IFAC

INTERNATIONAL FEDERATION OF AUTOMATIC CONTROL

WARSZAWA 1969

Stability

Fourth Congress of the International Federation of Automatic Control
Warszawa 16–21 June 1969

TECHNICAL SESSION 20

Organized by
Naczelna Organizacja Techniczna w Polsce

INTERNATIONAL FEDERATION OF AUTOMATIC CONTROL

Stability

TECHNICAL SESSION No 20

FOURTH CONGRESS OF THE INTERNATIONAL FEDERATION OF AUTOMATIC CONTROL
WARSZAWA 16 – 21 JUNE 1969

Organized by
Naczelna Organizacja Techniczna w Polsce

Contents

Wydawnictwa Czasopism Technicznych NOT - Polska

Zakład Poligraficzny WCT NOT. Zam. 116/69.

ON THE NUMERICAL CONSTRUCTION OF LIAPUNOV FUNCTIONS

G.Arienti, C.Sutti, G.P.Szegö

Università di Milano

Milano, Italy

Introduction

In this paper we shall present a method for the numerical construction of Liapunov functions [1,2,3] $v=\varphi(x)$, which characterize completely the stability properties of the equilibrium point $x=o$ of the differential equations

1.1 $\dot{x} = f(x), \qquad f(o) = o.$

This method is not based upon the classical stability theory of Liapunov, but on the recent theory of extension developed by N.P.Bhatia, G.P.Szegö and G.Treccani [3,1,16] and in particular on the local extension theorem (2.6).

In this new theory it is not required to analyze the geometrical properties of the surfaces $\varphi(x)$ =const and those of the surface $\varphi(x)$ = const which is tangent to the surface $\psi(x)=o$, where $\psi(x)=\langle \text{grad}\,\varphi(x), f(x)\rangle$, as it is required in the classical theory.

The interest of the method that we propose lies in the fact that it allow to make the best possible stimate of the region of asymptotic stability of the point x=o: A ({o}), by means of nonhomogeneous polynomial forms of order m for system of degree n. In addition it allows to identify the case in which A ({o}) is compact from the case in which A ({o}) is not compact and from the case in which A ({o}) is the whole space (i.e. x = o is globally asymptotically stable). In addition the computation gives as a byproduct two parameters which allow an easy approximate estimate of the geometrical properties of A ({o}).

The problem of the numerical construction of Liapunov functions has been investigated by many authors [4-8]. For instance, Margolis and Vogt [4], following the Zubov's theory [9], have realized a numerical program of generating Liapunov function for the second order system:

$$
1.2 \qquad \begin{aligned} x_1 &= f_1(x_1, x_2) \\ x_2 &= f_2(x_1, x_2) \end{aligned}
$$

where $f_1(x_1,x_2)$ an $f_2(x_1,x_2)$ are polinomial forms, $f_1(o,o)= f_2(o,o) = o$ and the point $x=o$ is locally asymptotically stable.

By the Zubov's theory, in order to estimate the stability region, it is sufficient to integrate the following linear partial differential equation

$$
1.3 \qquad \frac{\partial \varphi}{\partial x_1} f_1(x_1,x_2) + \frac{\partial \varphi}{\partial x_2} f_2(x_1,x_2) = \vartheta(x_1,x_2)\left[\varphi(x_1,x_2)-1\right]
$$

where $\vartheta(x_1,x_2)$ is an arbitrary positive definite quadratic form and f_1, f_2 are defined by the system (1.2). A corresponding procedure can be based upon the more general theory developed by Szegö [5], where insted of equation (1.3) the equation:

$$
\langle \text{grad} \quad \varphi(x), f(x) \rangle = \vartheta(x) \cdot \beta(\varphi(x))
$$

must be solved, where $\vartheta(x)$ is a real positive definite function along the solution of the fixed equation and $\beta(\varphi): R^1 \to R^1$ is a real function for which the integral

$$
\int_0^{\varphi} \frac{d\sigma}{\beta(\sigma)}
$$

exists.

In 1962 Rodden [6] by following the Zubov's theory again, resumes the Margolis's and Vogt's paper, and improved the numerical aspects. He assumes as Liapunov function a polinomial form of m-degree $\varphi_m(x)$. The analysis is developped through three steps:

a) Research of the ϑ^* surface on which $\langle \text{grad}\, \varphi_m(x), f(x)\rangle = 0$ and $\varphi_m(x)$ changes sign.

b) determination of the points of tangency between $v=\varphi_m(x)$ and ϑ^* and search of the corrisponding value $c^*_m = \varphi_m(x^*)$

c) Analysis of the surface $\varphi_m(x) = c^*_m$

These works by Margolis and Vogt and by Rodden have some shortcooming: they all follow the Zubov's classical theory, thus all proposed methods depend upon the choice of the arbitrary function $\vartheta(x)$ and they are significant only in the case n=2 .

The paper by Weissenberg [7] overcomes some of the difficulties of the works by Margolis and Vogt and by Rodden. He studies the asympotic stability region of the point x = o for discontinuous systems in which the Zubov's theory is not available. He is the first to present the problem of construction Liapunov function as the one of the best possible polynomial estimate of the region of attraction. G.Geiss [8] finally impruves the method of Weissenberger in a work on the best estimate of a region of asymptotic stability by means of quadratic forms. This procedure is based upon the maximization over the elements of the positive definite matrix H of the hypervolume Γ^2 defined by the level surface of a quadratic form x'Hx, tangent to the set $\psi(x)=0$. This is a max-min problem i.e. the search of

$$\max_{H>0} \Gamma^2 \left(\min_{\psi(x)=0} x'Hx\right)$$

Notice that $\psi(x) = \langle \text{grad}\,(x'Hx), f(x)\rangle$. In this case the hypervolume Γ^2 can be explicitely related to the coefficients of the matrix H. This method of Geiss is not generalizable to the case of polynomial forms $\Pi_m(x)$ of degree m>2, since in this case the hypervolume Γ^m cannot be explicitely related to the coefficients polynomial form $\Pi_m(x)$ and there do not exist ways of testing if a polynomial form $\Pi_m(x)$ is positive definite are not.

The local extension theorem 2.6 allows us to overcome this diddiculty and under the very reasonable hypothesis that the equilibrium point x=o of the equation (1.1) is (locally) asymptotically, stable or (locally) completely unstable, to identity numerically region in which those stability properties hold.

2. The Extension Theorems.

In this section the theoretical foundations of the extension thery will be given .For semplicity only the case of the stability properties of equilibrium points will be considered.The same results hold for the more general case of invariant sets, with compact neighborhoods.

In the sequel, when not otherwise stated, capital Roman letters will denote matrices of sets, small Roman letters vectors (notable exceptions t=time, k, h, v and w which are scalars), small greek letters scalars.In what follows R^n denotes the euclidean n-space.

If $M \subset R^n$ is a set, we shall denote with $\overline{M}$, $\mathcal{C}M$, ∂M and M its clusure, complement boundary and interior respectively. $S[x,\delta]$, $S(x,\delta)$ and $H(x,\delta)$ will denote, the closed sphere, the open sphere and the spherical hyper surface with center x and radius $\delta > o$.

The extension theorems can be presented in various frameworks: for the flow defined by abstract dynamical systems in suitable spaces, as well as for the case of ordinary differential equations satisfying suitable conditions. In this work we shall present the results for the case of the ordinary differential equation (1.1) where $f : R^n \rightarrow R^n$ is such that conditions for global existence and uniqueness of solutions of the equation 1.1 are satisfied, or in other word equation 1.1 defines a dynamical systems. It must be pointed out that with a suitable, rather heavy, mathematical mashinery [15] the same extension theorems hold for the mode general case in which equation 1.1 has global existence of solutions, but not necessarely uniqueness [16].

2.1 Global extension theorem.

Let $v = \varphi(x)$ and $w = \psi(x)$ be real valued functions defined on the space R^n. Let

i) $v = \varphi(x) \in \mathscr{C}^1$

ii) $\varphi(o) = o$

iii) $\psi(x) = \langle \text{grad } \varphi(x), f(x) \rangle$

iv) for all sequences $\{x^n\} \subset R^n$, if $\psi(x^n) \rightarrow o$, then $x^n \rightarrow o$

v) the differential equation $\dot{x} = f(x)$, $f(o) = o$ defines a dynamical systems.

Then whatever the local stability properties of $x = o$ may be, they are global.

Proof. We shall prove this theorem for the particular case in which $x = o$ is (locally) asymptotically stable. The proof for the other cases is similar. To fix the ideas let then $\psi(x) < o$ for $x \neq o$.

Let $A(\{o\})$ be the region of attraction of the critical point $x = o$. It is well known [2] that $A(\{o\})$ is an open invariant set and such that there exists a real number. $\delta > o$ such that $S[\{o\}, \delta] \subset A(\{o\})$.

We want to prove that $A(\{o\}) = R^n$, or equivalently that $\partial A(\{o\}) = \emptyset$.

Let

$$2.2 \qquad \nu = \min \{\varphi(x) : x \in \partial S[\{o\}, \delta]\}$$

We claim that for all $x \in \partial A(\{o\})$, $\varphi(x) \geq \nu$. In fact, from the hypothesis made on $\varphi(x)$ it follows that $\varphi(x(x^o,t))$ is a strictly decreasing function of t for all $x^o \neq o$. Now, if for some $y \in \partial A(\{o\})$ it were $\varphi(y) < \nu$ it would be possible to find an $x^o \in A(\{o\})$, $x^o \notin S(\{o\}, \delta)$ such that $\varphi(x^o) < \nu$. As $x^o \in A(\{o\}) \setminus S[\{o\}, \delta]$, there is a $\tau > o$ such that $x(x^o,\tau) \in H[\{o\}, \delta]$. Then $\nu \leq \varphi(x(x^o,\tau)) < \varphi(x^o) < \nu$ for $\varphi(x(x^o,t))$ is strictly decreasing. This is absurd.

Since $\varphi(x) \geq \nu$ for $x \in \partial A(\{o\})$, we have by (v) that $\psi(x) < 0$ for $x \in \partial A(\{o\})$.

Let

2.3 $$-\mu = \sup\{\psi(x) : x \in A(\{o\})\}.$$

By (iv) $\mu > 0$ since $\partial A(\{o\})$ is bounded away from $\{o\}$. Let now $x^o \in \partial A(\{o\})$, then $x(x^o,t) \in \partial A(\{o\})$ for $t \geq 0$, since $\partial A(\{o\})$, is invariant. Then

2.4 $$\varphi(x(x^o,t)) = \varphi(x^o) + \int_{t_o}^{t} \psi(x(x^o,\tau))d\tau \leq \varphi(x^o) - \int_{t_o}^{t} \mu d\tau = \varphi(x^o) - \mu(t-t_o)$$

which shows that $\lim_{t \to +\infty} \varphi(x(x^o,t)) = -\infty$, which is absurd, since we have proved that for all $x \in \partial A(\{o\})$, $\varphi(x) \geq \nu$. This contradiction shows that $\partial A(\{o\}) = \emptyset$ and proves the theorem.

2.5 Remark

Condition (iv) of theorem 2.1 is equivalent to saying that the functions $w = \psi(x)$ (positive or negative) definite in the space R^n

2.6 Local extension theorem

Let $\dot{x} = f(x)$ be a dynamical systems. Let $v = \varphi(x)$ and $w = \psi(x)$ be real valued functions defined in R^n such that

i) $\varphi(x) \in \mathscr{C}^1$

ii) $\varphi(o) = o$

iii) $\psi(x) = \langle \text{grad}\, \varphi(x), f(x) \rangle$

iv) $x^c \in R^n \cup \{\infty\}, x \neq o$, such that grad $\varphi(x^c) = o$

v) $N(\mu) = \{ x\ R^n : \varphi(x) < \mu \}$

vi) $N_c(\mu)$ the component of $N(\mu)$ which contains $x = o$

vii) $\beta^c > o$ is a real number such that $x^c \in \partial N_c(\beta^c)$.

viii) $\psi(x) \neq o$ for $x \in N_c(\beta^c) \setminus \{o\}$

ix) for $\{x^n\} \subset \overline{N_c(\beta^c)}$, $\psi(x^n) \to o$ implies $x^n \to \partial N_c(\beta^c)$

x) $x = o$ is (locally) asymptotically stable.

Then

2.7 $\quad N_c(\beta^c) \subset A(\{o\})$

In addition if

xi) $\psi(x) \neq o$ for $x \in \partial N_c(\beta^c)$,

then

2.8 $\quad \overline{N_c(\beta^c)} \subset A(\{o\})$.

Proof. From the hypothesis x) it follows that there exists a real number $\delta > o$, such that for all $x \neq o$ with $\|x\| < \delta$ it is (to fix the ideas)

2.9 $\quad \varphi(x) > o$ and $\psi(x) < o$.

Let now

2.10 $\quad \nu = \min \{ \varphi(x) : \|x\| = \delta \}$,

then there exist a component $N_c(\nu/2)$ of $N(\nu/2)$ which is compact and such that $N_c(\nu/2) \subset S[\{o\}, \delta]$. If $\beta \geq \frac{\nu}{2}$, then $N_c(\beta)$ will contain $N_c(\nu/2)$.

Clearly for $o < \varepsilon < \nu$ the set $N_c(\varepsilon)$ is bounded.

From the hypothesis made it follows that $\overline{N_c(\beta)}$ is a positively invariant set for $\beta < \beta^c$.

We shall now prove that for $\beta < \beta^c$, the set $\overline{N_c(\beta)}$ is compact.

Let

2.11 $\quad -k = \sup \psi(x)$ for $x \in \overline{N_c(\beta)} \setminus N_c(\nu/2)$

From the hypothesis (viii) and (2.9) it follows that

2.12 $\quad k > o$.

Let now $\quad x^o \in \partial N_c(\beta)$. It follows that

2.13 $\varphi(x(x^o,t)) = \varphi(x^o) + \int_0^t \psi(x(x^o,\tau))\, d\tau \leq \varphi(x^o) - \int_0^t k\, d\tau =$
$= \varphi(x^o) - k\,t < \beta^c - kt$.

Let now

$$\tau = (\beta^c - \nu/2)/k.$$

For $t \geq \tau$ all trajectories starting from all points $x \in \partial N_c(\beta)$ belong to $N_c(\nu)$ since $\overline{N_c(\beta)}$ is positively invariant. Suppose now that $\partial N_c(\beta)$ is not compact, and consider a sequence $\{x^n\} \subset \partial N_c(\beta)$; $\|x^n\| \to +\infty$, such that no subsequence of $\{x^n\}$ is convergent. On the other hand let

$$y^n = x(x^n, T) \in \overline{N_c(\nu)}.$$

Since the subsequence $\{y^n\}$ is contained in the compact set $\overline{N_c(\nu)}$ it contains a convergent subsequence, which, for semplicity sake we assume to coincide with $\{y^n\}$. Let now

$$y^n \to y^o \in \overline{N_c(\nu)}.$$

Let

$$y(y^n,t) = x\,(x^n, t + \quad)$$

where

$$y\,(y^n, o) = y^n.$$

Since the given system $\dot{x} = f(x)$ is a dynamical systems, then $y\,(y^n,.) \to y(y^o,.)$ uniformly, on each compact intervall. Then it is, in particular, $y\,(y^n, -\tau) = x^n \longrightarrow y(y^o, -\tau)$ Thus it cannot be $\|x^n\| \to +\infty$. Then for $\beta < \beta^c$ both $\partial N_c(\beta)$ and $\overline{N_c(\beta)}$ are compact.

Let now $x^o \in N_c(\beta^c)$. Let $\varphi(x^o) = \beta$ for $x^o \in \overline{N_c(\beta)}$. Since $\overline{N_c(\beta)}$ is positively invariant and compact, it is $\Lambda^+(x^o) \neq o$. In addition $\Lambda^+(x^o) = \{o\}$, since if $y \in \Lambda^+(x)$, $\psi(y) = o$ [3]. Then $N_c(\beta^c) \subset A\ (\{o\})$.

Assume now that, in addition, condition (xi) is satisfied. We notice that this does not rule out that there exist a sequence $\{y^n\} \subset \partial N_c(\beta^c)$, such that $\psi(y^n) \to o$ as $\|y^n\| \to +\infty$.

We shall prove that in this case $\overline{N_c(\beta^c)} \subset A(\{o\})$, while $\partial A(\{o\}) \cap \partial N_c(\beta^c) = \emptyset$.

Let $\partial A(\{o\}) \cap \partial N_c(\beta^c) \neq 0$ and let $y \in \partial A(\{o\}) \cap \partial N_c(\beta^c)$. Then the positive semitrajectory $\gamma^+(y)$ through y is such that $\gamma^+(y) \subset \partial A(\{o\}) \cap \overline{N_c(\beta^c)}$. Now, from what has been seen before it cannot exist $\tau > o$ such that $y\tau \in N_c(\beta^c)$, thus $\gamma^+(y) \subset \partial A(\{o\}) \cap \overline{N_c(\beta^c)}$.

Now $\varphi(x) = \beta^c$ for $x \in \partial N_c(\beta^c)$, then for all $x \in \gamma^+(y)$ it would be $\varphi(x) \equiv \beta^c$ and hence $\psi(y) \equiv o$ for all $x \in \gamma^+(y) \subset \partial N_c(\beta^c)$, which is against the hypothesis xi).

