, w_r(\eta)\, R_{Z_r}(s + \tau - \eta)\, d\tau\, d\eta , \qquad (r = 1, 2, \ldots, n).$$

Applying Fourier's transformation to the equations /3.7/and /3.8/ we obtain

$$(3.9) \quad \varphi(\omega) - \bar{\varphi}_m(\omega)\, \alpha_m(\omega) - \varphi_r(\omega)\, \bar{\alpha}_r(\omega) + \psi_{rm}(\omega)\, \alpha_m(\omega)\, \bar{\alpha}_r(\omega) = 0, \qquad (r \neq m;\ r, m = 1, 2, \ldots, n)$$

and

$$\varphi(\omega) - \bar{\varphi}_r(\omega)\, \alpha_r(\omega) - \varphi_r(\omega)\, \bar{\alpha}_r(\omega) + \psi_{rr}(\omega)\, \alpha_r(\omega)\, \bar{\alpha}_r(\omega) =$$

$$(3.10) \quad = \psi_{Z_r}(\omega)\, \alpha_r(\omega)\, \bar{\alpha}_r(\omega), \qquad (r = 1, 2, \ldots, n).$$

As $\sum_{r=1}^{n} P_r/t/ = 0$, the following relations between correlation functions and spectral densities must hold

$$(3.11) \quad \sum_{r=1}^{n} R_{P_r f}(s) = 0 \qquad \text{and} \qquad \sum_{r=1}^{n} \varphi_r(\omega) = 0,$$

$$(3.12) \quad \sum_{r=1}^{n} R_{P_r P_m}(s) = 0 \qquad \text{and} \qquad \sum_{r=1}^{n} \psi_{rm}(\omega) = 0 \qquad (m = 1, 2, \ldots, n).$$

Now, the way of solving equations /3.9/ and /3.10/with regard to the unknown quantities $\alpha_r(\omega)$ and $\psi_{Z_r}(\omega)$ will be shown. To simplify, we then omit the symbol ω . Let's put

$$(3.13) \quad \beta_r = \frac{1}{\alpha_r} \qquad \text{and} \qquad \beta = \sum_{r=1}^{n} \beta_r .$$

The relations /3.11/ and /3.12/ can be rewritten in the form

$$(3.14) \quad \sum_{\substack{r=1 \\ r \neq m}}^{n} \varphi_r = -\varphi_m \qquad \text{and} \qquad \sum_{\substack{r=1 \\ r \neq m}}^{n} \psi_{rm} = -\psi_m \qquad (m = 1, 2, \ldots, n).$$

In order to eliminate ψ_{rm} from /3.9/, we divide /3.9/ and /3.10/ by $\bar{\alpha}_r\, \alpha_m$ and, using /3.13/, we obtain

(3.15) $$\varphi\cdot\beta_m\bar{\beta}_r-\varphi_r\beta_m-\bar{\varphi}_m\bar{\beta}_r+\psi_{rm}=0 \qquad (r\neq m;\ r,m=1,2,\ldots,n)$$

and

(3.16) $$\varphi\cdot\beta_r\bar{\beta}_r-\varphi_r\beta_r-\bar{\varphi}_r\bar{\beta}_r+\psi_r=\psi_{z_r} \qquad (r=1,2,\ldots,n).$$

Thereupon, adding for every fixed m equations /3.15/ for $r\neq m$; $r=1,2,\ldots,n$ and making use of relations /3.14/, we obtain

(3.17) $$\left(\beta_m-\frac{\bar{\varphi}_m}{\varphi}\right)\left(\bar{\beta}_m-\frac{\varphi_m}{\varphi}\right)-\beta\left(\beta_m-\frac{\bar{\varphi}_m}{\varphi}\right)+\frac{\psi_m\varphi-\varphi_m\bar{\varphi}_m}{\varphi^2}=0 \qquad (m=1,2,\ldots,n).$$

Knowing $\varphi, \varphi_m, \psi_m$ we can solve the set of equations /3.17/ with regard to the unknown quantities β and β_r. The solution will be given in the next paragraphs.

It should be noticed that also equation /3.16/ can be rewritten as follows

(3.18) $$\left(\beta_m-\frac{\bar{\varphi}_m}{\varphi}\right)\left(\bar{\beta}_m-\frac{\varphi_m}{\varphi}\right)+\frac{\psi_m\varphi-\varphi_m\bar{\varphi}_m}{\varphi^2}=\frac{\psi_{zm}}{\varphi} \qquad (m=1,2,\ldots,n).$$

After subtraction of /3.17/ from /3.18/ we have

(3.19) $$\beta\left(\beta_m-\frac{\bar{\varphi}_m}{\varphi}\right)=\frac{\psi_{zm}}{\varphi} \qquad (m=1,2,\ldots,n).$$

4. Determination of the modulus of the transform of the joint weight function. The share function. Discriminant of the system

The right side of the equation /3.19/ must be real and non-negative, because it is the ratio of spectral densities of processes of m-th system power demand to those of frequency. It follows that for the real ω there exists real and non-negative function b_m, which satisfies

(4.1) $$b_m\beta=\beta_m-\frac{\bar{\varphi}_m}{\varphi}.$$

b_m will be called the share function of the m-th system. It is

real for all the real ω .

Summing up the expressions in /4.1/ for m = 1,2,...,n and taking into account that $\sum_{m=1}^{n} \varphi_m = 0$ we obtain

$$\sum_{m=1}^{n} b_m = 1 . \tag{4.2}$$

Let's denote

$$h_m \stackrel{def}{=} \frac{\varphi \cdot \psi_m - \varphi_m \bar{\varphi}_m}{\varphi^2} \qquad (m = 1,2,\ldots,n). \tag{4.3}$$

Function h_m will be called the discriminant of m-th system. It is real for real ω .

Now, the method of determining the modulus of β and b_m on the ground of discriminants h_m /m = 1,2,...,n/ will be given.

Using /4.1/ and /4.3./ we can rewrite /3.17/ in the form

$$|\beta|^2 b_m^2 \left(1 - \frac{1}{b_m}\right) + h_m = 0 \qquad (m = 1,2,\ldots,n). \tag{4.4}$$

From /4.4/ and /4.2/ it follows, that for every real $h_m / \omega / \geq 0$. Solving the equation /4.4/ with regard to b_m we obtain

$$b_m = \frac{1}{2}\left(1 + \delta_m \sqrt{1 - \frac{4h_m}{|\beta|^2}}\right) \qquad (m = 1,2,\ldots,n). \tag{4.5}$$

Summing up the equations /4.5/ for m = 1,2,...,n and taking into account /4.2/ the following result is obtained

$$(n-2)|\beta| + \sum_{m=1}^{n} \delta_m \sqrt{|\beta|^2 - 4h_m} = 0 \tag{4.6}$$

where δ may take the values +1 or -1.

This way, only one equation with one unknown quantity $|\beta|$ for each ω is obtained. Having found $|\beta|$ by e.g. regula falsi method, b_m can be determined from equation /4.5/. It is however, necessary to choose, the appropriate sign of δ_m. It

can be proved that a necessary and sufficient condition to have a solution of the set of equations /4.5/ for each ω there must be

$$2h_{m_o} < \sum_{m=1}^{n} h_m \quad \text{where} \quad h_{m_o} = \max_m h_m \,. \tag{4.7}$$

If /4.7/ is fulfilled and

$$(n-2)\sqrt{h_{m_o}} \geqslant \sum_{m=1}^{n} \sqrt{h_{m_o} - h_m} \,, \tag{4.8}$$

then $\delta_m = -1$ for $m = 1,2,\ldots,n$.

If, however, /4.7/ is fulfilled and, besides

$$(n-2)\sqrt{h_{m_o}} < \sum_{m=1}^{n} \sqrt{h_{m_o} - h_m} \tag{4.9}$$

then $\delta_{m_o} = 1$ and $\delta_m = -1$ for $m = 1,2,\ldots,n;\ m \neq m_o$.

Thus the method to determine $|\alpha| = \frac{1}{|\beta|}$ and b_m has been given. Then the transforms α and α_m /m = 1,2,...,n/ will be determined.

5. Determination of the transform of the joint weight function

The weight function w/t/ = 0 for $t < 0$. Let us put

$$\alpha(\omega) = \int_0^\infty w(t)\, e^{-i\omega t}\, dt \tag{5.1}$$

where $\omega = u + iv$ is any complex number for which the integral /5.1/ exists. Let's suppose

$$\int_{-\infty}^{\infty} \frac{|\ln|\alpha(\omega)||}{1+\omega^2}\, d\omega < \infty \,. \tag{5.2}$$

It is known, that the function α /u + iv/ will be analitical and bounded for each $v < 0$. It will enable us to determine this function, if its modulus on the real axis is known[8]. The solution is given by the equation

$$\alpha(u) = |\alpha(u)|\,(\cos q(u) + i \sin q(u)) \tag{5.3}$$

where

$$q(u) = -\frac{2}{\pi}\int_0^\infty \frac{u \ln|\alpha(x)|}{u^2 - x^2}\,dx . \tag{5.4}$$

6. Determination of the step response functions of the individual systems

Having found the transform of the joint weight function of the interconnected power systems and taking into account relations /4.5/, the transform α_m of the weight functions of individual systems can be determined from formula /4.1/, which can be rewritten in the form

$$\alpha_m = \frac{1}{\frac{b_m}{\alpha} + \frac{\varphi_m}{\varphi}} \qquad (m = 1,2,\ldots,n). \tag{6.1}$$

It may be noticed, that having found α and α_m and applying /3.13/, the spectral density φ_{z_m} of power demand can be determined from the relation /3.19/.