2.10 Remark

The global extension theorem 2.1 and the local extension theorem 2.6 do not cover the whole spectrum of situations; for instance it is not clear what is the behaviour of the flow and of the level lines of the real-valued function $v = \varphi(x)$, in the case in which the condition (iv) of theorem 2.6 is not satisfied, but there does not exist any point $x^c \in R^n$, $x^c \neq o$, such that grad $\varphi(x^c) = o$. It is clear that this condition alone is not enough to conclude global asymptotic stability.

§ 3 Numerical method for generating Liapunov functions.

The numerical method that it has been developed has the following features:

i) it allows to distingush between the case of global and that of local asymptotic stability

ii) it allows, in the case of local asymptotic stability to make a distintion between the case in which the closure of the region of asymtotic stability $\overline{A(\{o\})}$ is not compact and the case in which $\overline{A(\{o\})}$ may be compact.

iii.) it allows to find the best possible estimate of the set A ({o}) by means of non-homogeneous polynomial forms of arbitrary high order

iv) it is valid for systems of order n.

v) it is based upon the local extension theorem 2.9 and therefore it does not require an analysis of the geometrical properties of the function $v = \varphi(x)$.

v i)in the case it which x=o is not globally asymptotically stable, it given two parameters which are very useful to characterize the set A ({o}), i.e.,the radiuses of the spheres which are inscribed and circumscribed to the set $N_c(\beta^c) \subset A(\{o\})$.

The construction of the Liapunov function in the regular case (when x=o is not a critical case) starts with the quadratic form

3.1 $$v = \varphi_2(x) = x' H x$$

which may be obtained, for instance, for the system of linear approximation

3.2 $$\dot{x} = J x$$

obtained from the given system 1.1, where J is the Jacobian matrix of f(x) computed in the neighbourhood of x =o and therefore with constant coefficients.

The total time derivative of 3.1 with respect to the system 3.2 is

3.3 $$(x) = x' (J'H + HJ) x .$$

Let

3.4 $$J' H + H J = - C$$

where C an arbitrary positive definite matrix. From it is possible to compute H.

Since x = o is locally asymptotically stable H is positive definite.

Compute now $\psi(x)$ with respect to the equation 1.1

3.5 $$\psi^2(x) = f'(x)Hx + x'H f(x).$$

This function, is such that the point x=o is an isolated point of the set

3.6 $$P_2 = \{ x \in R^n : \psi^2(x) = o \}$$

We have now two situations either $P_2 = \{o\}$ and then x =o is globally asymptotically stable or $Q_2 = P_2 \setminus \{o\} \neq \emptyset$ .In this latter case one consider the set $N_c(\beta^c)$ defined in theorem 2.6, i.e. the set

3.7 $$\min \{ N_c(\beta) : x \in Q_2 \}$$

One considers then the sphere S_i inscribed in $\partial N_c(\beta^c)$ with center in x = o and changes the coefficients of H in such a way as to maximize the radius of S_i.After this maximization one considers again P_2, if $P_2 = \{c\}$ (i.e. if the radius of S_i is infinite), the problem is solved (i.e. x=o is globally asymptotically stable), if not, one repeats the procedure considering now the function :

3.8 $$v = \varphi_2(x) + \varphi_3(x) + \varphi_4(x) = \pi_4(x)$$

i.e. the nonhomogeneous polynomial form of furth order.
One goes on in this fashion by considering nonhomogeneous polynomial of increasing even order until either $P_m = \{o\}$, or the increment of the radius is less then a given quantity $\varepsilon > o$.
In this case one can conclude that (numerically) x=o is not globally asymtotically stable. In this case by using the coefficients of π_m ,which have been computed one can construct the set $N_c(\beta^c) \subset A(\{o\})$, i.e. che set min $N_c(\beta)$:$x \in Q_m$, where $Q_m = P_m \setminus \{o\}$. The set $N_c(\beta^c)$ is the best estimate of A ({o}) .

Next it is desired to analyse the properties of $N_c(\beta^c)$ in order to distinguish between the case in which A ({o}) is not compact ($\overline{N_c(\beta^c)}$ is not compact) and the case in

which $\overline{A(\{o\})}$ is compact in the chosen numerical approximation ($\overline{N_c(\beta^c)}$ is compact).
For that the radius of the sphere S_c, with center in x=o, circumscribed to the set $\partial N_c(\beta^c)$ is maximized over the coefficients of the polynomial form. If this maximization problem does not have a solution, then we shall conclude that $\overline{A(\{o\})}$ is not compact, while in the apposite case that $\overline{N_c(\beta^c)} \subset A(\{o\})$ is compact. Notice that in the whole procedure the use of polynomial forms allows us to conclude that $N_c(\beta^c)$ is always compact set without having to analize the behaviour of $\psi(x)$ at infinity.
The analytical problem of the maximization of the radius ρ_i of the sphere $S_i(\{o\},\rho_i)$, which is inscribed in the set $\partial N_c(\beta^c)$, which is tangent to the set Q_m as the minimization on the space R^{N_m} of the N_m coefficients of the nonhomogeneous polynomial form of m the order $\Pi_m = \varphi(A_m;x)$, $A_m \subset R^{N_m}$, where

3.9 $$N_m = \sum_{2}^{m} \frac{(n-1+r)!}{r(n-1)!\ r!}$$

of the functional

3.10 $$-\min\left\{\|x\| + k_2\left[\varphi(A_m,x) - \min_x\left(\varphi(A_m,x) + k_1\frac{[\psi(A_m,x)]^2}{\|x\|}\right)\right]^2\right\}$$

where k_1 and k_2 are penalization constants.
The analytical problem of the maximization of the radius ρ_c of the sphere $S_c(\{o\},\rho_c)$ circumscribed to $\partial N_c(\beta^c)$ is on the other hand reduced to the minimization on the space R^{N_m} of the functional

3.11 $$-\max\left\{\|x\| - k_2\left[\varphi(A_m,x) - \min\left(\varphi(A_m,x) + k_1\frac{[\psi(A_m,x)]^2}{\|x\|}\right)\right]^2\right\}$$

4. Numerical aspects

In this section we shall analyse the numerical problems involved with the minimization of the functional (3.10) the same conclusions hold also for functional (3.11).

Let x =o be locally asymptotically stable equilibrium point of the system (1.1)

First of all we have to find the smallest set $\varphi(x)=\beta>0$ wich has a point of contact with the surface $\psi(x)=0$.

The problem is that of computing

4.1 $\min\limits_{x\in R^n} \varphi(A,x) = \beta^c$, subject to $\psi(A,x) = 0$.

The numerical solution of this constrained minimum problem is obtained by solving the unconstrained problem:

4.2 $\beta^2 = \min\limits_{x} \xi(A,x,k_1) = \min\limits_{x} \left[\varphi(A,x) + k_1 \frac{[\psi(A,x)]^2}{\|x\|}\right]$

where $\|x\|$ is introduced to avoid the trivial solution, and the penality constant k_1 is chosen to assume satisfaction to a prescribed accuracy of the constraint $\psi(A,x) = 0$ (Courant [12]).Next we have to find the radius ρ of the sphere inscribed in the set $\varphi(A,x) = \beta^c$ and with center in x=o, that is we have to compute :

4.3 $\min\limits_{x\in R^n} \|x\| = \min\limits_{x\in R^n} \sum\limits_{i=1}^{n} (x^i)^2$, subject to $\varphi(A,x) - \beta^c = 0$

Again the constrained minimum problem is solved as the following unconstrained problem:

4.4 $\min\limits_{x\in R^n} \{\|x\| + K_2 [\varphi(A,x) - \beta^c]^2$

where k_2 is the penalisation constant.

For the search of the radius R of the sphere circumscribed to the set $\partial N_c(\beta^c)$ and with center in x = o, we compute instead :

4.5 $\max\limits_{x\in R^n} \{\|x\| - K_2 [\varphi(A, x) - \beta^c]^2 =$

$-\min\limits_{x\in R^n} \{-\|x\| + k_2 [\varphi(A,x) - \beta^c]^2$

Finally we modify the coefficients of the Liapunov function in order to increase the radius of the inscribed sphere, that is we solve the following max min problem:

4.6 $$\max_{A\in R^{N_m}}\{\min_{x\in R^n}\{\|x\| + K_2[\varphi(A,x) - \beta^c]^2\}\} =$$
$$= -\min_{A\in R^{N_m}}\{-\min_{x\in R^n}\{\|x\| + K_2[\varphi(A,x) - \beta^c]^2\}\}$$

By solving the max-min problem 4.6 and 4.2, the problem of the stability region is solved.

The computational procedure will then be as follows:

a) choose A

b) calculate β^c via (4.2)

c) calculate the radius ρ of the inscribed sphere via 4.4

d) modify A in direction of larger radius of the inscripted sphere.

e) return to b

This procedure is repeated as the degree m of Liapunov function increases.

5 . Numerical results

Along the lines presented in the previous section a programm in FORTRAN IV has been written. This program is valid for systems of order n and arbitrary high polynomial approximation. We shall next give some simple numerical results obtained with this program on the second order system,

5.1 $$\dot{x} = Bx + f(x)$$

where $\dot{x} = (x_1, x_2)$, $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 \\ 0{,}04 \end{pmatrix} \cdot x_1^3$,

for which the search has been stopped at the second order approximation :

5.2 $$\varphi(x) = \gamma x_1^2 + \delta x_1 x_2 + \sigma x_2^2 .$$

The function $\varphi(x)$ corrisponding to 5.1 and 5.2 becomes:

5.3 $\psi(x) = 0{,}04\,\delta\, x_1^4 + 0{,}08\,\sigma\, x_1^3\, x_2 - \delta\, x_1^2 + (2\gamma - 2\sigma - \delta) x_1 x_2$

$+ (\delta - 2\sigma)\, x_2^2 .$

In the problem 4.2 and 4.4 we have programmed the Fletcher* and Powell* method 10 . In our opinion, this algorithm is the most powerful for minimizing functions, of which the analytical gradient is known.
On the other hand for searching the optimal coefficients of the Liapunov's function, it has bean used the Powell's method[11] which minimizes a function by modifying one variable at the time.
In both methods the minimum is searched (in all interations) along suitable directions.

The efficiency of the algorithm depends upon the accuracy of the onedimensional search program which has been used. The improve this accuracy it has been adopted the method of the "golden section" together with an automatic search of the minimum along a fixed direction.In programming we has to carefully avoid to fall into the isolated point $x = o \in P = \{x \in R^n;\ \psi(x) = o\}$. We have assumed as initial value of the coefficients of the Liapunov function:

$\gamma = 3.0$, $\delta = 1.0$, $\sigma = 2.0$

for which $\rho = 3.445$

The complete iteractions have been five (that is fifteen iteration of the coefficients γ, δ, σ) and the computational time has been 24 minutes with an IBM 7040 digital computer.
The results have been the following

1st iteration : $\gamma = 1.900$, $\delta = 1.383$, $\sigma = 1.400$, $\rho = 4.272$

2nd iteration : $\gamma = 1.800$, $\delta = 1.489$, $\sigma = 1.100$, $\rho = 4.330$

3rd iteration: $\gamma = 1.700$, $\delta = 1.489$ $\sigma = 0.948$, $\rho = 4.354$

4th iteration : $\gamma = 1.600$, $\delta = 1.457$, $\sigma = 0.848$, $\rho = 4.370$

5th iteration : $\gamma = 1.574$, $\delta = 1.482$, $\sigma = 0.848$, $\rho = 4.373$

§ 6 Conclusions.

This method that we propose for the numerical construction of Liapunov functions, based upon the extension theorems provides, in principle, the complete solution of the stability problem of the equilibrium point x=o of the system 1.1. The method is non been extended to a more complete analysis of the set $\partial A(\{o\})$ in the case in which $\overline{A}(\{o\})$ is compact. The method proposed has of course, to pay something for its generality, and this something is essentially the rather long computation time. If one is interested one in quadratic approximations, for which it is possible an explicit computation of the volume the solution of the problem would be much faster. Our actual research now purely numerical and it is aiming to reduce the computation time, by testing different one-dimensional search procedures, by an exaluation of the effect of the precision of the onedimensional search and its speed of solution of the complete max-min problem.

References

1) G.P.Szegö "Sulla costruzione numerica delle funzioni di Liapunov: I Teoria" Rapporto Università di Milano, IFUM 057/TR-Dicembre 1967.

2) G.P.Szegö, G.Arienti, C.Sutti:"Sulla costruzione numerica delle funzioni di Liapunov: II Calcolo". Rapporto Università di Milano, IFUM 058/TR,Milano-Febbraio 1968.

3) N.P.Bhatia e G.P.Szegö :Dynamical systems:stability theory and applications" Lecture Notes in Mathematics vol. 35. Springer-Verlag, New York-Heidelberg-Berlin 1967.

4) S.G.Margolis and W.G.Vogt: "Control engineering applications of V.I.Zubov's construction procedure for Liapunov functions" IEEE Trans.on Automatic Control, vol. AC-8,N° 2, April 1963, pp.104-113.

5) G.P.Szegö :On a new differential equation for the stability analysis of time-invariant control systems" SIAM Journ. Control.-Vol. 1 (1962)pp.63/75.

6) J.J.Rodden:"Applications of Liapunov stability theory, Lockheed Missile and Space Co., L.M.S.C/A603476,April 1964.

7) S.Weissenberger:"Stability analysis of relay-control systems via the direct method of Liapunov". NASA-CR-320 Washington, D.C.Ottobre 1965.

8) G.Geiss:"Estimation of the domain of attraction" Grumman Research Dept.RM-316 J.N.York,Marzo 1966.

9) V.I.Zubov:"The Methods of A.M.Liapunov and their applications" Noordhoff, Groninger, 1964.

10) R.Fletcher and M.I.D.Powell:"A Rapidly Convergent Descut Method for Minimization" The Comp.Journal, Vol. 6, pag. 163,1963.

11) N.J.D.Powell: "An efficient method for finding the minimum of a function of several variables without calculating derivatives" the Comp. Journal, vol. 7, pag. 155, 1962.

12) R.Courant:"Calculus of Variations"Courant Institute, New York.

13) D.J.Wilde:"Optimum Seeking Methods",Prentice, Hall, Englewood Cliffs, New York 1965

14) J.F.R.Gantmacher:"The theory of matrices" Chelsea New York 1959

15) G.P.Szegö, G.Treccani:" Sistemi dinamici senza unicità". Rapporto Università di Milano, Tech.Rep.IFUM-059/TR, Milano, Febbraio 1968.

16) G.P.Szegö,G.Treccani: "Teoremi di estensione e teoremi del tipo di Rolle", Rapporto Università di Milano, Tech. Rep. IFUM-062/TR,Milano, Marzo 1968.

FINITE TIME STABILITY IN CONTROL SYSTEM SYNTHESIS*

W. L. Garrard
Department of Aeronautics and Engineering Mechanics
University of Minnesota
Minneapolis, Minnesota, USA

1. Introduction

This study is concerned with the application of the concept of finite-time stability to control system synthesis. The results developed are applicable to dynamical systems governed by the set of vector differential equations**

$$\frac{dx}{dt} \equiv \dot{x} = f(x,t) + Bu \ , \quad x(t_o) = x_o \qquad (1)$$

where x, the state, is a n-vector, and u, the control, is a m-vector. B is a n×m constant matrix, $f(x,t)$ is a n-vector such that the norm of $f(x,t)$ is bounded in the domain of the state space and in the time interval of interest, and x_o is the initial state.

Numerous studies concerning the application of classical Liapunov stability theory to the selection of u for systems described by (1) have appeared in recent years.[1-7] In general, the techniques presented in these studies result in the selection of a u such that (1) is asymptotically stable. That is, all solutions of (1) eventually belong to an arbitrary small neighborhood containing the origin provided the initial state belongs to some domain which also contains the origin.

Less restrictive results are often required. For example, in many cases of practical interest it is only necessary to select u in such a way as to guarantee that x belongs to

* This work was supported by National Science Foundation Grant GK-3273.

** It is assumed, as usual, that unique continuous solutions of (1) exist.

some given set during a finite interval of time (e.g. the problem of maintaining a rocket within a given neighborhood of some nominal trajectory during transfer of the rocket from a neighborhood of a point A to a neighborhood of another point B). Recently Weiss and Infante[8,9] developed a qualitative theory of finite-time stability for systems described by (1). Precise definitions were formulated and sufficient conditions were given for various types of finite-time stability. The approach taken by Weiss and Infante was one of analysis rather than synthesis, and the norm of Bu was assumed to be less than or equal to some given value over the time interval of interest.

The present work is focused on the problem synthesizing systems which exhibit finite-time stability. Given that x_o belongs to a specific set, conditions are established which are sufficient to guarantee that $x(t)$ belongs to some given set for a specified interval of time. The control u may then be selected in such a way as to satisfy these conditions. Thus the theorems developed appear to be of practical use in control system synthesis. Illustrative examples are presented.

2. Notation, Definitions, and Problem Formulation

If X is the state space for (1), then let $\|x\|$ denote the Euclidean norm of x and let

$$B(a) = \{x \in X; \|x\| < a\} , \quad \overline{B}(a) = \{x \in X; \|x\| \le a\} ,$$

and $\tau = [t_o, t_o + T)$ where $t_o, T \in R^1$. Also $V(\|x\|)$ is a continuous scalar function which has continuous first partial derivatives with respect to x , and $V^a = V(\|x\| = a)$. Furthermore, $\frac{dV}{dt} \equiv \dot{V} = \langle V_x, \dot{x} \rangle$ where $V_x = \text{grad } V(\|x\|)$ and $\dot{x}$ is given by (1).*

Definition 1: System (1) is stable with respect to the

* The symbol $\langle y, z \rangle$ denotes the scalar product of the vectors y and z .

set $(\alpha, \beta, t_o, T, \|\cdot\|)$, $\alpha < \beta$ if for any $x(t)$, $\|x_o\| < \alpha$ implies $\|x(t)\| \le \beta$ for all $t \epsilon \tau$.

The numbers α, β, t_o, and T are specified a priori in a given problem.

Definition 2: System (1) is <u>quasi-contractively stable</u> with respect to $(\alpha, \beta, \gamma, t_o, T, \|\cdot\|)$, $\alpha < \beta < \gamma$ if for any $x(t)$, $\|x_o\| < \alpha$ implies system (1) is <u>stable</u> in the sense of Definition 1 with respect to $(\alpha, \gamma, t_o, T, \|\cdot\|)$ and there exists a $t_2 \epsilon \tau$ such that $\|x(t)\| \le \beta$ for all $t \epsilon (t_2, t_o + T)$.

Definition 3: System (1) is <u>contractively stable</u> with respect to $(\alpha, \beta, \gamma, t_o, T, \|\cdot\|)$, $\beta < \alpha < \gamma$ if for any $x(t)$, $\|x_o\| < \alpha$ implies system (1) is <u>stable</u> in the sense of Definition 1 with respect to $(\alpha, \gamma, t_o, T, \|\cdot\|)$ and there exists a $t_2 \epsilon \tau$ such that $\|x(t)\| \le \beta$ for all $t \epsilon (t_2, t_o + T)$.

Definition 4: System (1) is <u>unstable</u> with respect to $(\alpha, \beta, t_o, T, \|\cdot\|)$, $\alpha < \beta$ if for any $x(t)$, $x_o \epsilon \{B(\beta) - \overline{B}(\alpha)\}$ implies the existence of a $t_1 \epsilon \tau$ such that $\beta \le \|x(t_1)\|$.

The definitions presented above are very similar to those of Weiss and Infante.[9]

The problem considered in this paper is that of establishing conditions which guarantee stability (quasi-contractive stability, contractive stability, instability) of system (1). As demonstrated in the illustrative examples, these conditions can be used in the selection of a u which guarantees the proper variety of finite-time stability or instability.

3. <u>Theorems on Finite-Time Stability</u>

Theorem 1: System (1) is stable in the sense of Definition 1 if for all $t \epsilon \tau$ and all $x \epsilon \{\overline{B}(\beta) - B(\alpha)\}$

(a) $< V_x, f > + < V_x, Bu > \leq \frac{V^\beta - V^\alpha}{T}$

(b) $V^a > V^b$, $\|a\| \geq \|b\|$ for all $a, b \in \{\overline{B}(\gamma) - B(\alpha)\}$.

Proof: Let $x(t)$ be an arbitrary trajectory of (1) such that $\|x(t_o)\| < \alpha$ and assume there exists a $t_2 \in \tau$ such that $\|x(t_2)\| = \beta$ and a $t_1 \in \tau$ such that $\|x(t_1)\| = \alpha$.* Then

$$V(\|x(t)\|) = V^\alpha + \int_{t_1}^{t} (< V_x, f > + < V_x, Bu >) d\sigma . \quad (2)$$

From condition (a)

$$V(\|x(t_2)\|) \leq V^\alpha + \int_{t_1}^{t_2} \left(\frac{V^\beta - V^\alpha}{T} \right) dt , \quad (3)$$

and since $t_2 - t_1 < T$,

$$V(\|x(t_2)\|) < V^\beta . \quad (4)$$

From condition (b), (4) implies $\|x(t_2)\| < \beta$. This contradicts the original hypothesis that $\|x(t_2)\| = \beta$. Thus, there does not exist a $t_2 \in \tau$ such that $\|x(t_2)\| = \beta$, and therefore $\|x(t)\| < \beta$ for all $t \in \tau$.

Theorem 2: System (1) is quasi-contractively stable if for all $t \in \tau$ and all $x \in \{\overline{B}(\gamma) - B(\alpha)\}$

(a) $< V_x, f > + < V_x, Bu > \leq \frac{V^\gamma - V^\alpha}{T}$

(b) $V^a \geq V^b$, $\|a\| \geq \|b\|$ for all $a, b \in \{\overline{B}(\gamma) - B(\alpha)\}$

(c) $< V_x, f > + < V_x, Bu > \leq \frac{(V^\beta - V^\delta)(V^\gamma - V^\alpha)}{KT\,(V^\gamma - V^\delta)}$

for $t \in [t_1, t_2)$ where $0 < K \leq 1$, $\beta \leq \delta < \gamma$, $t_1 = \left(\frac{V^\delta - V^\alpha}{V^\gamma - V^\alpha} \right) T + t_o$, and $t_2 = \left(\frac{V^\delta - V^\alpha + K(V^\gamma - V^\delta)}{V^\gamma - V^\alpha} \right) T + t_o$,

(d) $< V_x, f > + < V_x, Bu > \leq 0$, $t \in [t_2, t_o + T)$.

---

* If such a t_1 does not exist, no control is needed.

Proof: Since $\alpha < \beta < \gamma$, $\frac{(V^\beta - V^\delta)(V^\gamma - V^\alpha)}{KT\ (V^\gamma - V^\delta)} \le 0 \le \frac{V^\gamma - V^\alpha}{T}$

conditions (a) and (b) guarantee stability from Theorem 1. Let $x(t)$ be an arbitrary trajectory of (1) such that $\| x(t_o) \| < \alpha$ and assume there exists a $t' \varepsilon \tau$ such that $\| x(t') \| = \alpha$ where $t' < t_1$. Then

$$V(\| x(t_1) \|) \le V(\| x(t') \|) + \int_{t'}^{t_1} (\frac{V^\gamma - V^\alpha}{T}) dt \quad (5)$$

or

$$V(\| x(t_1) \|) \le V^\alpha + \frac{V^\gamma - V^\alpha}{T}(t_1 - t') \le V^\alpha + \frac{V^\gamma - V^\alpha}{T}(t_1 - t_o) \quad (6)$$

since $t_o \le t'$, and if the value of t_1 given in condition (c) is substituted into (6), it can be seen that $V(\| x(t_1) \|) \le V^\delta$. Furthermore

$$V(\| x(t_2) \|) \le V(\| x(t_1) \|) + \int_{t_1}^{t_2} \frac{(V^\beta - V^\delta)(V^\gamma - V^\alpha)}{KT\ (V^\gamma - V^\delta)} dt \quad (7)$$

and

$$V(\| x(t_2) \|) \le V^\delta + \frac{(V^\beta - V^\delta)(V^\gamma - V^\alpha)}{KT\ (V^\gamma - V^\delta)}(t_2 - t_1) \ . \quad (8)$$

If the value of t_2 given in condition (c) is substituted into (8), it can be seen that $V(\| x(t_2) \|) \le V^\beta$. This implies $\| x(t_2) \| \le \beta$. Finally from condition (d)

$$V(\| x(t) \|) = V(\| x(t_2) \|) - \int_{t_2}^{t} b d\sigma , \qquad t \varepsilon [t_2, t_o + T) \quad (9)$$

where $0 < b$, thus

$$V(\| x(t) \|) \le V^\beta - b(t - t_2) \quad (10)$$

which implies $\| x(t) \| \le \beta$ for $t \varepsilon [t_2, t_o + T)$. Therefore conditions (a), (b), (c), and (d) guarantee quasi-contractive stability for system (1).

Theorem 3: System (1) is contractively stable if for all $t \varepsilon \tau$ and all $x \varepsilon \{\overline{B}(\gamma) - B(\beta)\}$ the conditions of Theorem 2 are satisfied.

Proof: The proof is identical to that of Theorem 2 but in this case $\beta < \alpha$.

Theorem 4: System (1) is unstable if for all $t \varepsilon \tau$

(a) $(< V_x, f > + < V_x, Bu >) > \frac{V^\beta - V^\alpha}{T}$

(b) $V^a \geq V^b$, $\|a\| \geq \|b\|$ for all $a, b \varepsilon \{X - \overline{B}(\alpha)\}$.

Proof: $V(\|x(t)\|) > V(\|x(t_o)\|) + \frac{V^\beta - V^\alpha}{T}(t - t_o)$, and for $t = t_o + T$, $V(\|x(t_o + T)\|) > V^\beta$ which implies the existence of a $t_1 \varepsilon \tau$ such that $\|x(t_1)\| \geq \beta$.

As with most synthesis techniques based upon classical Liapunov stability theory, the problem of determining a u which satisfies any of the above theorems over the domain of interest is not trivial and may often tax the ingenuity of the investigator. A suggested procedure is to assume

$$u = g(\omega(t), x, t) \tag{11}$$

where $\omega(t)$ is a p-dimensional vector, $p \leq n$, which is to be selected in such a way as to satisfy the conditions specified in the theorem of interest. If $\omega(t)$ belongs to a compact set Ω in R^p , the problem of selecting $\omega(t)$ could be considered as a problem in nonlinear programming.[10] That is, select $\omega(t)$ in such a way that

$$\min_{\omega \varepsilon \Omega} \left\{ \max_{\substack{\alpha \leq \|x\| \leq \beta, \\ t_o < t < t_o + T}} (< V_x, f > + < V_x, g >) \right\} \leq C(t) \tag{12}$$

where $C(t)$ is a piecewise constant function specified by the conditions of the theorem of interest. Of course for a specific g there is no guarantee that a ω which satisfies (12) exists.

4. Illustrative Examples

Two illustrative examples are considered.

Example I: A linear system

Consider the system governed by

$$\dot{x} = A(t)x + Iu \,, \qquad u = P(t)x \tag{13}$$

where A is a n x n matrix, I is the identity matrix, and P is a n x n matrix to be determined in such a way as to insure the stability of (13) in the sense of Definition 1. Furthermore, let $V = \langle x,x \rangle$ and $(\beta/\alpha)^2 = k > 1$.

From Theorem 1, P must be selected such that

$$\underline{x}'[A + P]\underline{x} \le \underline{x}' \frac{[1 - 1/k]}{2T} I\underline{x} \tag{14}$$

where $\langle \underline{x},\underline{x} \rangle = \beta^2$. Inequality (14) is satisfied if

$$P = -A + \frac{[1 - 1/k]}{2T} I \,. \tag{15}$$

From (15), it can easily be seen that

$$x'[A + P]x \le \underline{x}' \frac{[1 - 1/k]}{2T} I\underline{x}$$

for all $\langle x,x \rangle \le \beta^2$. Therefore from Theorem 1, the control $u = Px$ where P is given by (15) stabilizes (13) over the interval $[t_o, t_o + T)$. This may easily be verified for $k \ge e$ by taking the derivative of $\langle x,x \rangle$ and using (13) and (15).

Example II: Van der Pol's equation

Consider a second-order system governed by

$$\dot{x}_1 = x_2 \,, \qquad \dot{x}_2 = (2 - 0.5x_1^2)x_2 - x_1 + u \tag{16}$$

and let $\alpha = 1$, $\beta = 2$, $t_o = 0$, and $T = 2$. By numerical integration it can be shown that for $u = 0$, $[x_1(t^*) + x_2(t^*)]^{\frac{1}{2}} > 2$ where $t^* < 2$ for certain initial conditions (e.g. $x_1(0) = 1$,

$x_2(0) = 0)$. For stability, Theorem 1 yields

$$(2. - 0.5x_1^2)x_2^2 + ux_2 \leq 0.75 \tag{17}$$

for $V = \langle x,x \rangle$. If $u = ax_2$, it can be seen that $a = - 1.81$ satisfies (17) over the domain of interest.

For quasi-contractive stability where $\alpha = 1$, $\beta = \delta = 1.1$, $\gamma = 2$, $K = .5$, $t_o = 0$, $T = 2$. Theorem 2 yields

$$(2. - 0.5x_1^2)x_2^2 + ux_2 \leq 0.75 \tag{18}$$

for $0 \leq t < 0.14$, and

$$(2. - 0.5x_1^2)x_2^2 + ux_2 \leq 0 \tag{19}$$

for $.14 \leq t < 2$. If $u = a(t)x_2$, it can be seen that $a(t) = - 1.81$ for $0 \leq t < 0.14$ and $a(t) = - 2$ for $0.14 \leq t < 2$ satisfies (18) and (19) over the domain of interest.

It should also be noted that for $u = a(t)x_2$ it is impossible to obtain an $a(t)$ which satisfies Theorem 3 for contractive stability. Typical unstable, stable, and quasi-contractively stable trajectories for Example II are presented in Fig. 1, and the variation of the norms with time are presented in Fig. 2.

5. Conclusions

The theorems developed in this paper appear to be of practical use in the synthesis of control laws which guarantee finite-time stability for dynamical systems governed by ordinary differential equations in which the control enters linearly. The results obtained are much less restrictive than those given by classical Liapunov stability theory. Furthermore, the theorems for finite time-stability presented herein can be applied more easily than those given elsewhere in that very simple Liapunov functions may be used (e.g. in both examples the Euclidean norm was used as a Liapunov function).

As with other techniques based on Liapunov stability theory, the problem of selecting a control which satisfies the theorems developed is non-trivial and is an open area for further research.

References

1. Kalman, R. E., and J. E. Bertram, "Control System Analysis and Design via the Second Method of Liapunov," ASME J. Basic Eng., 82, pp. 371-393, 1960.

2. Geiss, G., "The Analysis and Design of Nonlinear Control Systems via Liapunov's Direct Method," Grumman Aircraft Corp. Rept. RTD-TDR-63-4076, 1964.

3. Johnson, G. W., "Synthesis of Control Systems with Stability Constraints via the Second Method of Liapunov," IEEE Trans. Auto. Control, AC-9, pp. 380-385, 1964.

4. Nahi, N. E., "On the Design of Optimal Systems via the Second Method of Liapunov," IEEE Trans. Auto. Control, AC-9, pp. 274-275, 1964.

5. Grayson, Lawrence P., "The Status of Synthesis Using Liapunov's Method," Automatica, 3, pp. 91-121, 1965.

6. Monopoli, R. V., "Synthesis Techniques Employing the Direct Method," IEEE Trans. Auto. Control, AC-10, pp. 369-370, 1965.

7. Kushner, Harold J., Stochastic Stability and Control, Academic Press, New York, 1967.

8. Weiss, Leonard, and E. F. Infante, "On the Stability of Systems Defined Over a Finite Time Interval," Proc. Nat'l Acad. Sci., (USA) 54, pp. 44-48, 1965.

9. Weiss, Leonard, and E. F. Infante, "Finite Time Stability Under Perturbing Forces and on Product Spaces," IEEE Trans. Auto. Control, AC-12, pp. 54-59, 1967.

10. Rosen, J. B., "Controllable Stability and Equivalent Nonlinear Programming Problem," in Nonlinear Differential Equations and Nonlinear Mechanics, (J. F. LaSalle and S. Lefschetz, eds.), Academic Press, New York, pp. 366-376, 1963.