Next, the weight functions w/t/ and w_m/t/ can be calculated applying the inverse Fourier's transform

$$w_m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha_m(\omega)\, e^{i\omega t}\,d\omega \tag{6.2}$$

which permits finding the step response functions

$$W_m(t) = \int_0^t w_m(\tau)\,d\tau . \tag{6.3}$$

Thus, the problem of determination of the step response functions W/t/ and W_m/t/ based upon the correlation functions R_f, R_{P_r}, $R_{P_r f}$/r = 1,2,...,n/ has been theoretically solved.

7. Final considerations

It is easy to see that the frequency bias $\overline{\nu}$ of a power system, defined as $K = \lim_{t\to\infty} \frac{1}{W/t/}$ which implies $K = \lim_{\omega\to 0} \frac{1}{\alpha/\omega/}$ can be determined from the equation /4.6/ for $\omega = 0$. Thereupon, the K_m /m = 1,2,...,n/ can be calculated from the

equation /4.1/, where $K = \lim_{\omega \to 0} \beta/\omega$ /, after previous determination of the b_m functions from equation /4.5/, also for $\omega = 0$. From the definition of the Fourier's transform follows that the spectral density for $\omega = 0$ is the integral of the correlation function. Then, if the frequency bias is to be found, the appropriate integrals of the correlation functions should be estimated. The mathematical model, based on the estimation of integrals of correlation functions may be called a model "with memory". Its particular case is a model which can be called a model "without memory" in which only the values of correlation functions for $\tau = 0$ are used, /i.e. the proper variances and covariances/[3]. This model can be used only in the case of homogeneous systems, i.e. when the integrals of the correlation functions of frequency, exchange powers and cross correlation functions between these processes are aproximately proportional to the appropriate variances and covariances with equal coefficient of proportionality for all the systems.

Finally, we want to add that for two interconnected systems /n = 2/, the problem stated in this paper can be solved only after additional assumptions. In the papers[3,4] /see bibliography/ the method of determination of the K coefficient in such cases is presented. There is no problem to generalize this method for the determination of the transfer functions on the basis of the theory described in this paper.

8. Numerical example

In order to verify the presented method, calculations of transfer functions were performed on the basis of measurements in the interconnected systems of Poland and East Germany - 1 /see fig.2/, Czechoslovakia - 2, Hungary - 3, Rumania and West Ukraina - 4. The measurements were carried out in october 1965. The period of measurement was $T = 1000$ sec, $\Delta t = 0{,}5$ sec, the number of sample data $N = 2000$. The balance of exchange power of each system was centered by subtraction of the so-called moving average[6], and the process of frequency too. The period of the moving average was $T_r = 50$ sec. Fig.3 shows graphs of the corrected correlation functions between these processes. In fig.4 the spectral densities and in fig.6 and 7 the cross

spectral densities are presented. In fig. 8 the modulus of the transform of the joint weight function of the interconnected systems $\alpha(\omega)$ and phase characteristic $q/\omega/$ are given. In fig. 9, the frequency transfer function of the interconnected systems is shown. At the end, fig. 10 presents the step response functions of each system /curves 1 - 4/ and that for interconnected system /curve 5/.

9. Conclusions

Presented method to determine the transfer function will give correct results only if the estimation of all the used functions is properly made [1,2,8]. E.g. if the moving average method for mean of process estimation is applied, suitable correction of the correlation functions will be necessary. The proper method of estimation of spectral densities, $h_m/\omega/$ and $q/\omega/$ functions will also be necessary. This method will be presented in further publications.

A more detailed analysis of method simplifying assumptions is also required. Because of the numerical way of determination of all the statistical functions, the proper choice of a range of function determination as well as of the intervals between the values of the argument of the measured functions is necessary.

The other procedure of transfer functions determination is also applicable. The modulus of the transform of joint weight function can be aproximated with a suitable function and then the transfer function determined in an analitical way.

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Symboles

n	- number of systems
r, m	- indices
τ, η, s, x	- variables of integration
$\bar{\beta}$	- value conjugate to complex value
$\lvert\beta\rvert$	- modulus of β
$f(t)$	- centered process of the frequency
$P_m(t)$	- centered process of exchange power balance of m-th system
$E(X)$	- operator of random variable x
$R_f(s)$	- atocorrelation function of the frequency
$R_{P_r P_r}(s)$	- autocorrelation function exchange power balance of r-th system
$R_{P_r P_m}(s)$	- cross correlation function of exchange power balances between r-th and m-th systems
$R_{Z_m}(s)$	- autocorrelation function of power demand of m-th system
$R_{P_r f}(s)$	- cross correlation function between exchange power of r-th system and frequency
$\varphi(\omega)$	- spectral density of frequency

$\psi_m(\omega)$ - spectral density of exchange power balance of m-th system

$\psi_{rm}(\omega)$ - cross-spectral density between exchange power balances of r-th and m-th systems

$Re\,\psi_{rm}(\omega)$ - real part of $\psi_{rm}(\omega)$ function

$Im\,\psi_{rm}(\omega)$ - imaginary part of $\psi_{rm}(\omega)$ function

$\varphi_z(\omega)$ - spectral density of power demand

$\varphi_m(\omega)$ - cross-spectral density between exchange power balance of m-th system and frequency