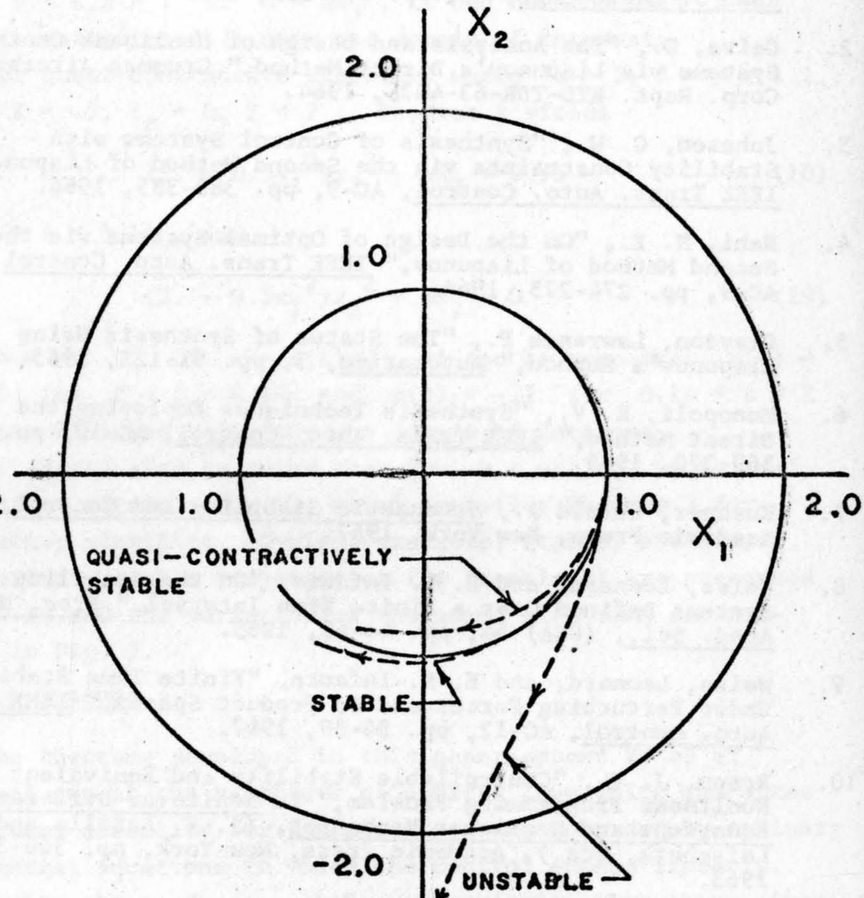


Fig. 1 Van Der Pols Eq. Phase - Plane Plot

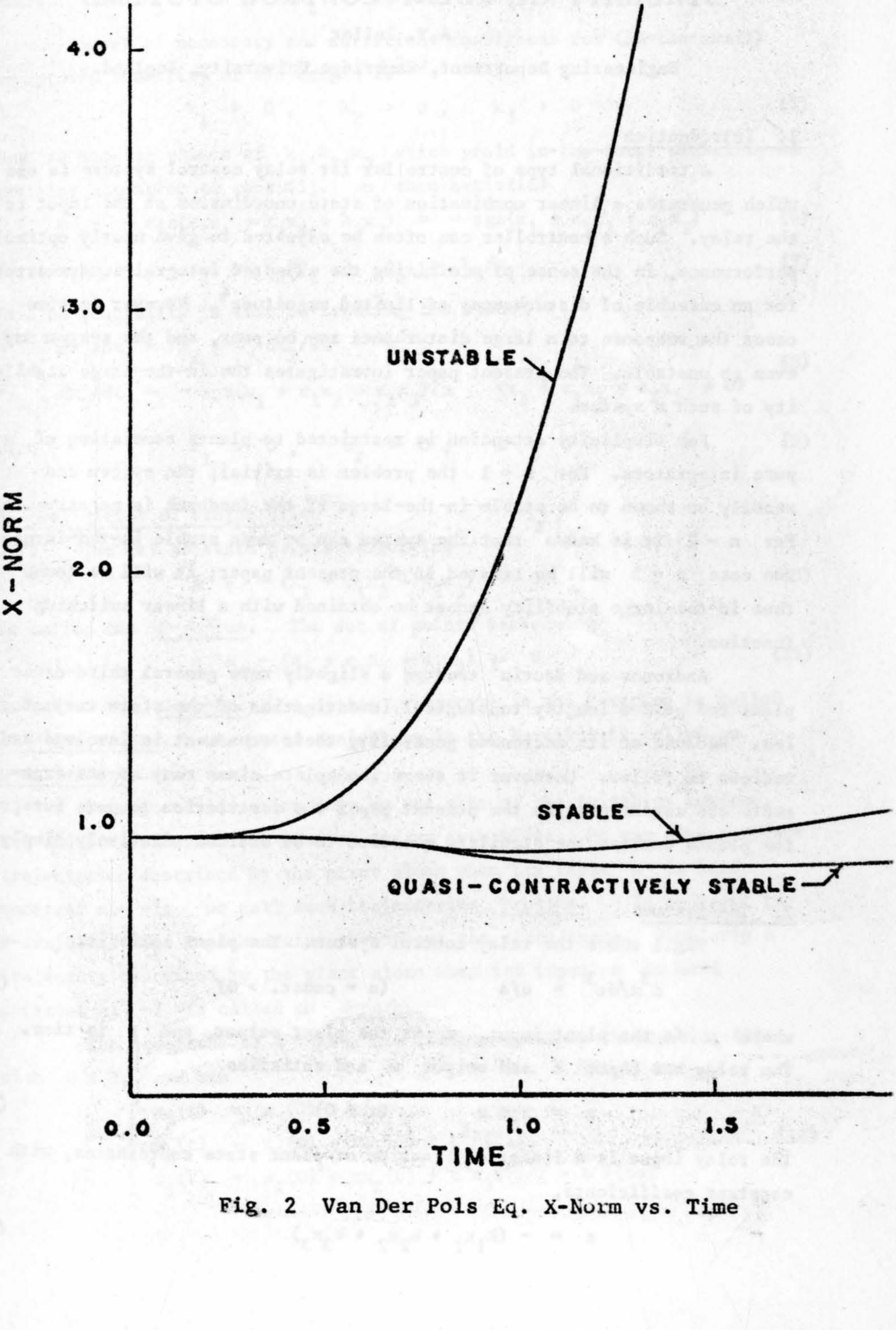


Fig. 2 Van Der Pols Eq. X-Norm vs. Time

STABILITY OF RELAY CONTROL SYSTEMS

A.T. Fuller

Engineering Department, Cambridge University, England.

1. Introduction

A traditional type of controller for relay control systems is one which generates a linear combination of state coordinates at the input to the relay. Such a controller can often be adjusted to give nearly optimal performance, in the sense of minimizing the expected integral-square-error for an ensemble of disturbances of limited magnitude[4]. However in some cases the response to a large disturbance may be poor, and the system may even go unstable. The present paper investigates the in-the-large stability of such a system.

For simplicity attention is restricted to plants consisting of n pure integrators. For $n = 1$ the problem is trivial; the system can readily be shown to be stable in-the-large if the feedback is negative. For $n = 2$ it is known[4] that the system can be made stable in-the-large. The case $n = 3$ will be treated in the present paper; it will be found that in-the-large stability cannot be obtained with a linear switching function.

Andronov and Bautin[1] treated a slightly more general third-order plant and gave a lengthy topological investigation of the state trajectories. Because of its increased generality their treatment is involved and tedious to follow. Moreover it seems incomplete since many of its arguments are curtailed. In the present paper the restriction to pure integrator plants enables the stability question to be decided relatively simply.

2. The system

Fig.1 shows the relay control system. The plant satisfies

$$d^3x/dt^3 = u/a \qquad (a = \text{const.} > 0) \tag{1}$$

where u is the plant input, x is the plant output, and t is time. The relay has input z and output u and satisfies

$$u = \text{sgn}.z \qquad (z \neq 0) \tag{2}$$

The relay input is a linear combination of plant state coordinates, with constant coefficients,

$$z = -(k_1x_1 + k_2x_2 + k_3x_3) \tag{3}$$

where $x_1 \equiv x$, $x_2 \equiv dx/dt$, $x_3 \equiv d^2x/dt^2$ **(4)**

A set of necessary and sufficient conditions for (in-the-small) asymptotic stability of the origin is[2,3]

$$k_1 > 0, \quad k_2 > 0, \quad k_3 > 0 \quad (5)$$

Thus in seeking values of k_1, k_2, k_3 which yield in-the-large stability we restrict attention to case (5). u then satisfies

$$u = -\operatorname{sgn}(k_1x_1 + k_2x_2 + k_3x_3) = -\operatorname{sgn}(x_1 + c_1x_2 + c_2x_3) \quad (6)$$

where

$$c_1 = k_2/k_1, \quad c_2 = k_3/k_1 \quad (7)$$

Thus no generality is lost in treating the system

$$\begin{cases} dx_1/dt = x_2, \quad dx_2/dt = x_3, \\ dx_3/dt = -\operatorname{sgn}(x_1 + c_1x_2 + c_2x_3)/a \quad (x_1 + c_1x_2 + c_2x_3 \neq 0) \end{cases} \quad (8)$$

with

$$c_1 > 0, \quad c_2 > 0, \quad a > 0 \quad (9)$$

3. Definition of trajectories

The set of state points satisfying

$$z \equiv -(x_1 + c_1x_2 + c_2x_3) > 0 \quad (10)$$

is called the P-region. The set of points satisfying

$$z \equiv -(x_1 + c_1x_2 + c_2x_3) < 0 \quad (11)$$

is called the N-region. A system trajectory in the P-region is called a P-trajectory. A system trajectory in the N-region is called an N-trajectory.

P-trajectories exist only on one side of the switching surface, and are properties of the control system as a whole. We can also consider trajectories described by the plant alone when its input u is held constant at +1; we call such trajectories P-paths. In contrast to P-trajectories, P-paths exist throughout the state space. Similarly a trajectory described by the plant alone when its input u is held constant at −1 is called an N-path.

The equations of a P-path are readily found by integrating (1) with $u \equiv 1$, and are

$$\begin{cases} x_3(t) = x_3(0) + t/a \\ x_2(t) = x_2(0) + tx_3(0) + t^2/(2a) \\ x_1(t) = x_1(0) + tx_2(0) + t^2x_3(0)/2 + t^3/(6a) \end{cases} \quad (12)$$

4. Sliding motion

If, at a point of the switching surface, the directions of motion along the trajectories on each side of the switching surface are not away from the switching surface, sliding motion occurs. Sliding motion is idealized as motion in the switching surface (e.g.[4]), i.e. is defined as satisfying

$$x_1(t) + c_1x_2(t) + c_2x_3(t) \equiv 0 \tag{13}$$

Differentiation of (13) and use of (8) shows that the sliding motion obeys the linear differential equations

$$dx_2/dt = x_3, \quad dx_3/dt = -(x_2 + c_1x_3)/c_2 \tag{14}$$

A trajectory described during sliding is called an S-trajectory.

The sliding region can be found as follows. The switching surface is represented by

$$z(x_1,x_2,x_3) = 0 \tag{15}$$

where

$$z(x_1,x_2,x_3) \equiv -(x_1 + c_1x_2 + c_2x_3) \tag{16}$$

In the P-region and near the switching surface z is positive. Hence the trajectory there approaches or is parallel to the switching surface if

$$dz/dt \leq 0 \tag{17}$$

i.e. if

$$x_2 + c_1x_3 + c_2/a \geq 0 \tag{18}$$

from (8) and (16). Similarly, in the N-region and near the switching surface, the trajectory approaches (or is parallel to) the switching surface if

$$x_2 + c_1x_3 - c_2/a \leq 0 \tag{19}$$

Thus the sliding region consists of the points of the switching surface where (18) and (19) hold simultaneously, i.e. where

$$|x_2 + c_1x_3| \leq c_2/a \tag{20}$$

In other words the sliding region is the strip of the switching plane between the pair of parallel planes

$$x_2 + c_1x_3 + c_2/a = 0, \quad x_2 + c_1x_3 - c_2/a = 0 \tag{21a,b}$$

5. Periodic motion without sliding

To establish that system (8) is unstable in-the-large it suffices to show that the system trajectories possess a cycle, i.e. a closed loop (e.g. a limit-cycle). The simplest type of cycle which might exist is a symmetrical one consisting of a P-trajectory and an N-trajectory. We call such a cycle a PN-cycle, and we begin by seeking the values of c_1 and c_2 for which a PN-cycle exists. A PN-cycle exists if there is a point (X_1,X_2,X_3) on the switching plane such that

(i) there is a P-path from (X_1,X_2,X_3) to the symmetrically opposite point $(-X_1,-X_2,-X_3)$,

(ii) there is an N-path from $(-X_1,-X_2,-X_3)$ to (X_1,X_2,X_3),

(iii) the P-path in (i) is a P-trajectory (as defined in §3),

(iv) the N-path in (ii) is an N-trajectory.

5.1 Existence of the P-path and the N-path

Let us find when condition (i) is satisfied. From (12), the P-path starting at (X_1,X_2,X_3) goes through $(-X_1,-X_2,-X_3)$ at time $t = T > 0$ if

$$\begin{cases} -X_1 = X_1 + TX_2 + T^2X_3/2 + T^3/(6a) \\ -X_2 = X_2 + TX_3 + T^2/(2a) \\ -X_3 = X_3 + T/a \end{cases} \tag{22}$$

Also, since (X_1,X_2,X_3) is on the switching plane

$$X_1 + c_1X_2 + c_2X_3 = 0 \tag{23}$$

Solving (22) and (23) we find

$$X_1 = \sqrt{3c_2^3}/a, \quad X_2 = 0, \quad X_3 = -\sqrt{3c_2}/a \tag{24}$$

and

$$T = 2\sqrt{3c_2} \tag{25}$$

Thus condition (i) is satisfied when the switch point (X_1,X_2,X_3) is given by (24). By symmetry condition (ii) is also satisfied then.

5.2 Existence of the P-trajectory and the N-trajectory

Let us find when conditions (i) and (iii) are both satisfied. This is so if the P-path described by (12) with initial conditions (24) is in the P-region for all $0 < t < T$; i.e. if

$$z(x_1(t),x_2(t),x_3(t)) > 0 \qquad (0 < t < T) \tag{26}$$

(12), (16), (24) and (25) yield, after some algebra,

$$z(x_1(t),x_2(t),x_3(t)) = t(T - t)(6c_1 - T + 2t)/(12a) \tag{27}$$

Throughout $0 < t < T$ the first two factors on the right of (27) are positive, and the third factor is positive if

$$6c_1 - T \geq 0 \tag{28}$$

i.e. (26) is satisfied if (28) is. (28) is equivalent to

$$c_2 \leq 3c_1^2 \tag{29}$$

in view of (25). Thus when (29) holds, conditions (i) and (iii) are satisfied by point (24); and by symmetry (ii) and (iv) are also satisfied.

5.3 Existence of the PN-cycle

We have found that when the controller coefficients satisfy inequality (29) there exists a PN-cycle, with switch points (X_1,X_2,X_3) and $(-X_1,-X_2,-X_3)$ given by (24) and half-period T given by (25). Thus when (29) holds the system is unstable in-the-large.

6. Periodic motion with sliding

Let us investigate the remaining case

$$c_2 > 3c_1^2 \tag{30}$$

In this case the point with coordinates (24) is inside the sliding strip (as follows on substituting values (24) in (20)), and so can no longer be a switch point of a PN-cycle. However there might exist instead a cycle involving sliding. The simplest type of such a cycle would be a symmetrical one consisting of a P-trajectory, an S-trajectory, an N-trajectory, and an S-trajectory. We call the latter a PSNS-cycle. Our aim is to find whether a PSNS-cycle exists when (30) holds.

Fig.2 shows the projection of the sliding region in the (x_2,x_3) plane. The left edge of the strip is represented by the straight line L, which is given by (21a), and the right edge by line R, given by (21b). The P-trajectory of a PSNS-cycle can begin only on the left edge, since there the P-trajectory is parallel to the switching plane, whereas at other points in the sliding region the P-trajectory is towards the sliding region. Thus a PSNS-cycle exists if there is a point U on the left edge such that

(i) the P-path from point U meets the switching plane at a point V, and V is in the sliding region,

(ii) the P-path in (i) is a P-trajectory,

(iii) the S-trajectory from point V meets the right edge of the sliding region at a point W,

(iv) the point W is symmetrically opposite the point U, i.e. the coordinates (W_1,W_2,W_3) of W equal minus the coordinates (U_1,U_2,U_3) of U

Indeed if (i)-(iv) hold, what we may call a PS-trajectory exists from U to W, and by symmetry an NS-trajectory exists from W back to U. The PS- and NS-trajectories then form a PSNS-cycle.

6.1 Existence of the P-trajectory

Let us find when condition (i) is satisfied. Since U is on the switching plane and on the left edge (21a)

$$U_1 + c_1U_2 + c_2U_3 = 0 \quad \text{and} \quad U_2 + c_1U_3 + c_2/a = 0 \tag{31}$$

From (12), the P-path from starting point U reaches a point V at time $t = Y > 0$ if

$$\begin{cases} V_1 = U_1 + YU_2 + Y^2U_3/2 + Y^3/(6a) \\ V_2 = U_2 + YU_3 + Y^2/(2a) \\ V_3 = U_3 + Y/a \end{cases} \tag{32}$$

If V is on the switching plane

$$V_1 + c_1V_2 + c_2V_3 = 0 \tag{33}$$

Elimination of U_1, U_2, V_1, V_2, V_3 from (31)-(33) yields

$$Y^3 + 3(aU_3 + c_1)Y^2 = 0 \tag{34}$$

Thus the P-path has only two points in the switching plane, namely at times $Y = 0$ (corresponding to the starting point U) and

$$Y = -3(aU_3 + c_1) \tag{35}$$

(35) implies that if

$$aU_3 + c_1 < 0 \tag{36}$$

the P-path returns to the switching plane after leaving U; i.e. if (36) holds, point V exists.

Let us find when V is in the sliding strip. From (31), (32) and (35), point V satisfies

$$\begin{cases} V_2 = (3/2)aU_3^2 + 5c_1U_3 + (9/2)c_1^2/a - c_2/a \\ V_3 = -2U_3 - 3c_1/a \end{cases} \tag{37}$$

V is in the sliding strip if V satisfies (20), i.e. if

$$|V_2 + c_1V_3| \leq c_2/a \tag{38}$$

Equations (37) simplify (38) to

$$(aU_3 + c_1)^2 \leq (4/3)c_2 \tag{39}$$

Inequalities (36) and (39) hold simultaneously if

$$-2\sqrt{c_2/3} \leq aU_3 + c_1 < 0 \tag{40}$$

Thus if (40) holds, V exists and is in the sliding strip, i.e. condition (i) is satisfied. Moreover, condition (ii) is then satisfied because V, being in the sliding strip, is approached by a P-trajectory, which therefore coincides with the P-path which approaches V.

6.2 Existence of the S-trajectory to the strip edge

Let us find when condition (iii) is satisfied. The characteristic equation of the sliding motion (14) is

$$c_2p^2 + c_1p + 1 = 0 \tag{41}$$

and, because of (9) and (30), has complex characteristic roots with negative real parts. Hence the S-trajectories have a stable focus at the origin.