$b_m(\omega)$ - share function of m-th system

$h_m(\omega)$ - discriminant of the m-th system

$W_m(t)$ - step response function of m-th system

$w_m(t)$ - weight function of m-th system

$\alpha_m(\omega)$ - Fourier's transform of weight function of m-th system

$|\alpha(\omega)|$ - reciprocal of $|\beta(\omega)|$

$W(t)$ - step response function of interconnected systems

$\alpha(\omega)$ - Fourier's transform of w/t/ function

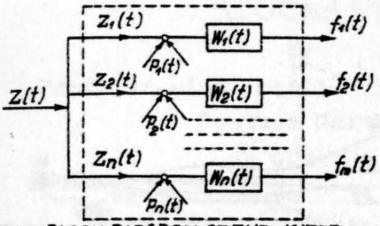

FIG.1. BLOCK DIAGRAM OF THE INTER-CONNECTED POWER SYSTEMS

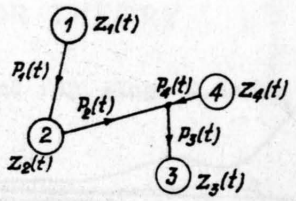

FIG.2 SCHEMATIC DIAGRAM OF THE INTER-CONNECTION OF FOUR POWER SYSTEMS

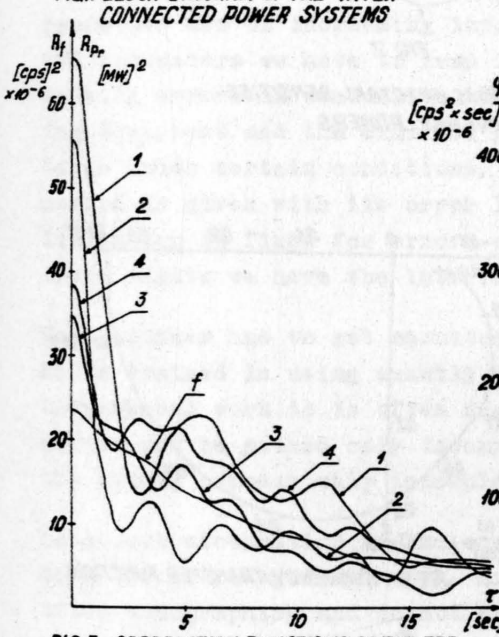

FIG.3. CORRELATION FUNCTIONS OF THE FREQUENCY AND THAT OF THE EXCHANGE POWER FLOWS

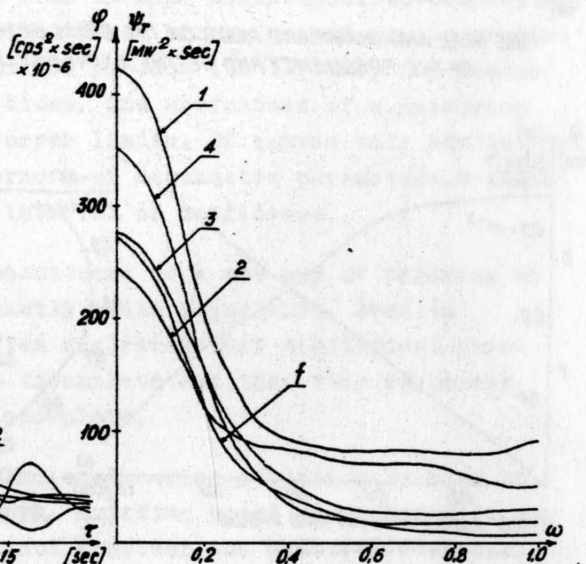

FIG.4. FUNCTION OF THE SPECTRAL DENSITY OF THE FREQUENCY AND THAT OF THE EXCHANGE POWER FLOWS

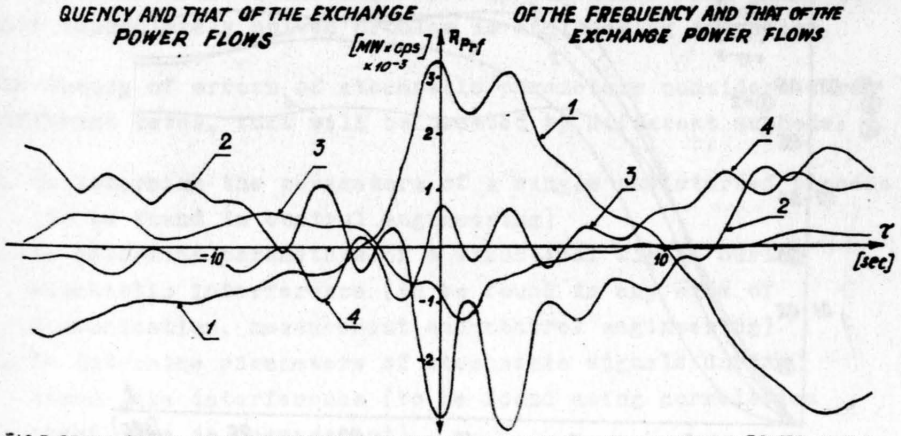

FIG.5. CROSS-CORRELATION FUNCTIONS OF THE FREQUENCY AND OF EXCHANGE POWER FLOWS OF POWER SYSTEMS

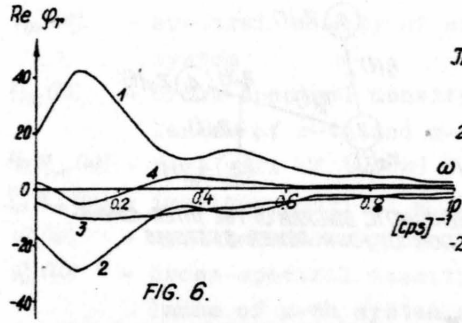

FIG. 6.

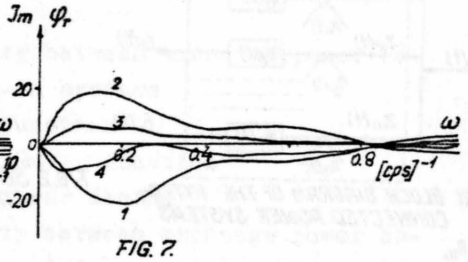

FIG. 7.

THE REAL AND IMAGINARY PARTS OF THE CROSS SPECTRAL DENSITIES OF THE FREQUENCY AND OF THE EXCHANGE POWERS

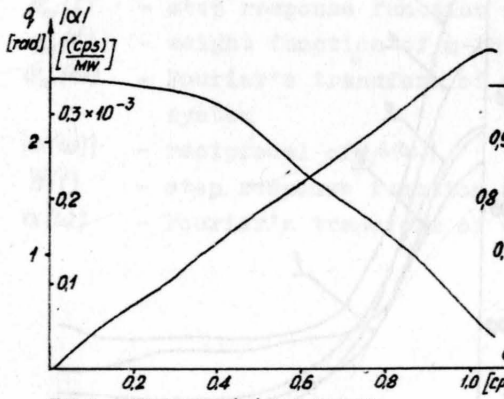

FIG. 8. AMPLITUDE |α| AND PHASE CHARACTERISTICS

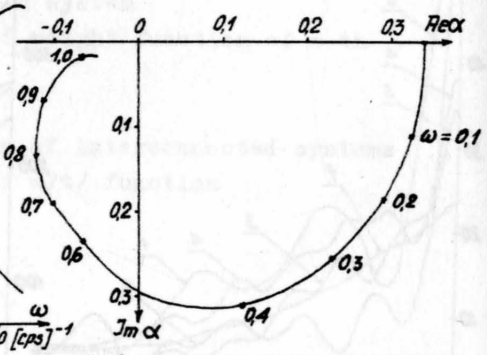

FIG. 9. FREQUENCY TRANSFER FUNCTION

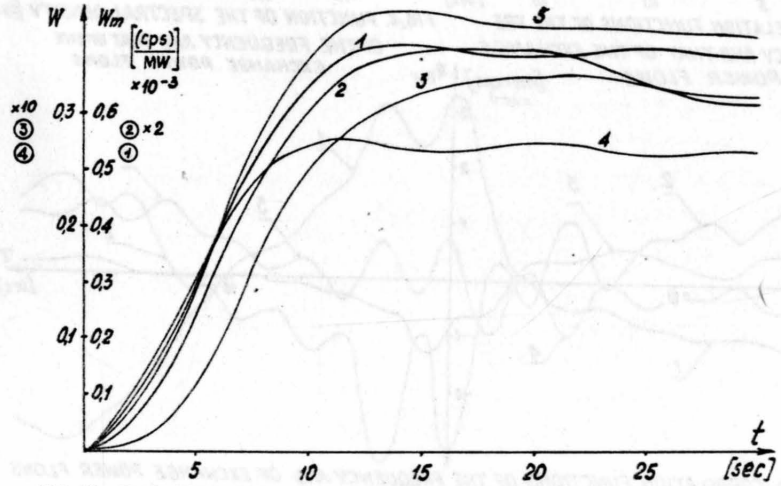

FIG. 10. THE STEP RESPONSE FUNCTIONS OF THE POWER SYSTEMS

STOCHASTIC ERROR THEORY

by

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GDR

In the field of communication, measurement and control stochastic processes are of increasing importance. Using stochastic signals and parameters we have to keep in mind their specific properties causing errors in evaluation. These errors depend on the measuring equipment and the examined process itself. They may become large under certain conditions. The usefulness of a measuring method is given with its error limits. Of course only statistical limits can be fixed for errors of stochastic parameters. Between these limits we have the interval of confidence.

The engineer has to get accustomed to a new way of thinking as he is trained in using exactly valid algorithms. Even in theoretical work it is often neglected that statistical parameters can be gained only incomplete and that they represent the actual process only incomplete.

To absorb stochastics in the engineering education we have to do a lot of pedagocical work. Existing books on mathematics are often monographies and do not consider the spectral representation accustomed to the engineer. My discourse will point to this incompletely solved problem in engineering education.

The theory of errors of stochastic parameters considers three different cases, that will be treated by different methods:

1. To determine the parameters of a single undisturbed process (to be found in control engineering)
2. to determine parameters of a sinusoidal signal during stochastic interference (to be found in any area of communication, measurement and control engineering)
3. to determine parameters of stochastic signals during stochastic interference (to be found using correlation techniques in measurement).

Statements on parameters of stochastic processes are usually based on the presumption that the process is stationary in time and ergodic moreover. To be stationary means the statistical parameters to be invariant with time. To be ergodic means, the statistical mean values of an ensemble can be determined by observing the mean value in time of one parameter only, the measuring time being infinite. In practical work this condition cannot be accomplished. The theory of errors of stochastic parameters is therefore usually engaged with errors in gaining mean values in short-time-measurements. These errors depend on the measuring time, the measuring method and the stochastic process itself.

The most important parameters of stochastic processes are the probability distribution and the compound-probability. Often a Gaussian distribution can be used as a model for the real distribution:

a) superimposing several not Gaussian processes we get a close approximation of a Gaussian distribution. W. Giloi proved this in superimposing several processes with constant distribution (fig.1). The special curves for n = 1 to n = 4 originate from the n-fold convolution of the rectangular distribution. We see that superimposing only four rectangular distributions gives a good approximation of a Gaussian distribution.

b) Transferring stochastic signals through a first-order-phase-lag-circuit (low-pass-system) we have the same result by integration, i.e. the approximation of a Gaussian distribution.

Assuming a Gaussian distribution as a mathematical model is therefore usually correct, as most automatic control systems have a low upper cutoff-frequency.

The Gaussian distribution can be expressed as

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \qquad \text{with } M_1 = (\overline{x(t)}) = 0$$

In the Gaussian distribution values outside x = become rather small. With x = σ we have 68% of all values inside this interval,

with $x = 2\sigma$ we have 95,5%, and with $x = 3\sigma$ 99,7%. Low-pass-filtering eliminates short strong pulses, i.e. amplitudes differing from the mean value.

Often the probability distribution is not sufficient for a complete description of a stochastic process. Describing it by the autocorrelation function (AKF) $\psi_{xx}(\tau)$ is often more useful. The values $\psi_{xx}(0)$ and $\psi_{xx}(\infty)$ determine the Gaussian distribution:

$$\sigma = \sqrt{\psi(0)} \qquad \text{and} \qquad \mu = \sqrt{\psi(\infty)}$$

Fig.2 demonstrates $\psi(\tau)$ for several fluctuating processes with the same probability distribution p(x). The AKF $\psi_{xx}(\tau)$ goes steady or oscillating to zero.

For a small correlation duration $\tau_0 = \frac{1}{\psi(0)} \int \psi(\tau)\, d\tau$ we have a spectrum of large extent. For a complete description of a stochastic process we need besides the dispersion a second parameter, the correlation time τ_0 or the bandwidth of the spectrum. The exact representation of $\psi_{xx}(\tau)$ is not always needed, this being the basis of the error-theory in stochastic measurements (Lit.3,4). This is especially valid for coloured noise, i.e. for fluctuation processes with inconstant spectral power density.

We can calculate the power density from the AKF by a Fourier-transform

$$S(f) = \int_{-\infty}^{+\infty} \psi(\tau)\, e^{-j\omega\tau}\, d\tau = \mathcal{F}\{\psi(\tau)\}$$

In fig.2 we see that only a process with extremely small correlation duration will have a constant power density.

In the following considerations we take into account only fluctuating prccesses with limited bandwidth and with constant spectral power density within this range.

Now we come to the determination of statistical parameters by measuring average values

$$\overline{z(t)} = M\{z(t)\} = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{+\infty} z(t)\, dt$$

Hereinz(t) = f $\{x(t)\}$ is a function of the fluctuation process

x(t). We have to distinguish different cases:

$z(t) = x(t)$	determination of linear mean value
$z(t) = x^2(t)$	" " square " "
$z(t) = x(t) \cdot y(t)$	" " product " "
$z(t) = x(t) \cdot x(t-\tau)$	" " autocorrelation function
$z(t) = x(t) \cdot y(t-\tau)$	" " crosscorrelation "

Using an analogue computer we get the block diagram fig.3, connecting multiplying and integrating devices.

The mean time values may be examined in the timedomain or as well in the frequency domain. The mentioned mathematical and computing model is useful for almost any problem concerning stochastic processes. The input signal x(t) may be the sum of a determined signal s(t) or a stochastic signal n(t) combined with stochastic noise r(t):

1) $x(t) = n(t)$ stochastic signal
2) $x(t) = s(t) + r(t)$ determined signal with stochastic interference
3) $x(t) = n(t) + r(t)$ stochastic signal with stochastic interference.

In order to analyse a signal with interference it is useful to have the undisturbed signal y(t) as second input at the analogue computer.

We assume the dc-components of the input signals to be zero:

$$\overline{x(t)} = 0 \qquad \text{or} \qquad \varphi_{xx}(\infty) = \overline{x(t)}^2 = 0$$

At the output of the analogue computer we nevertheless always get a dc-component, depending on the input signal z(t):

$$\overline{z(t)} = \overline{x^2(t)} = \sigma^2 \qquad \text{or} \qquad \overline{z(t)} = \varphi_{xx}(\tau)$$

This dc-component is to be evaluated. The ac-component of z(t), i.e. $z(t) - \overline{z(t)}$ is the error, the object of our theory. The better we suppress this component by a low-pass filter (integration network), the better will be the accuracy.

The ac-component of the output-signal z(t) can be determined by the spectral power density $S_{zz}(f)$ or the AKF $\varphi_{zz}(\tau)$. The total ac power we find by integrating

$$\int_{-\infty}^{+\infty} S_{zz}(f)\,df = \sigma_{zz}^2 = \psi_{zz}(0)$$

and call this variance σ^2 of z(t), a 4. order momentum. The output signal of the analogue computer being a momentum of the 2. order with respect to the input signal x(t).

As mentioned, we have to suppress the ac-component of z(t) in order to get a small error. In integrating measuring systems this is done by low-pass filter, the cutoff-frequency being as low as possible. At the output of this filter we have the residual noise, determining the final accuracy.
The total power of the residual noise can be calculated from

$$\sigma_{RR}^2 = \int_{-f_{gr}}^{+f_{gr}} S_{zz}(f)\,df$$

The power density distribution $S_{zz}(f)$ of the fluctuating process at the analogue multiplier's output is given as convolution of the input signals spectral power densities.

Operating in the frequency domain we have a short and clearly to understand error calculation. Working with technical systems, we have to pay attention to a restriction: a real low-pass filter has a finite attenuation factor in the pass band as well as in the stop band. We have to take into account the transfer function H(f) and get

$$\sigma_{RR}^2 = \int_{-\infty}^{+\infty} |H(f)|^2 S_{zz}(f)\,df$$

In fig.3.1 we see the characteristics of some different, idealized low-pass filters. Sometimes we use an ideal integration filter with an exactly limited storage time (fig.3.1,b). The transfer function, gained by Fourier transform, will then be the split function. This type of filter is preferred when operating in the time domain and here we use the correlation functions instead of the spectral power densities.

Again we have to distinguish two steps:

a) calculating the ac-components at the multiplier's output
b) calculating the ac-components behind the integrator.

In fig.4 we have a block-diagram for making clear the steps of calculation. We repeat: With the input signals x(t) and y(t) we have behind the delayline at the input of the analogue computer x(t) and y(t- τ). At the multiplier's output we get $z(t, \tau) = x(t) \cdot y(t-\tau)$. The dc-component of this expression being the crosscorrelation function (KKF) , which is disturbed by an ac-component

$$z(t) - \overline{z(t)} = z(t,\tau) - \psi_{xy}(\tau) = \tilde{z}(t,\tau)$$

where the variable τ is the correlation spacing between x(t) and y(t) to be examined.

As you see, the ac-component is no longer determined by a power density $S_{zz}(f)$, but by its Fourier transform

$$\mathcal{F}\{S_{zz}(f)\} \cdot \psi_{\tilde{z}\tilde{z}}(\nu,\tau)$$

which is the AKF of z(t). In order to do this we need another time delay . In fig.4 you see how this AKF could be measured by a second correlator.

The AKF of the ac-component $\tilde{z}(t)$ is a 4. order momentum, the expectation value of four combined fluctuation parameters:

$$\psi_{zz}(\nu,\tau) = E(X_1, X_2, Y_1, Y_2) = M_4$$

In order to calculate it, we have to take the four-dimensional compound probability

$$M_4 = \iiiint X_1 X_2 Y_1 Y_2 \cdot \rho(X_1, X_2 Y_1 Y_2)\, dX_1 dX_2 dY_1 dY_2$$

All parameters having a Gaussian distribution we get the AKF ψ_{zz} at the multiplier's output

$$\psi_{zz}(\nu,\tau) = \psi_{xy}^2(\tau) + \psi_{xy}(\nu+\tau)\,\psi_{yx}(\tau-\nu) + \psi_{xx}(\nu)\cdot\psi_{yy}(\nu)$$

The part $\psi_{xy}^2(\tau) = \bar{z}^2 = \psi_{zz}(\infty)$ is the wanted measurement signal, while the other two parts give the resulting error $\tilde{\psi}_{zz}(\nu,\tau)$. (Lit.5,6,7,8).

For not correlated input signals or signals with a large correlation spacing we get the ac-component at the multiplier's output: $\tilde{\psi}_{zz}(\nu,\tau) = \psi_{xx}(\nu) \cdot \psi_{yy}(\nu)$

the Fourier transform of which is the above mentioned spectral power density.

The AKF at the integrators output ψ_{RR} is to be calculated from the input ψ_{zz} by convolution, as the integrator is a linear system:

$$\psi_{RR}(\tau) = \int_{-\infty}^{+\infty} \psi_{hh}(\vartheta)\, \psi_{zz}(\tau - \vartheta)\, d\vartheta$$

$\psi_{hh}(\vartheta)$ is the autocorrelation of the weighting function h(t) of the integrator.

As approximation we use the ideal integrator with exactly limited integration time:

$$h(t) = \begin{cases} \frac{1}{T} = \text{const} & 0 < t < T \\ 0 & \text{outside this interval} \end{cases}$$

The transfer function is then

$$H(j\omega) = si\left(\frac{\omega T}{2}\right)$$

and the power transfer function

$$|H(j\omega)|^2 = si^2\left(\frac{\omega T}{2}\right)$$

From this we get the above mentioned AKF of the weighting functior by Fourier transform:

$$\psi_{hh}(\vartheta) = \frac{1}{T^2}(T - \vartheta) = \frac{1}{T}\left(1 - \frac{|\vartheta|}{T}\right) \qquad \text{within } -T \leq \vartheta < +T$$

Setting = 0 (i.e. without second delay) we get the total ac-power behind the integrator

$$\sigma_{RR}^2 = \psi_{RR}(0) = \int_{-\infty}^{+\infty} \psi_{hh}(\vartheta) \cdot \tilde{\psi}_{zz}(\vartheta, \tau)\, d\vartheta$$

Substituting the AKF of the weighting function we get the well known formula of Davenport (Lit.9):

$$\sigma_{RR}^2 = \frac{1}{T}\int_{-T}^{+T} \left(1 - \frac{\vartheta}{T}\right) \tilde{\psi}_{zz}(\vartheta, \tau)\, d\vartheta$$

This expression is not exactly valid when using a real low-pass RC-filter, as these have no exactly limited pass band, but the additional error may be corrected.
In practical applications often some presumptions are not met. So additional errors occur.

If for example the fluctuation process is not stationary, which is necessary for measuring statistical parameters, we may store the quasi stationary signals during a short time interval and repeat them periodically for evaluation. If the correlation function within this interval can be determined analytically, for example $\varphi(\tau) = \sigma^2 e^{-\beta(\tau)}$, we can calculate conditions for an arbitrary small mean square error.

Until now we used in limited intervals continuous signals, but in practical applications sometimes only samples of the process are given in such an interval, which causes a loss of information. So the error at the output will increase. The additional error from sampling has been calculated by Schweizer (Lit.10) and results in +4% for example when sampling in steps of 1/10 the time constant of the integrator.

The considerations in evaluating the correlation function can be transferred without difficulties to the evaluation of the spectral power density distribution. At the second input of the multiplier we only connect a band-pass instead of a delay line.

Another application is the examination of transfer-functions with sinusoidal test signals and stochastic interference. In the time domain we need the measuring setup of fig.5.
If the integrator is acting only during n periods of the test signal, we get an error in the correlation function. Beyond that we find an error in the output signal caused by the stochastic interference:

$$\varphi_{xy}(\tau) = \varphi_{xy}(\tau)^{*} + \frac{1}{nT_s} \int_0^{nT_s} r(t) \cdot x(t+\tau)\, dt$$

An algorithm for this error usable for engineers is given in Lit.3.

In the third application we use stochastic test signals and consider the effects of stochastic interference. Good results can be obtained by correlation if we have the undisturbed test signal besides the examined signal. If that is not possible we have to evaluate the autocorrelation. The separation of the various effects depends on the difference of the statistical parameters of both processes. A common theory of errors for this case has

not yet been found. For some special cases we nevertheless can fix conditions to be met for a given mean square error.