In Fig.2 one of the spiral S-trajectories touches the line R. This is confirmed algebraically in Appendix 1, where it is shown that the point of tangency A has coordinates

$$A_2 = (c_2 - c_1^2)/a\,, \quad A_3 = c_1/a \tag{42}$$

The S-trajectory traced backwards from A meets line L at some point G (Fig.2), as shown in Appendix 2. The field of S-trajectories above trajectory GA must meet line R, since the trajectories spiral clockwise and do not cross one another. Therefore, if we can show that V is in this field, condition (iii) will be established.

From (37), when U_3 varies in the range (40), V describes a parabolic locus from point B, with coordinates

$$B_2 = (c_1^2 - 4c_1\sqrt{c_2/3} + c_2)/a\,, \quad B_3 = (-c_1 + 4\sqrt{c_2/3})/a \tag{43}$$

to point D with coordinates

$$D_2 = (c_1^2 - c_2)/a\,, \quad D_3 = -c_1/a \tag{44}$$

This locus is represented by the dashed line BD in Fig.2. B and D are on lines R and L respectively, as may be checked from (43) and (44).

We have to show that at least part of the locus BD is in the field of S-trajectories above GA. Thus it will suffice to show that B is above A, i.e.

$$B_3 > A_3 \tag{45}$$

But from (30), (42), (43), inequality (45) does indeed hold. Thus if U_3 satisfies (40), condition (iii) is satisfied.

6.3 Existence of symmetrically opposite points U and W

It remains to find when condition (iv) is satisfied. When U_3 increases in range (40), U traces the segment FD of line L, where D and F have ordinates

$$D_3 = -c_1/a, \quad F_3 = -(c_1 + 2\sqrt{c_2/3})/a \tag{46}$$

V then traces locus BD. Let C be the first point at which locus BD crosses trajectory GA (Fig.2. That BD does cross GA is confirmed in Appendix 2.) Let U then be at point E in the segment FD. Thus when U traces FE, V traces BC and W traces BA.

Let us demonstrate the continuity of the movement of W from B to A as U traces FE. As U varies, the movement of V from B to C is continuous, in view of (37). Thus the S-trajectory from V changes location continuously. Also, because trajectories from neighbouring points remain close, the S-trajectories from neighbouring points V cut line R in neighbouring points W. (It is assumed here that the S-trajectories are not tangential to R except at A. This assumption is justified in Appendix 1.) Hence W moves continuously from B to A as U traces FE.

Let U* be the reflection of U in the origin, i.e. let U* have coordinates $(-U_1, -U_2, -U_3)$. The reflection of the segment EF traced by U is the segment E*F* traced by U* on R (Fig.2). Segment E*F* is inside segment AB, as is shown in Appendix 3. Also, when U traces FE, U* traces F*E* continuously and, as we saw above, W moves continuously from B to A. Therefore at some stage W passes U*. W is then symmetrically opposite U, so that condition (iv) is satisfied.

6.4 Existence of the PSNS-cycle

Conditions (i)-(iv) have now been shown satisfied when (30) holds. Thus when the controller coefficients satisfy inequality (30) the system has a PSNS-cycle and is consequently unstable in-the-large.

7. Conclusions

We have found that a relay control system with a triple-integrator plant cannot be made stable in-the-large with a linear switching function. It is conjectured that the same is true when the plant consists of n pure integrators with $n \geq 3$, and when the relay is replaced by a saturator. In such cases one can perhaps obtain a simple near-optimal controller by using a switching function which is linear near the origin where the state point is usually to be found, and nonlinear elsewhere so as to yield stable response to occasional large disturbances[4].

8. Appendix 1. Existence of point A

From (21a) the slope of line R is

$$dx_3/dx_2 = -1/c_1 \tag{47}$$

From (14) the slope of the S-trajectory is

$$dx_3/dx_2 = -(x_2 + c_1 x_3)/(c_2 x_3) \tag{48}$$

Solving (21a), (47) and (48), we find that an S-trajectory touches R at and only at a point A given by

$$x_2 = (c_2 - c_1^2)/a, \quad x_3 = c_1/a \tag{49}$$

9. Appendix 2. Existence of points G and C

From (21) the quantity y defined by

$$y \equiv x_2 + c_1 x_3 \tag{50}$$

measures the displacement of (x_2, x_3) from the line parallel to L and R and through the origin. y(t) along an S-trajectory is readily calculated by solving (14), and for starting point A is

$$y(t) = (c_2/a)\exp(\alpha t)\left[\cos(\beta t) - (\alpha/\beta)\sin(\beta t)\right] \tag{51}$$

where $\alpha \pm j\beta$ ($\alpha < 0$, $\beta > 0$) are the roots of (41). It is easily shown that when t decreases from zero, expression (51) at first decreases to a minimum value

$$y(-\pi/\beta) = -(c_2/a)\exp(-\pi\alpha/\beta) < -c_2/a \tag{52}$$

Thus for some value of $t < 0$, $y(t) = -c_2/a$, i.e. the S-trajectory traced backwards from A cuts L at some point G.

Similarly, the S-trajectory traced forwards from D does not cut R. Hence D is below S-trajectory GA. Since B is above GA (see §6.2) at least one point of locus BD is on GA. This is point C.

10. Appendix 3. Location of segment E*F*

Since E is below D, E* is above D* (which is the reflection of D). But D* coincides with A, since D is symmetrically opposite A. Hence E* is above A. Also, from (43), (46) and (30)

$$B_3 - F_3^* = 2(-c_1 + \sqrt{c_2/3})/a > 0 \qquad (53)$$

so that F* is below B. Hence E*F* is inside AE.

11. References

1. ANDRONOV, A.A. and BAUTIN, N.N. Dokl.ANSSSR, 43 (1944) 197; Izv. ANSSSR, OTN, No.3, p.3, No.6, p.54.

2. TSYPKIN, YA.Z. Theory of relay systems of automatic control (Moscow 1955; German translation, Munich 1958).

3. ANOSOV, D.V. Avt.i Telemech., 20 (1959) 135.

4. FULLER, A.T. International J. Control, 5 (1967) 197.

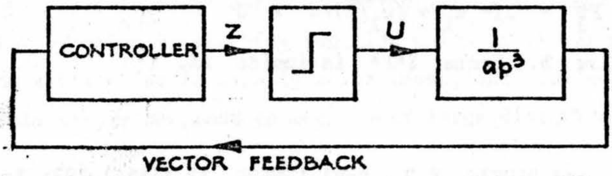
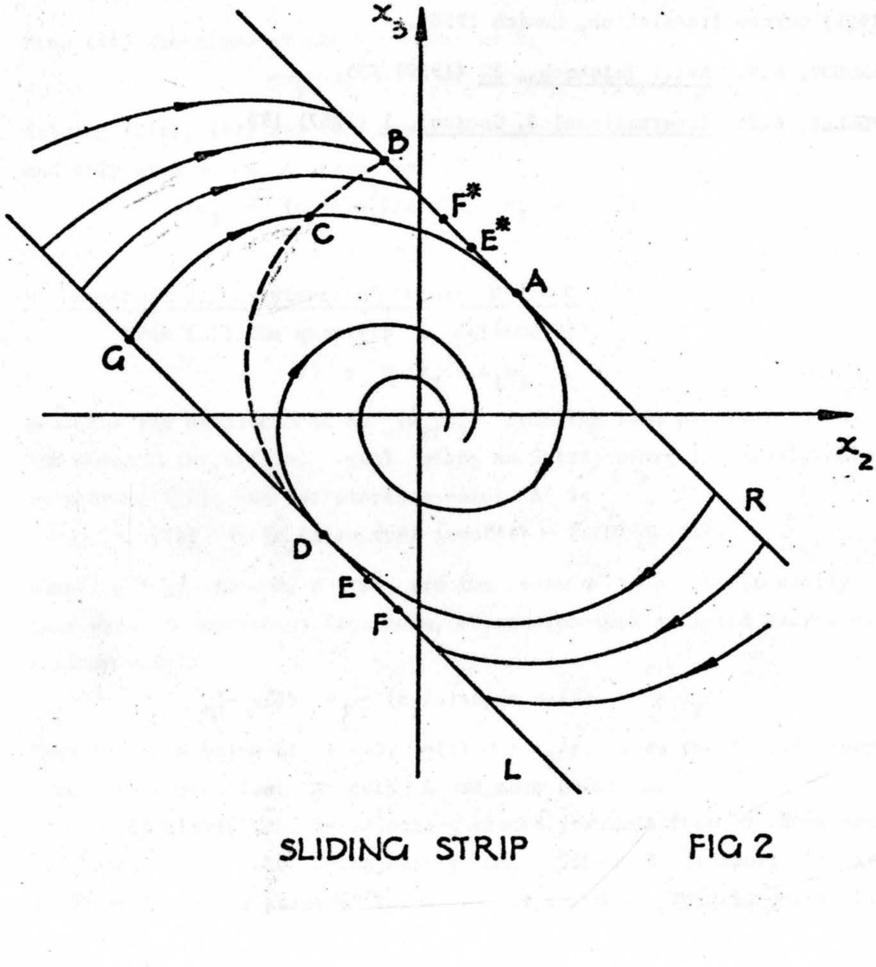


FIG 1
RELAY CONTROL SYSTEM



SLIDING STRIP FIG 2

APPROXIMATE DETERMINATION OF THE STABILITY DOMAIN FOR NONLINEAR SYSTEMS

by Eugenio Sarti

Centro Calcoli e Servomeccanismi, Facoltà di Ingegneria, Università di Bologna, Italy

Introduction

We consider nonlinear systems with an equilibrium point, the stability in the small of which is known. A precise determination of the stability domain $\underline{R}$, when possible, almost always implies hard computational problems. On the contrary, the domain $\underline{R}$ can easily be approximated from below by a subset $\underline{S}$ bounded by a Liapunov hypersurface(1): such an approximation appears to be of interest in most of practical cases. Wegrzyn et al.(2) suggested the use of Liapunov functions of the type " a quadratic form plus an integral of the nonlinearity". Here we propose a simple method, that uses these liapunov functions and is applied to a wide class of systems with one nonlinear element(3).

Determination of the region S

Let us consider the autonomous system, with one nonlinear element

$$\begin{aligned} \dot{x} &= Ax + By \\ y &= f(z) \\ z &= Cx \end{aligned} \qquad (1)$$

where $\underline{x}$ is the state vector, and $\underline{y}$, $\underline{z}$ are the scalar output and input variables of the nonlinear element. $\underline{B}$ is a column matrix, $\underline{C}$ a row matrix; all the eigenvalues of the square matrix $\underline{A}$ are supposed to have a negative real part, except one that is allowed to be identically zero[4]. The characteristic $\underline{f}(\underline{z})$ of the nonlinear element is supposed to be single-valued and piecewise continuous. We suppose, moreover, to be verified the existence and the uniqueness of the solution of (1), starting from any initial condition, at least in a region surrounding the origin.

Let us assume now that the characteristic $\underline{f}(\underline{z})$ lies within an absolute stability sector for the system (1), $(0,\bar{\underline{k}})$[5], for every $\underline{z}$ contained in an interval including the origin; and leaves the sector somewhere else (fig. 1):

$$\begin{gathered} f(0) = 0 \\ 0 < \frac{f(z)}{z} < \bar{k} \quad \forall \quad |z| < z_0 \\ \exists \; |z| \geq z_0 \; : \; \frac{f(z)}{z} \leq 0 \qquad \frac{f(z)}{z} \geq \bar{k} \end{gathered} \tag{2}$$

These formulae replace the well-known Lur'e condition for the absolute stability:

$$f(0) = 0 \qquad\qquad 0 < \frac{f(z)}{z} < \bar{k}$$

and the whole problem might be regarded, under this point of view, as an extension of the absolute stability problem.

Let now

$$V(x) = x.Lx + \beta\, F(z) \tag{3}$$

where

$$F(z) = \int_0^z f(\xi)\, d\,\xi$$

be a Liapunov function for the system (1) and for every characteristic $\underline{f}(\underline{z})$ contained in the sector $(0,\bar{\underline{k}})$. If the characteristic leaves the sector $(0,\bar{\underline{k}})$ so as to satisfy the condition (2), $\underline{V}(\underline{x})$ is still a Liapunov function for the whole region of the state space in which:

$$|C\,x| < z_0 \tag{4}$$

Let us consider a hypersurface

$$V(x) = h \tag{5}$$

such that every point inside satisfies (4). Such a hypersurface is Liapunov for the system (1): therefore it bounds a region S. Among the regions, that are obtained for different values of h, ve must find the widest one, for which (4) holds. Such a region is the required approximation of the stability domain.

Wegrzyn showed that for $\beta \geq 0$ the hypersurfaces (5) are equipotential, and h has the meaning of a generalized potential. The reader will see through the exposition, that the same holds even for $\beta < 0$; then ve have to find the largest value of h, for which (4) is satisfied. The equation

$$C\,x = \pm\, z_0 \tag{6}$$

represents a couple of hyperplanes in the state space; and the condition (4) has the geometrical meaning, that the hypersurface (5) must not intersect either of them.

It must be noted that the region S is not the widest region, that is obtainable as suggested by Wegrzyn. This is because the condition (4) does not supply the maximum h, such that the hypersurface (5) is Liapunov for the given system. The fact looks to be counterbalanced by the practical aspect of the method, that implies a fast routine computation on a medium-size computer, at least when the order of the system is not too high.

The procedure to determine the matrix L and the coefficient β of (3) is well-known[6]. Also the limit value of $\overline{k}$ results from it. Then h is obtained, in practice, by the condition that the hypersurface (5) be tangent to one of the hyperplanes (6). We look for the condition, $V(x) = h$ to be tangent to the hyperplane $Cx=+z_0$. A hyperplane tangent to (5) at a generic point $\underline{x}_0$ is represented by the equation:

$$\mathrm{grad}\ V\Big]_{x_0} \cdot (x-x_0) = 0$$

i.e.

$$[2\ L\ x_0 + \beta C^T f(Cx_0)] \cdot (x - x_0) = 0$$

In order that this hyperplane may be coincident with $Cx = z_0$, the following linear system must be solved:

$$Cx_0 = z_0$$

$$L\,x_0 + \frac{1}{2}\,\beta\ C^T f(z_0) = C^T \alpha$$

where α is a still indeterminate normalizing coefficient. By solving for α one obtains

$$\alpha = z_0 \frac{\det L}{\det M} + \frac{1}{2}\ \beta\ f(z_0)$$

where $\underline{M}$ is the matrix of the system, with the structure

$$M = \left\| \begin{array}{c|c} 0 & C \\ \hline -C^T & L \end{array} \right\|$$

We have then

$$h = V(x_0) = x_0 \cdot Lx_0 + \beta\ F(z_0)$$

and by substitution

$$h = \gamma\, z_0^{\,2} + \beta\ F(z_0)$$

where

$$\gamma = \frac{\det L}{\det M}$$

and also

$$\gamma = \frac{\det L}{\det(L + C^T C) - \det L}$$

Note that γ does not depend upon the nonlinear characteristic. Then (5) takes the general form:

$$x.Lx + \beta F(z) = \gamma z_o^2 + \beta F(z_o) \qquad (7)$$

where is, as known:

$$z = Cx$$

$$F(z) = \int_0^z f(\xi)\, d\xi$$

<u>Properties of the hypersurface (7)</u>

It is well to point out here some fundamental properties of the hypersurface defined by (7).

<u>(a) Point of tangency with the hyperplane</u> $Cx=+z_o$. For $z=z_o$ the equation (7) is reduced to:

$$x \,.\, Lx = \gamma\, z_o^2 \qquad (8)$$

then all the hypersurfaces (7), that are obtained for different nonlinear characteristics, are tangent to $Cx = z_o$ at the same point $\underline{x}_o$.

<u>(b) Intersection with the hyperplanes parallel to</u> $Cx = z_o$. Let

$$Cx = \bar{z} \qquad (9)$$

be the equation of the hyperplane, with $-\underline{z}_o < \bar{\underline{z}} < +\underline{z}_o$. The system of (7) and (9) must be solved: then (7) itself is reduced to a second degree equation, and we obtain a <u>generalized ellipse</u> (i.e. a hypersurface of order $\underline{n}$-2) in the state space. The orientation of its axes, and the rate of their length (that means: the position and the shape of the ellipse) do not depend on the nonlinear characteristic; the length itself, i.e. the extension of the ellipse, depends on the function of $\bar{\underline{z}}$:

$$F(z_o) - F(\bar{z})$$

that is represented by the shaded area in fig. 2. The extension of the ellypse increases with the area if β is positive.

(c) Let now $y = f(z)$ be an odd function of z and be included, for $|z| < z_o$, between two straight lines, $y = k_1 z$ and $y = k_2 z$. By substituting $k_1 z$ and $k_2 z$ for $f(z)$ in (7), two generalized ellipsoids are obtained:

$$V_1(z) = x.(L + \tfrac{1}{2}\beta k_1 C^T C)x = (\gamma + \tfrac{1}{2}\beta k_1) z_o^2$$

$$V_2(x) = x.(L + \tfrac{1}{2}\beta k_2 C^T C) x = (\gamma + \tfrac{1}{2}\beta k_2) z_o^2 \qquad (10)$$

They are tangent to the hyperplane $Cx = z_o$ at the point x_o. Let be, for instance, $\beta k_1 < \beta k_2$: then the second ellipsoid is contained in the first. As $f(z)$ is odd, then $F(z)$ is even and the hypersurface (7) is symmetric with respect to the origin. Under this condition it results immediately from (b) that the hypersurface (7) is completely contained in the first ellipsoid and contains the second one.