This lecture could only be a short review of problems and solutions in measuring stochastic parameters. We see that the correct application of the signal and system theory is necessary when determining stochastic parameters and errors. So sometimes the accustomed knowledge of engineers is surpassed and difficulties arise in the application of this technique.

Let me give a final remark. We get increasing accuracy with longer integration time (measuring time). This is a basic conclusion of information theory, noise can be reduced by longer observation time. This conclusion is often applied in control engineering and radio astronomy.

If we additionally need a good time resolution, for example in puls reflection systems this conclusion cannot be applied. This contradiction between long duration of signals and time resolution is solved by using puls-compression-techniques. Pulses with internal structure (coded pulses) are used and the receiver contains a matched filter. A matched filter is a linear system with a transfer function that is conjugate complex. with respect to the signal spectrum. The output signal of the matched filter equals the AKF of the signal. So this is an autocorrelator working in the time domain.

Coded signals are chosen which have only one short and high peak as AKF. This requires a larger pass-band for the signal spectrum. Coding is made by frequency modulation or by phase-shift-keying (pseudo random) of the carrier.

These methods will have an impact on measurement as could be seen at the last IMEKO - Congress in 1967 in this town (Warsaw). So new developments are to be expected. The main task is to transfer information without errors. The problems can be treated by the above examined case 3: Stochastic signals with stochastic interference. The matched filter improves the signal-noise ratio, but is useless for parameter determination. Error problems in this field are treated by the decision theory, another part of stochastic evaluation.

The integrating correlator in comparison with it is the universal tool for determination of stochastic parameters. The errors may be computed using the spectral power density distribution or the correlation functions.

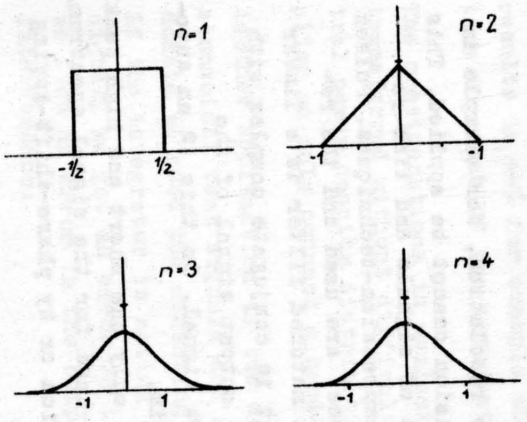

n=1
-1/2
1/2
n=2
-1
1
n=3
-1
1
n=4
-1
1

Bild 1

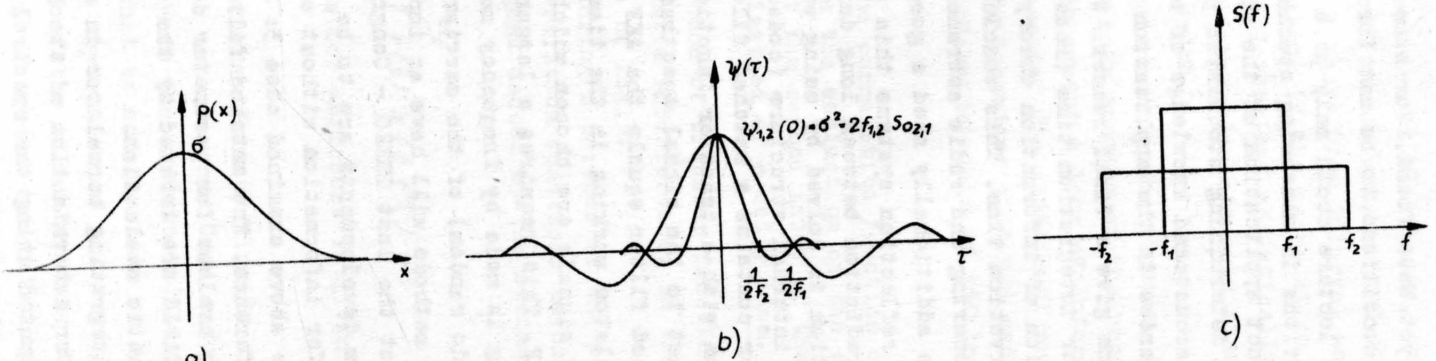
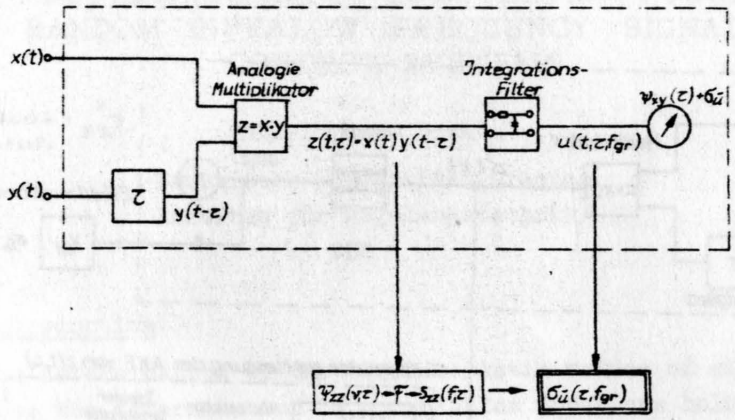

p(x)
б
x
a)
ψ(τ)
ψ1,2 (0) = б² = 2f1,2 So2,1
τ
1/2f2
1/2f1
b)
S(f)
-f2
-f1
f1
f2
f
c)

Bild 2



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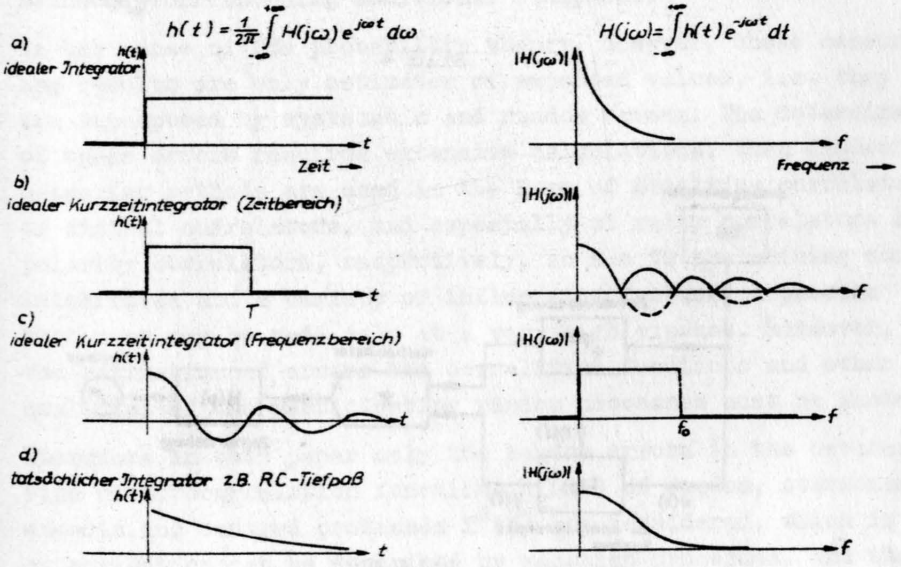

Bild 3.1

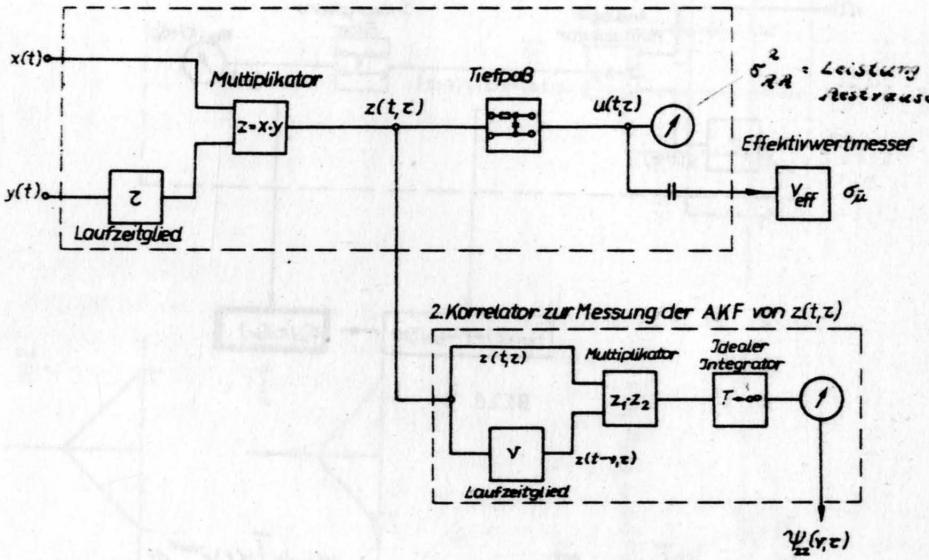
x(t)
y(t)
Multiplikator
z=x·y
τ
Laufzeitglied
z(t,τ)
Tiefpaß
u(t,τ)
Effektivwertmesser
V_eff
σ_u
2. Korrelator zur Messung der AKF von z(t,τ)
z(t,τ)
Multiplikator
Z1·Z2
Idealer Integrator
T→∞
V
z(t-ν,τ)
Laufzeitglied
Ψ_zz(ν,τ)