(d) Let h be increased. The ellipses, that are described in (b), expand, and therefore the whole hypersurface (7) is expanded. This happens whatever be the sign of β. This means that the Liapunov hypersurfaces $V(x) = h$ are equipotential even if β is negative;

Regions S bounded by ellipsoids

Let be $\beta = 0$ in (3), and consequently in (7). Then we obtain a value of the absolute stability limit, that is generally smaller then $\bar{k}$; as a consequence, the limit value z_o', beyond which the characteristic leaves the absolute a new coefficient γ are obtained. The equation (7) is reduced to the same form of (8):

$$x.L'x = \gamma' z'^2_0$$

Then the boundary of the region S is reduced to an ellipsoid, the shape of which does not depend on the nonlinear characteristic. It is clear that such a region S is a worse approximation of the domain R, then the region obtainble by assuming $\beta \neq 0$, i.e. by taking into account the nonlinear characteristic.

Regions T

Until now we dealt with the subsets S of the domain R, that are bounded by a Liapunov hypersurface: they are characterized by the fact that a trajectory of the system (1), starting from an internal point, can not leave them. More generally, we indicate now as T any set in the state space, from the elements of which the trajectories of the system are asymptotically stable: a trajectory starting from a set T may belong not entirely to it.

A region of the state space, that is fully contained in a region S, is of course a region T. A particular case has already been found: from the property (c) of the hypersurface (7) it appears that the ellipsoid (10) bounds a region T for the system (1). This means that whenever we know that f(z) is odd[7], and is bounded by a straight line (from upper or below, depending on the sign of β), we are able to approximate the stability boundary by an ellipsoid, even if the nonlinear characteristic is not entirely known.

In particular, if β is positive, the ellipsoid (8):

$$x \cdot L\, x = \gamma\, z_0^{\,2}$$

that is obtained from (10) when the line $y = k_2 z$ coincides with the axis of the ascissas, bounds a region T for all systems (1), the nonlinear characteristic of which is odd and satisfies (2). If β is negative, such a region T is bounded by the ellipsoid

$$x.(L + \frac{1}{2}\,\beta\,\bar{k}\,C^T C)\,x = (\gamma + \frac{1}{2}\,\beta\,\bar{k})\,z_0^{\,2} \,.$$

This type of approximation of R by an ellipsoid, bounding a region S or T, is still of interest because it is some kind of extension of the basic idea of absolute stabilty to systems that are locally stable: it gives a stability region, that is independent of the particular shape of the nonlinear characteristic.

First example: a second order system

The case of a second order system, although very particular, makes it possible to perform a very accurate analysis of the results, that is much more difficult for systems of a more complicated structure. We consider the system of fig. 3: by a suitable choice of the state variables the equations (1) are obtained, with

$$A = \begin{Vmatrix} -1 & 0 \\ 0 & 0 \end{Vmatrix} \qquad B = \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} \qquad C = \begin{Vmatrix} 1 & -1 \end{Vmatrix}$$

The system is absolutely stable in the sector $(0,\infty)$ with any value $\beta \geq 1$; it is easy to see that the area of every region S for such a system reaches a maximum for $\beta=2$, whatever the nonlinear characteristic may be. We consider the class of nonlinear characteristics of fig. 4: for $\beta = 2$ the value of $f(z)$ is supposed large enough, that the stability boundary is approximately the same as for $f(z) = \infty$.

The Liapunov hypersurfaces (7) are ellipses:

$$\frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 + k (x_1 - x_2)^2 = \frac{1}{4} + k \qquad (11)$$

Such ellipses, as obtained for some values of k, are compared in fig. 5 with the stability boundary. The ratio q between the area of the region S defined by (11), and the area of the corresponding region R, may be taken as a quality index. It is shown in fig. 6 as a function of k.

The nonlinear characteristics of fig. 7 have been also considered. The corresponding hypersurfaces (7), as shown in fig. 8, are no longer ellipses. They have been compared one to another for equal values of the intersection with the axis of the ascissas, $z_o = 1$, and for equal values of the area included between the characteristic and the axis of the ascissas: with a very good approximation, the same values of q have been obtained.

Regions T: the ellipse (11) bounds a region T for all the systems of fig. 3, the nonlinear characteristic of which is odd and has the property

$$\frac{f(z)}{z} \leq k \qquad \forall \qquad |z| < 1$$

In particular the circle

$$x_1^2 + x_2^2 = \frac{1}{2}$$

bounds a region T for the systems, the characteristic of which lies in the sector $(0,\infty)$ for $|z| < 1$. It may be easily noted that the same circle bounds the region S, that is obtained by supposing $s = 0$. In this case the absolute stability sector is reduced to $(0,1]$.

Second example: third order system

The same system as in fig. 3, but with a third order transfer function of the linear part, is considered. It is represented by

$$A = \begin{Vmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{Vmatrix} \qquad B = \begin{Vmatrix} -1 \\ 0 \\ 0 \end{Vmatrix} \qquad C = \begin{Vmatrix} 0 & 0 & 1 \end{Vmatrix}$$

It is absolutely stable in the sector $[0,8)$, with $\beta = 1$; for such a value two matrices L are obtained. We consider nonlinear characteristics of the same type of fig. 4, with $0 \leq k < 8$. The Liapunov hypersurfces are the ellipsoids

$$\frac{1}{4} x_1^2 + x_2^2 + \frac{1}{2} k x_3^2 = \frac{1}{2} k \tag{12}$$

$$\frac{1}{2} x_1^2 + 2 x_2^2 + (4+\frac{1}{2} k) x_3^2 - x_1 x_2 + 2 x_1 x_3 - 4 x_2 x_3 = \frac{4}{3} + \frac{1}{2} k \tag{13}$$

The ellipsoid (12) has the larger volume for k 8/3. As an example, in fig. 9 the intersection of (12) with the coordinate planes (computed for a value of k very close to 8) is compared with the corresponding intersection of the domain R.

In particular the ellipsoid (13), computed for $k=0$, bounds a region T for all the systems, the nonlinear characteristic of which is odd and belongs to the sector $[0,8)$ for $|z|<1$. If we suppose $\beta = 0$ the stability sector is reduced to $[0,4]$, and we get two ellipsoids, the second of which

$$x_1^2 + 3,618 x_2^2 + 4,000 x_3^2 - 3,236 x_1 x_2 + 3,236 x_1 x_3 - 6,472 x_2 x_3 = 1$$

has the larger volume. It is used to bound a region S for all the systems, the nonlinear characteristic of which lies in the sector $[0,4]$ for $|z|<1$.

The same system has been examined for different values of k and differently shaped nonlinear characteristics. The resulting ratio between the volumes of the regions S and R is approximately constant, provided that the characteristic leaves the stability sector quickly enough: this is because a smoothed characteristic in the region $\underline{z} = \sim \underline{z}_0$ produces a wider region $\underline{R}$, and therefore the quality of the approximation becomes worse.

In other words, at least for the present class of third order systems, the extension of the stability domain appears to depend more on the way how the nonlinear characteristic leaves the stability sector, and less on its behaviour in the interior of the sector itself.

Conclusion

A practical method has been shown, that follows directly from the concept of absolute stability. The main computational problem, that it involves, is to determine the matrix $\underline{L}$: this requires, as known, to solve on a digital computer a set of simultaneous algebraic equations. The quality of the results is often satisfactory, as shown in the examples.

Notes

(1) See, e.g., J.LaSalle, S.Lefschetz, Stability by Liapunov's direct method, Academic Press, New York, 1961, chapter 2.

(2) S.Wegrzyn, J.C.Gille, O.Paulinski, P.Vidal, The stability domain with respect to initial conditions, Proc. 3rd IFAC Conference, London 1966

(3) A short time after writing this paper, the author became acquainted with a communication by S.Weissenberger, Application of results from the absolute stability problem to the computation of finite stability domains (IEEE Trans. Vol. AC 13, p. 124, 1968) on the same subject, but

concerning with the particular case, where a quadratic form is used as a Liapunov function. This assumption, as shown in the text, leads to more restrictive and less approximate results.

(4) Under restrictive conditions, other cases may be considered, where some of the eigenvalues of $\underline{A}$ have a zero real part: see M.A.Aizerman, F.R.Gantmacher, Absolute stability of regulator systems, Holden-Day, San Francisco, 1964, chapter 4.

(5) The lower limit may be included if $\underline{A}$ is nonsingular; also the upper limit is often included in the interval, depending on the way followed to determine $\underline{k}$.

(6) M.A.Aizerman, F.R.Gantmacher, op.cit.

(7) This condition, $\underline{f}(\underline{z})$ to be odd, may be easily removed: it is sufficient, in order to do this, to determine the hyperplane $\underline{Cx} = -\underline{z}_1$, that is tangent to (7) on the opposite side of $\underline{Cx} = \underline{z}_0$ with respect to the origin. This means to solve for z_1 the equation

$$\gamma z_1^2 + \beta F(z_1) = \gamma z_0^2 + \beta F(z_0)$$

Then it is easy to prove that the hypersurface (10), with $\underline{z}_0$ replaced by $\underline{z}_1$, bounds a region $\underline{T}$.

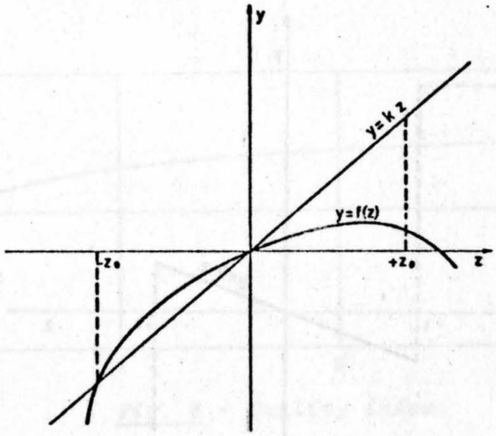


Fig. 1 - Nonlinear characteristic, satisfying the condition (1)

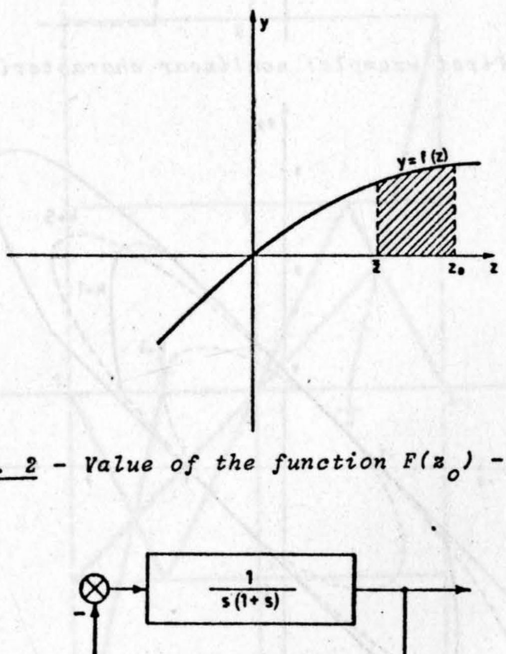


Fig. 2 - Value of the function $F(z_o) - F(\bar{z})$

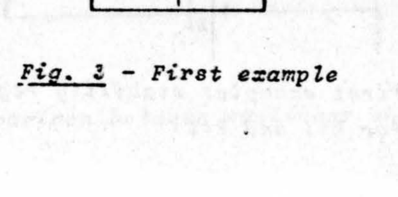


Fig. 3 - First example

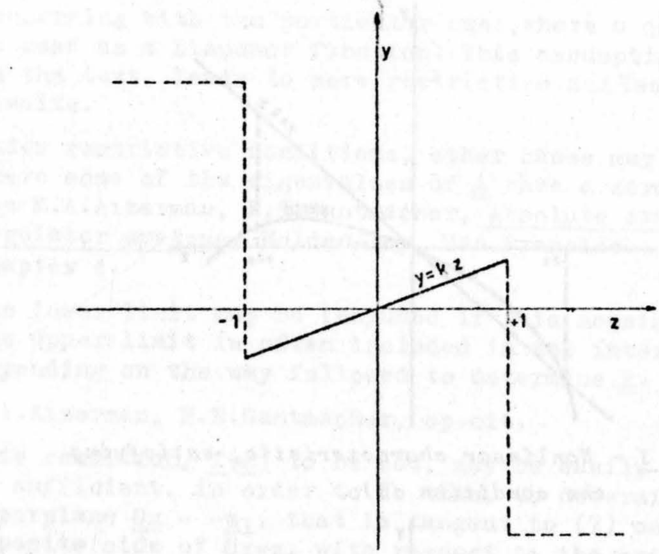


Fig. 4 - *First example: nonlinear characteristic*

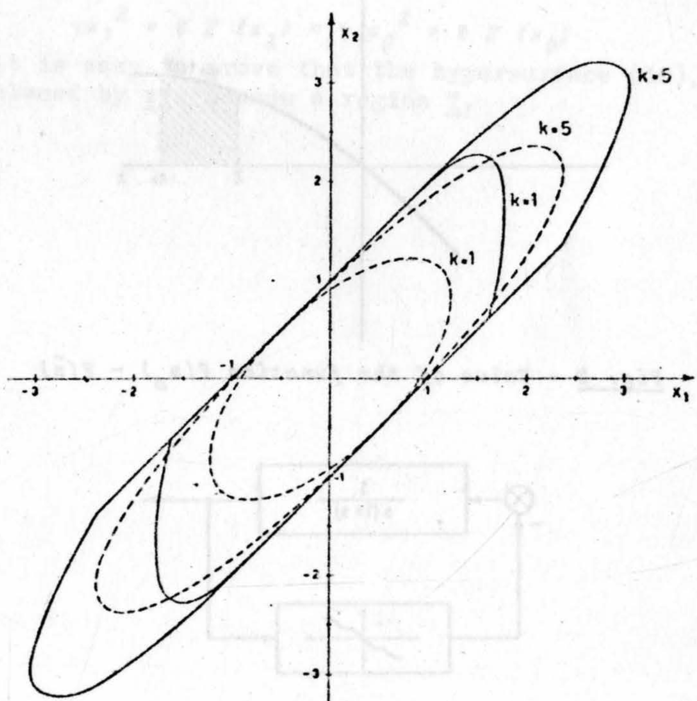


Fig. 5 - *First example: stability regions, for k=1 and k=5*

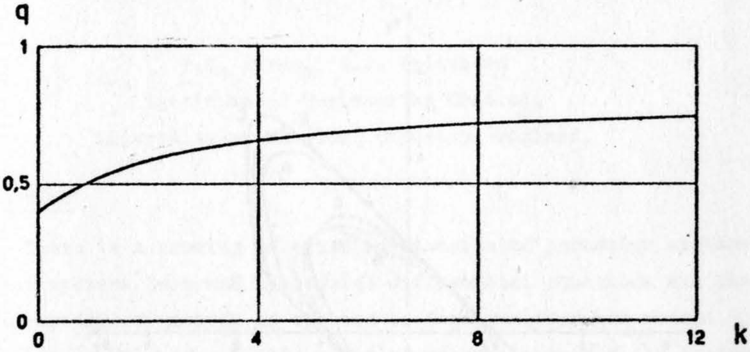


Fig. 6 - *Quality index*

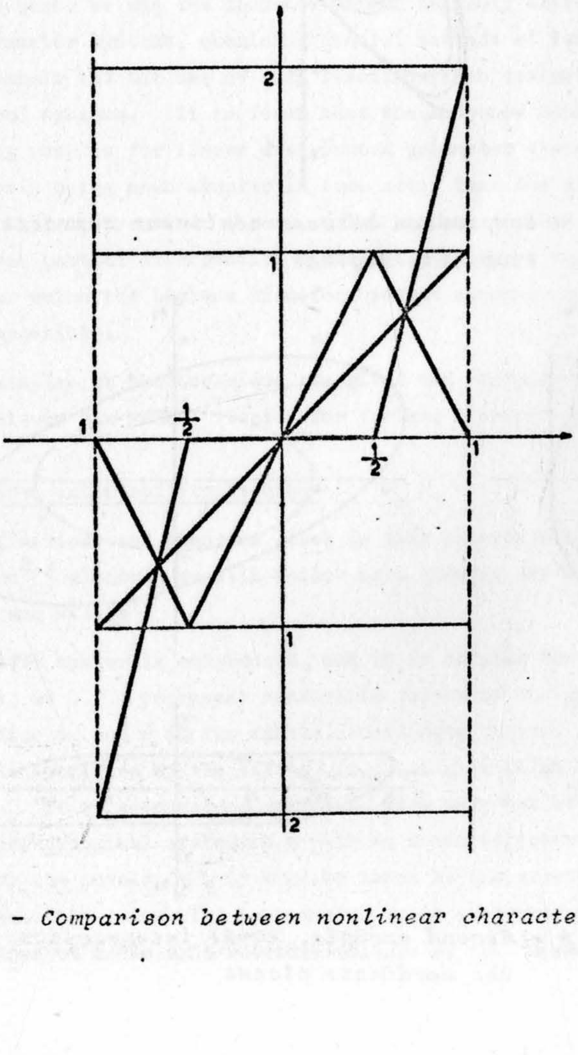
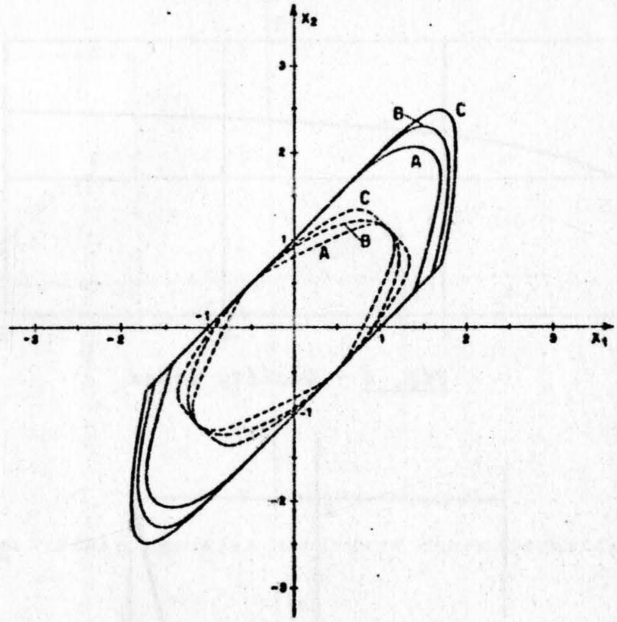
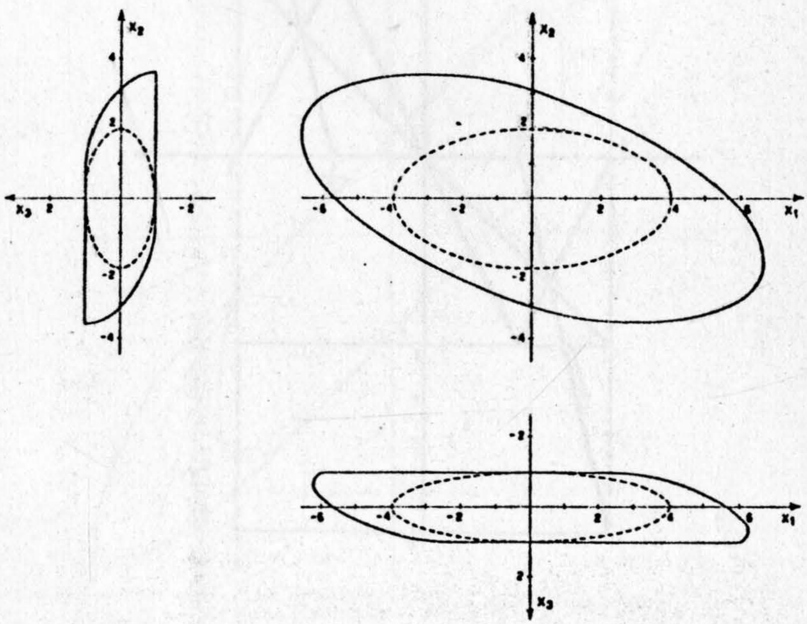


Fig. 7 - *Comparison between nonlinear characteristics*



<u>Fig. 8</u> - Comparison between nonlinear characteristics stability regions



<u>Fig. 9</u> - Second example, k=~8: intersections with the coordinate planes

ON THE CONSTRUCTION AND USE OF LIAPUNOV FUNCTIONALS

P.C. Parks, A.J. Pritchard
Institute of Engineering Control,
University of Warwick, Coventry, England.

Introduction

There is a growing interest in distributed parameter systems, i.e. dynamical systems governed by partial differential equations and their analysis by direct methods as opposed to finite difference models. Wang[1] has recently published a bibliography of papers in this field concerned with optimal control and stability.

In this paper we use the Liapunov method suitably extended to distributed parameter systems, examining general methods of constructing Liapunov functionals and the use of such functionals in designing feedback loops for control systems. It is found that the Liapunov method can give some interesting results for linear distributed parameter systems, the stability analysis being much simpler in some cases than the classical analysis using Laplace Transform theory. The Liapunov method can be extended to cover partial differential equations with space dependent coefficients for which the Laplace Transform method becomes much more difficult or impossible.

Four examples of the technique are given and comparisons with classical techniques are made. Topics for further research are suggested.

1. Basic Concepts and Stability Theorem

The definitions and theorems given in this section are essentially those of Movchan[2,3] although we will follow more closely the interpretation given by Knops and Wilkes[4].

A specific system is considered, and it is assumed that the variables $z_1(x, t)$, $z_2(x, t)$ .... represent measurable values of the physical quantities at time t, and x in the spatial coordinate vector. The state of the system is specified by the set $(z_1, z_2, \ldots, t)$ which will be written (ζ, t). If we associate a metric, ρ, with this set we may represent the motion of the dynamical system as a path in a metric space. One such path, denoted by the points (ζ^o, t) will be taken as the unperturbed path and all other possible paths will be referred to as perturbed paths. The metric, ρ may then be taken as a suitable measure of the "distance" of a point in the perturbed path from a point in the unperturbed path. A further metric, ρ_o will also be introduced, such that the metrics ρ and ρ_o are zero for $\zeta = \zeta^o$. The metric ρ_o will serve as a generalized measure of the initial disturbance and ρ will serve as a measure of the disturbance at subsequent times t.

Definition of Stability

The perturbed path originating at the point (ζ_o, t_o) is compared with all other possible paths originating in a neighbourhood of this initial point, and it is said to be stable with respect to the metrics ρ and ρ_o if

(i) the metric $\rho(\zeta, t)$ is a continuous function of t for any non-degenerate path,

(ii) Given any $\varepsilon_1 > 0$, a number $\delta_1(\varepsilon_1, t_o)(> 0)$ can be found such that for any perturbed path at initial instant t_o, the inequality $\rho(\zeta(t_o), t_o) < \varepsilon_1$ is satisfied provided that $\rho_o(\zeta(t_o), t_o) < \delta_1$ is also satisfied.

(iii) Given any number $\varepsilon_2 > 0$, a number $\delta_2(\varepsilon_2, t_o)(> 0)$ can be found such that for any perturbed path $\rho(\zeta, t) < \varepsilon_2$ for all t, provided that at the instant t_o $\rho_o(\zeta(t_o), t_o) < \delta_2$.

If δ_1 and δ_2 are independent of any initial instant t_1, the system is said to be uniformly stable, and if $\rho \to 0$ as $t \to 0$ the unperturbed path is said to be asymptotically stable.

Liapunov Stability Theorem

The unperturbed path will be uniformly asymptotically stable with respect to the metrics ρ, ρ_o if

(i) ρ is a continuous function of t for any non-degenerate path,

(ii) $\rho(\zeta, t)$ tends to zero as $\rho_o(\zeta, t)$ tends to zero, i.e. given $\varepsilon > 0$, there exists δ such that $\rho < \varepsilon$ whenever $\rho_o < \delta$.

(iii) On the subspace consisting of those points which satisfy $\rho(\zeta, t) < R$, where R is a real positive number, there exists a functional $V(\zeta, t)$ corresponding to a real number at each point (ζ, t) with the following properties:

(a) There exists a non-decreasing function $\gamma(\rho)$ such that $\frac{dV}{dt} \leqslant -\gamma(\rho) < 0$.

(b) There exists non-decreasing functions $\alpha(\rho)$, $\beta(\rho)$ such that $\beta(\rho) \geqslant V(\zeta, t) \geqslant \alpha(\rho) > 0$.

(c) The functional $V(\zeta, t)$ is continuous with respect to the metric ρ_o on the set of initial instants t_o. If $\rho \equiv \rho_o$ we may say that the system is uniformly asymptotically stable with respect to a single metric ρ.

Movchan has also extended Chetaev's Instability Theorem to continuous systems, details of which may be found in references[2,3,4].

2. Construction of Liapunov Functionals

It has been shown in the previous section that in order to apply Liapunov's Stability Theorem it is necessary to construct suitable Liapunov functionals. An obvious choice for such a functional is the energy of a dynamic system. Leray[5], Garding[6] and Peyser[7] by using energy integrals and inequalities have obtained estimates for the solution of the Cauchy problem of hyperbolic equations of several variables. Their work is closely related to the field of stability and the energy integrals they generate may be taken as Liapunov functionals. We first review their methods, which have been established rigorously for hyperbolic equations and then use similar methods to generate functionals for a wider class of operators.

(i) Method of Leray and Garding

Let L be a hyperbolic operator of order m. We multiply this operator by an operator, N, of order m-1, and express the product Lu Nu as a gradient q(u) plus a quadratic form Q(u) in derivatives of order less than m. If G is a lens shaped domain, bounded by space-like hypersurfaces B, we have

$$\iint_G Nu\, Lu\, dx_1\, dx_2 \ldots dx_n\, dt = \int_B q(u)\, dx_1\, dx_2 \ldots dx_n + \iint_G Q(u)\, dx_1\, dx_2 \ldots dx_n\, dt.$$

Leray and Garding[5,6] show that it is possible to choose $\int_B q(u)\, dx_1\, dx_2 \ldots$ to be positive definite if N is chosen so that its characteristic surfaces separate those of L.

Let $N(\xi)$, $N(\xi)$ be the characteristic forms of L and N and χ a direction which is space-like[8] for the operator L. Then we choose

$$N(\xi) = \frac{d}{d\lambda} L(\xi + \lambda\chi)\Big|_{\lambda=0}$$

We now take $\int_B q(u)\, dx_1\, dx_2 \ldots$ as the Liapunov functional, V, and since $Lu = 0$, we have

$$V = \int_B q(u)\, dx_1\, dx_2 \ldots\ldots dx_n = - \iint_G Q(u)\, dx_1\, dx_2 \ldots dx_n\, dt.$$

$$\frac{dV}{dt} = - \int_B Q(u)\, dx_1\, dx_2 \ldots dx_n$$

(ii) Method of Peyser

Consider a hyperbolic operator $L(u)$ of order m for a function $u(x, t)$ of two variables x, t and a solution u of $L(u) = 0$ in a slab $\Sigma: 0 \leq t \leq T$ and assume u has in Σ compact support. We choose an operator $N(u)$ from $L(u)$ by formally differentiating the expression $L(u)$ with respect to the symbol $\frac{\partial}{\partial t}$.

Peyser[7] shows that

$$\iint_{\Sigma} N(u)\, L(u)\, dx\, dt = \int \left[q(u, T) - q(u, 0)\right] dx + \iint_{\Sigma} Q(u)\, dx\, dt$$

where $Q(u)$ is a positive definite quadratic form. We take $\int q(u)\, dx$ as the Liapunov functional.

The methods outline above may now be generalised to other operators. We shall refer to the method as the P-Method and it will consist of multiplying the operator L by a suitable operator N and integrating by parts to obtain the Liapunov functional. The operator N may be chosen by either the method of Peyser or Leray and Garding. For systems governed by hyperbolic operators the resulting Liapunov functional will be positive definite, however for more general systems the functional will only be positive definite if some of the parameters of the system are suitably bounded. This leads, by making use of integral inequalities, to criteria for stability. Some of the more useful of these inequalities will be reviewed in the next section, but first we illustrate the methods for a single, one dimensional parabolic equation, and a pair of equations which govern the vibrations of a damped rotating shaft.

Diffusion Equation

We consider the equation

$$L(T) = T_t - kT_{xx} = 0 \text{ where } T = 0 \text{ at } x = 0, 1$$

and T_t, T_x denote partial derivatives of T with respect to t and x. The operator N is easily found by the P - Method. We have $N = T$, so that

$$\iint N(T)\, L(T)\, dx\, dt = \tfrac{1}{2} \int_0^1 T^2\, dx - k \iint TT_{xx}\, dx\, dt$$

$$= \tfrac{1}{2} \int_0^1 T^2\, dx + k \iint T_x^{\,2}\, dx\, dt$$

The Liapunov functional $V = \tfrac{1}{2} \int_0^1 T^2\, dx$

and $\frac{dV}{dt} = -k\int_0^1 T_x^{\,2}\, dx$

The Liapunov functional is the energy of the system and although we have only considered a relatively simple example here, the extension to more general systems of parabolic equations is easily carried through. Wang[9] and Hsu[10] have carried out a stability analysis of such systems by using the energy as the Liapunov functional.

Vibrations of a damped rotating shaft

The relevant equations are shown by Bishop[11] to be

$$v_{tt} - \Omega^2 v - 2\Omega u_t + k_1 v_{xxxx} + k_2 v_t - \Omega k_3 u = 0 \quad (1)$$

$$u_{tt} - \Omega^2 u + 2\Omega v_t + k_1 u_{xxxx} + k_2 u_t + \Omega k_3 v = 0 \quad (2)$$

where u, v are the deflection components with respect to moving axes and Ω, k_1, k_2, k_3 are taken to be positive constants. The above equations will be discussed in more detail in section 4, Example 1 where stability criteria will be obtained. Here we make use of the equations to illustrate techniques involved in constructing Liapunov functionals. We consider the case of pinned ends, when

$$u = v = u_{xx} = v_{xx} = 0 \text{ at } x = 0$$

By using the P-Method we easily find that the operator N is composed of $2v_t + k_2 V - 2\Omega u$ and $2u_t + k_2 u + 2\Omega v$. Multiplying equations (1) and (2) by the components of the operator N, adding and integrating by parts over the slab $0 \leq x \leq 1$, $t_o \leq t \leq T$, we obtain

$$\int_o^1 \int_{t_o}^T \Big[(2v_t + k_2 v - 2\Omega u)(v_{tt} - \Omega^2 v - 2\Omega u_t + k_1 v_{xxxx} + k_2 v_t - \Omega k_3 u)\Big] dx\, dt$$

$$+ \int_o^1 \int_{t_o}^T \Big[(2u_t + k_2 u + 2\Omega v)(u_{tt} - \Omega^2 u + 2\Omega v_t + k_1 u_{xxxx} + k_2 u_t - \Omega k_3 v)\Big] dx\, dt$$

$$= \int_o^1 \Big[\Big(v_t^2 + u_t^2 - 2\Omega u v_t + 2\Omega v u_t + k_2(v v_y + u u_t) + \Omega^2(u^2 + v^2) + k_1(v_{xx}^2 + u_{xx}^2)$$

$$+ k_2^2(u^2 + v^2)\Big]_{t_o}^T \Big] dx + \int_o^1 \int_{t_o}^T \Big[k_2(u_t^2 + v_t^2) + (2\Omega^2 k_3 - \Omega^2 k_2)(u^2 + v^2)$$

$$+ k_1 k_2(u_{xx}^2 + c_{xx}^2) + 2\Omega k^3(u_t v - u v_t)\Big] dx\, dt.$$

We have for the Liapunov functional, V

$$V = \int_o^1 \Big[v_t^2 + u_t^2 + 2\Omega(v u_t - u v_t) + k_2(u u_t + v v_t) + \Omega^2(u^2 + v^2)$$

$$+ k_2(u_{xx}^2 + v_{xx}^2) + \frac{k_2^2}{2}(u^2 + v^2)\Big] dx$$

and $\frac{dV}{dt} = - \int_0^1 \left[k_2(u_t^{\,2} + v_t^{\,2}) + 2\Omega^2(k_3 - k_2)(u^2 + v^2) + k_1 k_2(u_{xx}^{\,2} + v_{xx}^{\,2}) + 2\Omega k_3(u_t v - u v_t)\right] dx.$

The above examples illustrate the P-Method of generating Liapunov functionals. We have also used the method to regenerate many of the functionals which have been used in the published stability analyses of continuous systems by the Liapunov Method. The method seems applicable in fields varying from aeroelasticity[12,13] to hydrodynamics[14].

(iii) Other Methods

Other methods of constructing functionals have been suggested recently by Sirazetdinov[15] and Buis and Vogt[16]

3. Inequalities

In order to obtain stability criteria and relate the chosen functionals and metrics it will be necessary to use integral inequalities. In this section we review those inequalities which prove to be most useful to the examples which follow in Section 4. A more detailed treatment may be found in references[5, 17].

Schwarz's Inequality

$$\left\{ \int_a^b x(t)\, y(t)\, dt \right\}^2 \leq \int_a^b x^2(t)\, dt \int_a^b y^2(t)\, dt$$

where a, b may be finite or infinite.

Use of the Calculus of Variations

A typical problem of variational calculus is to find a minimum of

$$J(y) = \int_0^1 F(x, y, y_x, y_{xx})\, dx$$

for all functions y(x) which satisfy certain boundary conditions. The general theory tells us that if such a minimizing function exists it must satisfy Euler's equation:

$$F_y - \frac{d}{dx}\left(\frac{\partial F}{\partial y_x}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y_{xx}}\right) = 0$$

As an example, we consider

$$F = y_{xx}^{\,2} - \lambda^4 y^2$$

together with a variety of boundary conditions

(i) $y = 0, \quad y_{xx} = 0 \quad$ at $x = 0, 1$

(ii) $y = 0, \quad y_x = 0 \quad$ at $x = 0, 1$

(iii) $y = 0$ at $x = 0, 1 \quad y_x = 0$ at $x = 0, \quad y_{xx} = 0$ at $x = 1$

(iv) $y_{xx} = 0, \quad y_{xxx} = 0 \quad$ at $x = 0, \quad y = y_{xx} = 0$ at $x = 1$

Euler's Equation is $y_{xxxx} - \lambda^4 y = 0$

The eigenvalues λ^4 are easily computed for the various boundary conditions and are all positive and real. For each eigenfunction $Y(x)$ we have

$$J(Y) = 0$$

$$\therefore \int_0^1 y_{xx}^2 dx \geq \lambda_m^4 \int_0^1 y^2 dx$$

where λ_m is the minimum eigen value.