Bild 4

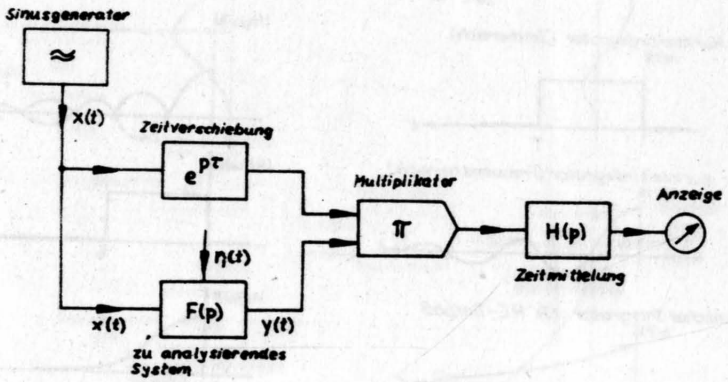
Sinusgenerator
x(t)
Zeitverschiebung
e^pτ
Multiplikator
Anzeige
H(p)
Zeitmittelung
η(t)
F(p)
x(t)
y(t)
zu analysierendes System

Bild 5

METHOD OF ESTIMATION OF RANDOM ERRORS IN THE DETERMINATION OF CORRELATION FUNCTIONS OF RANDOM INFRALOW FREQUENCY SIGNALS IN CONTROL SYSTEMS

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Institut für Regelungstechnik
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1. Introduction

In the determination of the characteristic values of signals and systems the determination of correlation functions holds a central position within the scope of the correlation theory of random processes. For their experimental determination nowadays are available special computers, so-called correlators, and digital computers, which have been adapted to their task by means of corresponding additional equipment.

In the sense of the probability theory, however, these measuring results are only estimates of expected values, i.e. they are superposed by systematic and random errors. The determination of these errors requires extensive calculations. When discrete measuring methods are used in the form of Stieltjes correlators or digital correlators, and especially of relay correlators or polarity correlators, respectively, so due to the arising nonlniearities and a variety of influencing factors, a precise statement may be made only at a very high expense. Moreover, for the estimation of errors the correlation functions and other qualities of the participating random processes must be known.

Therefore in this paper only the random errors in the determination of autocorrelation functions $R_x(\tau)$ of random, stationary, ergodic, and centred processes $X^o(t)$ are considered, which in approximation can be described by gaussian processes, and the power density spectrum $S_x(\omega)$ of which exists up to a radian frequency of $\omega = 0$. A method of their estimation is proposed. This method allows to indicate easily to be handled relations

for the determination of random errors for continuously and discontinuously operating analogous and discrete correlators for the time-shifts $\tau = 0$ and $\tau \to \infty$ or $\tau > \tau_K$, respectively, where τ_K is the correlation time.

2. A Method of Estimation of Random Errors in the Measurement of the Autocorrelation Function $R_x(\tau)$ of Random Infralow Frequency signals

Fig. 1 represents the general scheme of a continuously operating correlator. Since

$$w(T) = \frac{1}{T}\int_0^T z(t)\,dt = m_{zT} \quad . \tag{1}$$

and

$$M\left[M_{zT}\right] = M[Z] = m_z$$

the static relation between the expected values of the measured functions $R_y(\tau)$ and the correlation function to be measured $R_x(\tau)$, is represented by

$$R_y(\tau) = M\left[Q(X_1)Q(X_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(x_1)Q(x_2)\; w_{X_1X_2}(x_1,x_2;\varrho_x)dx_1dx_2. \tag{2}$$

Here $Q(x)$ describes the characteristics of symmetrical, coarsely quantizing equipment, $w_{X_1X_2}(\ldots)$ is the two-dimensional probability density function of the process $X(t)$, and T is the observation time. The standard deviation σ_{mzT} at the output of the short-time integration circuit according to [1] is defined as

$$\sigma^2_{mzT} = \frac{2}{T}\int_0^T (1 - \frac{\nu}{T})R_z(\nu,\tau)d\nu \tag{3}$$

with

$$R_z(\nu,\tau) = M\left[Q(X_1)Q(X_2)Q(X_3)Q(X_4)\right] - R_y^2(\tau) \tag{4}$$

and can serve as a measure of error for the quality of estimation m_{zT}. As a relative measure of error for the potential deviations of the estimation m_{zT} from the mean value m_z the following

relation is used

$$\delta = \varkappa \frac{\sigma_{mzT}}{m_z} \tag{5}$$

where $\varkappa$ is a coefficient indicating the confidence limit of the error δ. If the random variable M_{zT} is assumed to have a finite standard deviation σ_{mzT}, $\varkappa$ may be determined from the Tschebyscheff inequation [2]. Provided that to the distribution of M_{zT} applies the central-limit theorem, the demand of the Tschebyscheff inequation can be reduced and $\varkappa$ can be determined from a Gaussian distribution of M_{zT}.

From eqs. (3), (4) and (5) can be seen that the relative error δ with a given confidence limit $\varkappa$ is a function of the observation time T and the correlation function $R_z(\nu,\tau)$. The function $R_z(\nu,\tau)$ varies for the special measuring devices and generally results from a nonlinear transformation of the correlation function $R_x(\tau)$ of the signals to be analyzed. Hence δ in the first place depends on the structure of the power density spectrum $S_x(\omega)$ of the signals x(t). If discontinuous methods of measurement are used, the error δ additionally is a function of the sampling time Δt, since in this case the equipment of fig. 1 additionally contains a sampling device and a holding circuit corresponding to fig. 2.

For the investigation into random infralow frequency signals arising from industrial processes and having approximately a gaussian distribution, it may be desirable to estimate the expected measuring error <u>without</u> taking into account the complete course of the correlation function or of the power density spectrum, respectively, of these signals. For that reason in the proposed method the spectrum $S_x(\omega)$ is approximated by an equivalent spectrum.

2.1 <u>Approximation of the power density spectrum by equivalent spectra</u>

Provided that the power density spectra of infralow frequency signals exist up to $\omega = 0$, there exists a value $S_x(0) = S_{xo} \neq 0$. Hence the assumption of the equivalent spectra e.g.

$$S_x(\omega)_0 = \begin{cases} S_0 & \text{für } |\omega| \leqq \omega_0 \\ 0 & \text{für } |\omega| > \omega_0 \end{cases} \tag{6}$$

or

$$S_x(\omega)_1 = S_0 \, e^{-\frac{\pi \omega^2}{4 \omega_0^2}} \tag{7}$$

respectively, is justified [3,4], where ω_0 is the so-called effective noise bandwidth [5]. In order not to overestimate infralow frequency spectral fractions in spectra $S_x(\omega)$ with potentially arising resonance qualities, it is here proposed to determine ω_0 from

$$\omega_0 = \frac{1}{2} (\omega_{01} + \omega_{02}) \tag{8}$$

with

$$\frac{1}{2\pi} \int_{-\omega_{01}}^{\omega_{01}} S_x(0) \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) \, d\omega \tag{8a}$$

and

$$\frac{1}{2\pi} \int_{-\omega_{02}}^{\omega_{02}} \max_{0 \leqq \omega < \infty} \left\{ S_x(\omega) \right\} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) \, d\omega \tag{8b}$$

As can be seen from eqs. (6) and (7) the equivalent spectra $S_x(\omega)_0$ and $S_x(\omega)_1$, represented for error estimation, include only one parameter, namely the effective noise bandwidth ω_0. When ω_0 is known, on the base of these spectra for the special measuring methods approximate relations for the arising random error δ can be represented by

$$\delta \approx \text{æf} \left[\omega_0, T, \Delta t, Q(x) \right] \tag{9}$$

for $\tau = 0$ and $\tau \to \infty$ or $\tau > \tau_K$, respectively.

2.2 Experimental determination of the effective noise bandwidth ω_0

In [3] in contrast with [6] the proposition has been made to determine approximately the bandwidth ω_0 experimentally from

the zero crossings $\bar{n}_o$ of the random approximately gaussian signal x(t) through the axis $x = m_x$ by means of the relation indicated by[7]

$$\bar{n}_o = \frac{1}{\pi} \cdot \left[- \varrho_x''(0) \right]^{1/2} \tag{10}$$

Inserting the normalized autocorrelation functions $\varrho_x(\tau)_0$ and $\varrho_x(\tau)_1$, which correspond to eqs. (6) and (7), into eq. (10), we have

$$(o)\,\omega_o = \pi\sqrt{3} \cdot \bar{n}_o \tag{10a}$$

and

$$(1)\,\omega_o = \pi\sqrt{\frac{\pi}{2}} \cdot \bar{n}_o \quad . \tag{10b}$$

On certain conditions hence recorded noise signals can be evaluated with respect to the zero crossings $\bar{n}_o$, and with the help of eqs. (10a) and (10b) the effective noise bandwidths ω_o can be indicated in approximation. The relations are valid for the connection between ω_o and $\bar{n}_o$ of the equivalent spectra. Here, however, the value $\bar{n}_o$ of the actually present signal is inserted, so that the bandwidths, determined from eqs.(10a) or (10b), respectively, necessarily differ from those, which are found from eq. (8). Taking into consideration, however, that for the indicated class of spectra (section 2.1) the approximation relationships according to eq. (9) in general represent superior bounds and taking into account that $\delta \sim (\omega)_o^{-1/2}$ is valid and that $(o)\,\omega_o$ as well as $(1)\,\omega_o$ are always greater than ω_o according to eq. (8), with the exception of comparatively high resonance elevations, the use of eq. (10a) and (10b) in connection with eq.(9) leads to quite useful estimates of the error δ .

Regarding the approximate determination of the bandwidth ω_o from the zero crossings $\bar{n}_o$, it should be noted that $\bar{n}_o$ is calculated from the number ν_{oT} of passages during an observation time T. Hence $(\nu_{oT})/T$ is an estimation of $\bar{n}_o$. In order to hold given error limits for $\bar{n}_o$ with a certain confidence level, T or ν_{oT}, respectively, should be chosen correspondingly.

3. Error Estimation in Measurements of the Correlation Function $R_x(\tau)$

3.1 Error estimation in discontinuous measurements

In the discontinuous measurement of the correlation function one must be able to conclude with sufficient accuracy from the values $R_x(\lambda \Delta t), \lambda = 0; \pm 1; \pm 2; \ldots$ on the course of the function $R_x(\tau)$ for $\tau \neq \lambda \Delta t$. Therefore, the sampling time Δt, independently of the available observation time T, cannot be chosen any desired value. Provided that for a linear interpolation of the values $R_x(\lambda \Delta t)$ approximately 10 ... 15 measuring points are sufficient, for the permissible sampling time Δt applies the following relation, where the noise bandwidth ω_o is introduced

$$\Delta t < 0{,}3\,\pi/\omega_o \quad . \tag{11}$$

For such sampling times can be shown, however, that this implies a good coincidence of the spectrum $S_x^{(H)}(\omega)$ of the sampled and held signal $x_H(t)$ with the spectrum of the signal x(t) (see fig. 2). This means, however, that hence a separate error estimation in the discontinuous measurement of the correlation function $R_x(\tau)$ is not necessary.

Introducing into eq. (11) instead of the bandwidth ω_o the bandwidth $^{(o)}\omega_o$, determined from the zero crossings $\bar{n}_o$ by means of eq. (10a), experimentally from

$$\Delta t < 0{,}17/\bar{n}_o \tag{11a}$$

can be approximately determined the minimum required sampling time Δt.

3.2 Error estimation in continuous analogous measurements

When the observation time T is much greater than the correlation time τ_K, i.e. when $T \gg \tau_K$, for the random errors, referred to $R_x(0)$ it follows $\delta_{R0} = \delta_R\big|_{\tau = 0}$ and $\delta_{R\infty} = \delta_R\big|_{\tau \to \infty \text{ or } \tau > \tau_K}$, respectively, according to eq. (5), since the characteristics in fig. 1 are to be equated to unity Q(x) = 1, with eqs.(3) and

$$R_z(\nu,\tau) = R_x^2(\nu) + R_x(\tau+\nu)\, R_x(\tau-\nu) \tag{12}$$

for the considered class of random processes

$$\delta^2_{R0} \approx \varkappa^2 \cdot \frac{4}{T} \int_0^\infty \varrho_x^2(\nu)\, d\nu \tag{13}$$

and

$$\delta^2_{R\infty} \approx \varkappa^2 \cdot \frac{2}{T} \int_0^\infty \varrho_x^2(\nu)\, d\nu \tag{13a}$$

where $\varrho_x(\nu) = R_x(\nu)/R_x(0)$ is the normalized autocorrelation function of the signal $x(t)$.

Using for the approximate error estimation the equivalent spectrum $S_x(\omega)_0$ according to eq. (6), eqs. (13) and (13a) become

$$^{(c)}\delta^2_{R0} \approx \varkappa^2 \cdot \frac{2\pi}{\omega_0 T} \tag{14}$$

and

$$^{(o)}\delta^2_{R\infty} \approx \varkappa^2 \cdot \frac{\pi}{\omega_0 T} \tag{14a}$$

Introducing into eqs. (14) and (14a) instead of the bandwidth ω_0 the bandwidths $^{(o)}\omega_0$ or $^{(1)}\omega_0$, respectively, according to eqs. (10a) or (10b), determined from the zero crossings $\bar{n}_0$, the random error for $\tau = 0$ e.g. can be obtained for aperiodic spectra $S_x(\omega)$ from

$$\delta^2_{R0} \approx \varkappa^2 \cdot \frac{2\pi}{^{(o)}\omega_0 T} \tag{14b}$$

and for spectra with resonance qualities from

$$\delta^2_{R0} \approx \varkappa^2 \cdot \frac{2\pi}{^{(1)}\omega_0 T} \tag{14c}$$

in good approximation.

Example 1:

For the aperiodic spectrum

$$S_x(\omega) = \frac{S_o}{(1+\alpha_1^2\omega^2)(1+\alpha_2^2\omega^2)(1+\alpha_3^2\omega^2)}$$

with α_1 = 11,36 ms, α_2 = 4 ms and α_3 = 2,86 ms we receive after some algebra from eq. (13)

$$\delta_{RO}^2 \approx æ^2 \cdot 47{,}8 \cdot 10^{-3}\ s/T \quad .$$

The simulation of this spectrum on the analog computer with successive experimental evaluation of the zero crossings yielded $\bar{n}_o \approx 26{,}3\ s^{-1}$, and hence, using eqs. (10a) and (14b) as an approximation

$$\delta_{RO}^2 \approx æ^2 \cdot 43{,}9 \cdot 10^{-3}\ s/T \quad .$$

Example 2:

For the spectrum with resonance elevation

$$S_x(\omega) = S_o \frac{\frac{\omega^2}{\beta^2+\gamma^2} + 1}{\left[\frac{\omega^4}{(\beta^2+\gamma^2)^2} + 2\frac{\gamma^2-\beta^2}{(\beta^2+\gamma^2)^2}\omega^2 + 1\right](1+\alpha_2^2\omega^2)(1+\alpha_3^2\omega^2)}$$

with α_2 = 4 ms, α_3 = 2,86 ms, $\beta = 45{,}5 s^{-1}$ (36 s^{-1}) and $\gamma = 57\ s^{-1}$ (18 s^{-1}) from eq. (13) follows after some algebra

$$\delta_{RO}^2 \approx æ^2 \cdot 61{,}7 \cdot 10^{-3}\ s/T \quad \text{bzw.} \quad æ^2 \cdot 85{,}2 \cdot 10^{-3}\ s/T$$

The experimental evaluation of the zero crossings had as a result $\bar{n}_o \approx 24{,}3\ s^{-1}$ (16,9 s^{-1}), and hence from eqs. (10b) and (14c) as an approximation

$$\delta_{RO}^2 \approx æ^2 \cdot 65{,}5 \cdot 10^{-3}\ s/T \quad \text{bzw.} \quad æ^2 \cdot 93{,}8 \cdot 10^{-3}\ s/T$$

The validity of the proposed relations has been checked for the considered cases by experimentally determining the correlation functions with the help of the ISAC correlator and the digital computer ZRA-1.

3.3 Error Estimation in Continuous Discrete Measurements

3.3.1 Error estimation by using Stieltjes correlators

For a Stieltjes correlator the relation between the expected values of the measured function $R_{xy}(\tau)$ and of the correlation function $R_x(\tau)$ to be measured according to eq. (2) can be written

$$R_{xy}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 Q(x_2)\, w_{X_1X_2}(x_1,x_2;\varrho_x)\, dx_1\, dx_2 \,. \tag{15}$$

On the assumptions made we have

$$w_{X_1X_2}(x_1,x_2;\varrho_x) = \frac{1}{2\pi\sigma_x^2\sqrt{1-\varrho_x^2(\tau)}}\, e^{-\frac{x_1^2-2\varrho_x(\tau)x_1x_2+x_2^2}{2\sigma_x^2\left[1-\varrho_x^2(\tau)\right]}}$$

$$= \frac{1}{2\pi\sigma_x^2}\, e^{-\frac{x_1^2}{2\sigma_x^2}}\, e^{-\frac{x_2^2}{2\sigma_x^2}} \cdot \sum_{n=0}^{\infty} \frac{1}{n!}\, H_n\left(\frac{x_1}{\sigma_x}\right) H_n\left(\frac{x_2}{\sigma_x}\right) \varrho_x^n(\tau) \,. \tag{16}$$

$H_n(\lambda)$ define the Hermitian polynomials, e.g. [8].

From eqs. (15) and (16) it follows

$$R_{xy}(\tau) = K_1 \cdot R_x(\tau) \,, \tag{17}$$

where

$$K_1 = \frac{1}{\sigma_x^3\sqrt{2\pi}} \int_{-\infty}^{\infty} xQ(x)\, e^{-\frac{x^2}{2\sigma_x^2}}\, dx \tag{18}$$

is the coefficient of the equivalent statistical linearisation according to Booton [9].

The static error of the correlation function $R_x(\tau)$ expressed by

$$\frac{\Delta R_x(\tau)}{R_x(\tau)} = \frac{R_{xy}(\tau)-R_x(\tau)}{R_x(\tau)} = K_1 - 1 \tag{19}$$

can become zero for each value of the time shift τ, when an unequal level symmetrical quantizer is introduced. In [4] has been shown that for an optimal equal level quantizer with an even number of levels the error always becomes zero, while for an odd number an error remains, which, however, for 5 levels already is neglectable.

Since

$$R_z(\nu,\tau) = M\left[X_1Q(X_2)X_3Q(X_4)\right] - R_{xy}^2(\tau) \tag{20}$$

for the absolute random errors δ_{RSO} and $\delta_{RS\infty}$ of the Stieltjes correlator, which are related to $R_{xy}(0)$, at $\tau = 0$ or $\tau > \tau_K$, respectively, it follows from eqs. (5) and (3), on the assumption that the central-limit theorem is valid,

$$\delta^2_{RSO} \approx \varkappa^2 \cdot \frac{2}{TR^2_{xy}(0)} \int_0^\infty \left\{M\left[X_1Q(X_1)X_2Q(X_2)\right] - R^2_{xy}(0)\right\} d\nu \tag{21}$$

and

$$\delta^2_{RS\infty} \approx \varkappa^2 \cdot \frac{2}{TR^2_{xy}(0)} \int_0^\infty R_x(\nu)R_y(\nu)\, d\nu \ . \tag{21a}$$

Introducing into eq. (21) an equivalent characteristic

$$z_o = x\,Q(x) \tag{21b}$$

and using instead of the autocorrelation functions $R_z(\nu)_o$ and $R_y(\nu)$ in eq. (21a) expansions in a series based on eq. (16), for correlation functions $R_x(\tau)$ and $-1 < \varrho_x(\tau) \leqq 1$ of any desired shape apply the estimations

$$\frac{a_2}{\sqrt{2}\,K_1\,\sigma_x^2} < \left|\frac{\delta_{RSO}}{\delta_{RO}}\right| < \frac{\sigma_{zo}}{\sqrt{2}\,K_1\,\sigma_x^2} \tag{22}$$

and

$$1 < \left|\frac{\delta_{RS\infty}}{\delta_{R\infty}}\right| < \frac{\sigma_y}{K_1\,\sigma_x} \tag{22a}$$

In eqs. (22) and (22a) δ_{RO} and $\delta_{R\infty}$ represent the errors of the analogous correlator according to eqs. (13) and (13a), for K_1 eq. (18) is valid, and for a_2, σz_o and σ_y must be written

$$a_2 = \frac{1}{2\sigma_x\sqrt{\pi}} \int_{-\infty}^{\infty} x\,Q(x) \left[\left(\frac{x}{\sigma_x}\right)^2 - 1\right] e^{-\frac{x^2}{2\sigma_x^2}}\,dx\,, \tag{22b}$$

$$\sigma_{zo}^2 = \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 Q^2(x)\, e^{-\frac{x^2}{2\sigma_x^2}}\,dx - \left[\frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} xQ(x)\, e^{-\frac{x^2}{2\sigma_x^2}}\,dx\right]^2 \tag{22c}$$

and

$$\sigma_y^2 = \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} Q^2(x)\, e^{-\frac{x^2}{2\sigma_x^2}}\,dx \tag{22d}$$

According to the here proposed eqs. (22) and (22a) the expected random errors δ_{RSO} and $\delta_{RS\infty}$ of the Stieltjes correlator for a finite observation time T can be estimated in an uncomplicated manner, when the corresponding errors δ_{RO} and $\delta_{R\infty}$ of the analogous correlator (see section 3.2) as well as the standard deviation σ_x and the structure of the symmetrical quantizer are known. Here it is appropriate to choose for the relations δ_{RS}/δ_R the correspondent arithmetic mean values of the superior and the inferior bounds.

For the special case of the relay correlator, independently of the level of the characteristic Q(x), one obtains the estimations

$$0{,}5 < \left| \frac{\delta_{RRO}}{\delta_{RO}} \right| < 0{,}534 \qquad (23)$$

and

$$1 < \left| \frac{\delta_{RR\infty}}{\delta_{R\infty}} \right| < 1{,}253 \qquad (23a)$$

3.3.2 Error estimation when digital correlators are used

In place of eq. (15) to the description of the static relation for a digital correlator with coarse quantization (cf. fig.1) applies eq.(2). Only for the special case of the polarity correlator this dependency can be indicated analytically. Corresponding to eq. (19) the static error of the correlation function $R_x(\tau)$ is defined by

$$\frac{\Delta R_x(\tau)}{R_x(\tau)} = \frac{R_y(\tau) - R_x(\tau)}{R_x(\tau)} \qquad (24)$$

Contrary to the Stieltjes correlator with the digital correlator this error, due to the relation in eq. (2), can become zero only for a certain value of τ_0 of the time shift τ; hence an optimal dimensioning of the quantizer depends on τ. When $\tau_0 = 0$ is chosen, eq. (24) becomes

$$\frac{\Delta R_x(0)}{R_x(0)} = \frac{\sigma_y^2}{\sigma_x^2} - 1 = K_2^2 - 1 , \qquad (24a)$$

where K_2 represents the coefficient of the equivalent statistical linearisation according to Burt [10] and Kasakow [11].

Since for $R_z(\nu, \tau')$ eq. (4) is valid, according to eqs.(21) and (21a) for the absolute random errors δ_{RDO} and $\delta_{RD\infty}$ of the digital correlator can be written

$$\delta^2_{RDO} \approx \varkappa^2 \cdot \frac{2}{T\, R_y^2(0)} \int_0^\infty \left\{ M\left[q^2(X_1) q^2(X_2) \right] - R_y^2(0) \right\} d\nu \qquad (25)$$

and

$$\delta^2_{RD\infty} \approx \varkappa^2 \cdot \frac{2}{\pi R_y^2(0)} \int_0^\infty R_y^2(\nu)\, d\nu \qquad (25a)$$

Introducing into eq. (25) an equivalent characteristic

$$z_1 = Q^2(x) \qquad (26)$$

one obtains after considerations similar to those in section 3.3.1 the estimations

$$\frac{c_2}{\sqrt{2}\, K_2^2 \sigma_x^2} < \left| \frac{\delta_{RDO}}{\delta_{RO}} \right| < \frac{\sigma_{z1}}{\sqrt{2}\, K_2^2 \sigma_x^2} \qquad (27)$$

and

$$\frac{b_1^2}{K_2^2 \sigma_x^2} < \left| \frac{\delta_{RD\infty}}{\delta_{R\infty}} \right| < 1 \qquad (27a)$$

with

$$c_2 = \frac{1}{2\sigma_x\sqrt{\pi}} \int_{-\infty}^{\infty} Q^2(x) \left[\left(\frac{x}{\sigma_x} \right)^2 - 1 \right] e^{-\frac{x^2}{2\sigma_x^2}}\, dx\,, \qquad (27b)$$

$$\sigma_{z1}^2 = \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} Q^4(x)\, e^{-\frac{x^2}{2\sigma_x^2}}\, dx - \left[\frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} Q^2(x)\, e^{-\frac{x^2}{2\sigma_x^2}}\, dx \right]^2, \qquad (27c)$$

$$K_2^2 = \frac{1}{\sigma_x^3\sqrt{2\pi}} \int_{-\infty}^{\infty} Q^2(x)\, e^{-\frac{x^2}{2\sigma_x^2}}\, dx \qquad (27d)$$

and

$$b_1 = \frac{1}{\sigma_x^2\sqrt{2\pi}} \int_{-\infty}^{\infty} xQ(x)\, e^{-\frac{x^2}{2\sigma_x^2}}\, dx = \sigma_x K_1\,. \qquad (27e)$$

By means of the here proposed eqs. (27) and (27a) the random errors to be expected, δ_{RDO} and $\delta_{RD\infty}$, of the digital correlator can be estimated in an uncomplicated manner. Concerning the assumptions and the handling of these equations apply the remarks in section 3.3.1.

For the special case of the polarity correlator one obtains, independently of the level of the characteristic Q(x), the estimations

$$\frac{\delta_{RPO}}{\delta_{RO}} = 0 \tag{28}$$

and

$$0{,}636 < \left| \frac{\delta_{RP\infty}}{\delta_{R\infty}} \right| < 1 \quad . \tag{28a}$$

Using for the approximate estimation of the error $\delta_{RP\infty}$ the arithmetic mean value of the superior and inferior bounds, we have from eq. (28a)

$$\left| \frac{\delta_{RP\infty}}{\delta_{R\infty}} \right| \approx 0{,}82 \quad . \tag{28b}$$

Eq. (28b) represents a useful approximation. This is demonstrated by a comparison with the accurate calculation, carried out by van den Bos [12] with respect to the correlation function $R_x(\tau) = \sigma_x^2 \cdot e^{-\alpha\tau^2} \cdot \cos\beta\tau$, which has been checked by experiment by Veltman [13].

4. Summary

This paper proposes a method of approximate calculation of random errors in continuous or discontinuous analogous and discrete measurement of autocorrelation functions. The derived relations, which can be easily handled, for the characterization of the spectral properties of the signals to be analyzed contain the effective noise bandwidth. The latter can be determined for certain classes of spectra in good approximation from the zero crossings of recorded noise signals.

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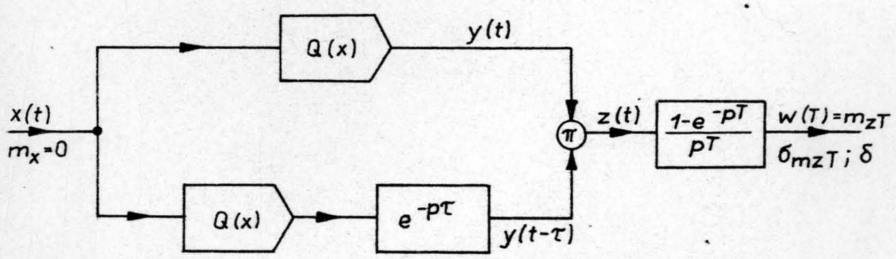

Fig. 1 Scheme of a correlator
for the continuous measurement of $R_x(\tau)$

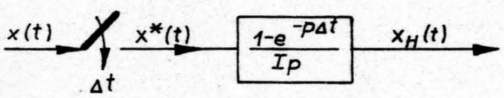

Fig. 2 Sampling and holding circuit
for methods of discontinuous measurement