Corresponding to boundary condition (i) we find $\lambda_m = \pi$.

(ii) " $\lambda_m = 4.73$.

(iii) " $\lambda_m = 3.927$.

(iv) " $\lambda_m = 2.365$.

4. Example I

Consider a damped, vibrating circular shaft shown in figure

The equations of motion are[11]

$$v_{tt} - \Omega^2 v = 2\Omega u_t + k_1 v_{xxxx} + k_2 v_t - \Omega k_3 u = 0$$
$$u_{tt} - \Omega^2 u + 2\Omega v_t + k_1 u_{xxxx} + k_2 u_t + \Omega k_3 v = 0$$

where $k_1 = \frac{EI}{A\rho}, \quad k_2 = \frac{b_e + b_i}{A\rho}, \quad k_3 = \frac{b_e}{A\rho}$

Ω is the imposed angular velocity of the shaft, A the cross-section area, ρ the density, b_i is the internal viscous damping, b_e the external viscous damping, EI the flexural rigidity, and u and v are the deflection components. The conditions of support of the bearings may correspond to clamped, pinned, free or sliding ends.

We consider the functional, derived in Section 2

$$V = \int_0^1 \Big[v_t^2 + u_t^2 + 2\Omega(vu_t - uv_t) + k_2(uu_t + vv_t) + \Omega^2(u^2 + v^2)$$
$$+ k_2(u_{xx}^2 + v_{xx}^2) + \frac{k_2^2}{2}(u^2 + v^2) \Big] dx,$$

and the metrics $\rho_0^2 = \rho^2 = \int^1 (u^2 + v^2)\, dx$

Then $\frac{dV}{dt} = - \int_0^1 \Big[k_2(u_t^2 + v_t^2) + 2\Omega^2(k_3 - k_2)(u^2 + v^2) + k_1k_2(u_{xx}^2 + v_{xx}^2)$

$+ 2\Omega k_3(u_t v - uv_t)\Big] \, dx$

Now $V = \int_0^1 \Big[\tfrac{1}{2}\big[u_t + (k_2u + \Omega v)\big]^2 + \tfrac{1}{2}\big[v_t + (k_2v - \Omega u)\big]^2 + \tfrac{1}{2}(v_t - \Omega u)^2$

$+ \tfrac{1}{2}(u_t + \Omega v)^2 + k_1(v_{xx}^2 + u_{xx}^2)\Big] \, dx$

then $V \geq k_1 \lambda_m^4 \rho^2$

We have $\frac{dV}{dt} \leq - \int_0^1 \Big[k_2 \Big[u_t + \frac{\Omega k_3}{k_2} v\Big]^2 + k_2 \Big[v_t - \frac{\Omega k_3}{k_2} u\Big]^2 +$

$+(2\Omega^2k_3 - \Omega^2k_2 - \Omega^2 \frac{k_3^2}{k_2} + k_1k_2\lambda_m^4)(u^2 + v^2)\Big] \, dx$

$\therefore \frac{dV}{dt} \leq - \gamma\rho^2,$

where $\gamma = 2\Omega^2k_3 - \Omega^2k_2 - \Omega^2 \frac{k_3^2}{k_2} + k_1k_2 \lambda_m^4$.

The system is therefore stable with respect to the metric ρ if $\gamma > 0$ i.e.

$$\frac{k_2}{k_2 - k_3} > \frac{\Omega}{\lambda_m^2\sqrt{k_1}}$$

i.e. $$\frac{b_e + b_i}{b_i} > \frac{\Omega}{\lambda_m^2\sqrt{k_1}}$$

This is a result obtained by Bishop[11] using a principal mode analysis. The values of λ_m corresponding to

(i)	pinned, pinned ends is		$\lambda_m = \pi$.
(ii)	clamped, clamped	"	$\lambda_m = 4.73$.
(iii)	sliding, pinned	"	$\lambda_m = 3.927$.
(iv)	free, pinned	"	$\lambda_m = 2.365$.

Example 2: Temperature control in a uniform insulated bar

Consider the temperature control problem shown in figure 2. The uniform insulated bar of unit cross-section is to be heated to a given temperature T_i by means of a heating element at the end $x = 0$. The heat input, h, is to be controlled by a suitable error actuated loop using the temperature measured at a suitable point in the bar.

The temperature $T(x, t)$ satisfies the partial differential equation

$T_t = k\, T_{xx}$, where k is the diffusivity, and the boundary conditions are

$$-K\, T_x \Big|_{x=0} = h, \text{ K being the thermal conductivity,}$$

and

$$T_x \Big|_{x=\ell} = 0.$$

Let us consider first the Liapunov functional

$$V = \int_{x=0}^{\ell} (T - T_i)^2\, dx.$$

This functional may be found formally by the method of Peyser on multiplying the partial differential equation for T by T and integrating with respect to t and x.

Now $$\frac{dV}{dt} = 2 \int_{x=0}^{\ell} (T - T_i)\, T_t\, dx = 2 \int_{x=0}^{\ell} (T - T_i)\, k\, T_{xx}\, dx$$

$$= \frac{2k}{K} (T - T_i)\Big|_{x=0} h - 2k \int_{x=0}^{\ell} (T_x)^2\, dx$$

on integrating by parts and using the boundary conditions. The form of $\frac{dV}{dt}$ suggests at once that

(i) the temperature should be measured at x = 0.

(ii) that h should be proportional to $-(T - T_i)\Big|_{x=0}$

By adding to V the additional term $\lambda\{(T - T_i)\Big|_{x=0}\}^2$ $\frac{dV}{dt}$ is found to have the additional term $2\lambda\{(T - T_i)\Big|_{x=0}\}\, T_t\Big|_{x=0}$ which suggests an alternative form for h with a derivative term

$$h = -G(T - T_i)\Big|_{x=0} - \frac{\lambda K}{k}\, T_t\Big|_{x=0}.$$

In both cases by use of the Schwarz inequality it may be shown that $\frac{dV}{dt} \leq -\gamma V$ where $\gamma > 0$ and that the system is asymptotically stable in terms of the norm ρ, where $\rho^2 = V$.

It might be argued that a better measure of the initial disturbance at t = 0 might be $\sup_{0<x\leq\ell} (T - T_i)^2$. This is admissable as a metric ρ_o since $\rho^2 = V \leq \rho_o \ell$.

Consider now the classical Laplace transform analysis of the first scheme in which $h = -G(T - T_i)\Big|_{x=0}$. The stability analysis involves consideration of the block diagram containing a transcendental transfer function in s as shown in Fig.3. We apply the Nyquist criterion to the open loop transfer function by considering a contour consisting of the imaginary axis indented at the origin and completed by a large semi-circle in the right-hand half plane. We are concerned essentially with the behaviour of $\frac{1}{\sqrt{s}} \cot h \sqrt{s}$ as s traverses this contour, there being no poles of the function inside. This behaviour is shown in Fig.4 and it can be seen that the (-1, 0) point will not be encircled even if G is increased. This analysis is considerably more involved however than the Liapunov method used above.

Example 3(i): Angular position control of a uniform shaft flexible in torsion

The wave equation is another familiar type of partial differential equation and is involved in the following feedback control problem which is illustrated in Fig.5.

The equation of motion of an element of the shaft yields the one dimensional wave equation for angular displacement $\theta(x, t)$

$$I \theta_{tt} = (G J \theta_x)_x = GJ \theta_{xx}$$

where I is the moment of inertia per unit length and GJ the torsion stiffness of the shaft. The boundary conditions are

$$T = -GJ\theta_x\Big|_o, \quad \theta_x\Big|_\ell = 0.$$

A suitable Liapunov functional is the total energy

$$V = \tfrac{1}{2} \int_{x=0}^{\ell} I(\theta_t)^2 \, dx + \tfrac{1}{2} \int_{x=o}^{\ell} GJ(\theta_x)^2 \, dx.$$

V can also be found by the method of Peyser, for which

$$\frac{dV}{dt} = \int_{x=o}^{\ell} \theta_t \, GJ \, \theta_{xx} + GJ \, \theta_x \, \theta_{tx} \, dx$$

$$= \Big[GJ \theta_t \, \theta_x \Big]_{x=o}^{\ell} = \theta_t\Big|_{x=o} T.$$

This suggests that T should be proportional to $-\theta_t\Big|_{x=o}$. However we require for closed loop control that $\theta(x, t) \to \theta_D$ and so we add to V an additional term $\frac{1}{2}\lambda_1(\theta_D - \theta\big|_{x=o})^2$. This yields in $\frac{dV}{dt}$ the additional term $\lambda_1(\theta_D - \theta\big|_{x=o})(-\theta_t\big|_{x=o})$. If now

$$T = \lambda_1(\theta_D - \theta\big|_{x=o}) - \lambda_2 \theta_t\Big|_{x=o}$$

then

$$\frac{dV}{dt} = -\lambda_2(\theta_t\Big|_{x=o})^2.$$ Now considering the norm

$$e = \{ \int^{\ell} (\theta\,)^2 + (\theta\,)^2 + (\theta\, - \theta)^2 \, dx \}^{\frac{1}{2}}$$

and using the Schwarz inequality to give

$$\int_0^{\ell} (\theta_x)^2 \, dx \geq \frac{2}{\ell^2} \int_0^{\ell} (\theta - \theta\Big|_{x=o})^2 \, dx$$

we have $\alpha\rho^2 \leq V \leq \beta\rho^2$ where

$$\alpha = \min \{\tfrac{I}{2}, \tfrac{1}{2}GJ - \varepsilon_1, \frac{\varepsilon_1}{\ell^2}, \tfrac{1}{2}\frac{\lambda_1}{\ell}\}, \; 0 < \varepsilon_1 < \tfrac{1}{2} GJ,$$

$$\beta = \max \{\tfrac{I}{2}, \frac{GJ}{2}, \frac{\lambda_1}{2}\} \Big/ \min \{\varepsilon_2, \frac{1 - \varepsilon_2}{\ell^2}, \tfrac{1}{2}\}, \; 0 < \varepsilon_2 < 1.$$

$\frac{dV}{dt} \not\equiv 0$ unless $\theta_t\Big|_{x=o} \equiv 0$ in which case $\theta = \theta_D$ everywhere.

The analysis suggests measuring θ and its time derivative at $x = 0$ and making the torque proportional to the position error plus velocity damping. Thus delays due to torsional waves travelling along the shaft are avoided.

The classical Laplace transform analysis once again is quite involved. The closed loop appears as in Fig.6. The transcendental transfer function looks superficially similar to the previous example, but with the important different that $\sqrt{s}$ is here replaced by s. This changes the pole pattern and we have to consider a contour as before but indented not only at $s = 0$ but also at $s = \pm \frac{r\pi i}{\ell\sqrt{\frac{I}{GJ}}}$, $r = 1, 2, 3 \ldots$ The essential behaviour is demonstrated by $\frac{1+s}{s} \coth s$ with $s = i\omega$ shown in Fig.7. The branches are connected by large semi-circles to the right so that the point $(-1, 0)$ is not in fact encircled. If the Laplace analysis is persued further to investigate feedback of θ and $\dot{\theta}$ from any point other than $x = o$ it will be found that such an arrangement will be unstable.

Example 3(ii): Extension to a non-uniform shaft

The Laplace transform analysis becomes difficult or impossible if I and GJ vary with x. The Liapunov analysis however still holds. We use the same functional, but we modify the definition of α and β to be

$$\alpha = \min \{\min_{0 \leq x \leq \ell} \frac{I}{2}, \left[\min_{0 \leq x \leq \ell} \tfrac{1}{2} GJ\right] - \varepsilon_1, \frac{\varepsilon_1}{\ell^2}, \tfrac{1}{4}\frac{\lambda_1}{\ell}\},$$

$$0 < \varepsilon_1 < \min_{0 \leq x \leq \ell} \tfrac{1}{2} GJ,$$

$$\beta = \max \{\max_{0 \leq x \leq \ell} \frac{I}{2}, \max_{0 \leq x \leq \ell} \frac{GJ}{2}, \frac{\lambda_1}{2\ell}\} \Big/ \min \{\varepsilon_2, \frac{1-\varepsilon_2}{\ell^2}, \tfrac{1}{2}\}$$

$$0 < \varepsilon_2 < 1.$$

Example 4: Normal acceleration control loop in a uniform flexible missile.

Let us now consider a uniform missile flexible in bending and controlled essentially by putting an angle of incidence of fins located at the centre of gravity. This is an idealisation of certain types of antiaircraft missile and is illustrated in Fig.8. If $y(x, t)$ is the bending displacement of the missile then the equation of motion for an element is

$$(EI\, y_{xx})_{xx} = -m\, y_{tt}$$

where EI is the bending stiffness and m the line density.
The boundary conditions are that $EI\, y_{xx} = (EI\, y_{xx})_x = 0$ at $x = -a$ and $x = a$, y, y_x, y_{xx} continuous at $x = 0$, and that $L = \left[(EI\, y_{xx})_x\right]_{0-}^{0+}$ where L is the normal force due to fin incidence, β.

Taking as a tentative Liapunov functional

$$V = \tfrac{1}{2}\int_{x=-a}^{a} EI\, (y_{xx})^2\, dx + m(y_t)^2\, dx \equiv \rho^2$$

we have

$$\frac{dV}{dt} = \int_{x=-b}^{a} EI\, y_{xx}\, y_{txx} + m\, y_t\, y_{tt}\, dx$$

$$= \int_{x=-a}^{a} EI\, y_{xx}\, y_{txx}\, dx + \int_{x=-a}^{0} y_t(-EI\, y_{xx})_{xx}\, dx + \int_{x=0}^{a} y_t(-EI\, y_{xx})_{xx}$$

$$= \int_{x=-a}^{a} EI\, y_{xx}\, y_{txx} + (EI\, y_{xx})_x\, y_{xt}\, dx + y_t\Big|_{x=0} L$$

$$= y_t\Big|_{x=0} L$$

This suggests that L should be proportional to $- y_t\Big|_{x=0}$. If a feedback loop with accelerometer feedback is to be used this suggests that the accelerometer be placed at $x = 0$ and the output integrated with respect to time to form $y_t\Big|_{x=0}$, thus leading to the loop shown in Fig.9. The Liapunov analysis may be easily extended to a non-uniform missile in the manner of Example 3(ii) using the same functional V.

While an exact Laplace transform analysis could once again be applied we now examine another commonly used technique, that of normal modes.

If we assume the motion is made up of a finite number of normal modes with mode shapes $f_i(x)$, frequencies ω_i and generalised co-ordinates q_i then Lagrange's equations for the circuit of Fig.9 with $n_D = 0$ are of the form

$$A_i\ddot{q}_i + B_i q_i = -K_1K_2(f_1(0)\dot{q}_1 + f_2(0)\dot{q}_2 + \dots + d_n(0)\dot{q}_n)f_i(0)$$

where

$$\omega_i^2 = \frac{B_i}{A_i}.$$ Considering the Liapunov function

$$V = \sum_{i=1}^{n} \tfrac{1}{2} A_i\dot{q}_i^2 + \tfrac{1}{2} B_i q_i^2$$

yields

$$\frac{dV}{dt} = \sum_{i=1}^{n} A_i\dot{q}_i\ddot{q}_i + B_i q_i\dot{q}_i = \sum_{i=1}^{n} -K_1K_2(f_1(0)\dot{q}_1 + \dots + d_n(0)\dot{q}_n)f_i(0)\dot{q}_i$$

$$= -K_1K_2\left(\sum_{i=1}^{n} f_i(0)\dot{q}_i\right)^2.$$

There would be no effective damping by L of a mode having a node at x = 0, but neither would L excite such a mode.

The suggested positioning of the accelerometer at the point where L acts appears to be new and contrasts strongly with the common practice of placing the accelerometer at or near nodes of the lower frequency modes. Considering an approximate open loop frequency response of Fig.9 using the n normal modes we obtain the open loop transfer function

$$K_1K_2 s\left\{\sum_{i=1}^{n} \frac{f_i^2}{A_i s^2 + B_i}\right\}.$$

Indenting a contour as before in the s-plane indented at the poles $s = \pm i\omega_i$ we obtain the Nyquist diagram indicated in Fig.10. If some damping had been included to modify the denominators to $A_i s^2 + C_i s + B_i$ the diagram would take the more familiar form of Fig.11. This shows that for an accelerometer placed at the same point that the force L is applied the circles or 'parasitic loops' all lie in the right-hand half plane and cannot therefore encircle (-1, 0). If the accelerometer were placed at a node of the fundamental mode this particular loop would be eliminated but other parasitic loops would lie in the left hand half plane, as shown in Fig.12 which corresponds to the transfer function

$$K_1K_2 s\left\{\sum_{i=1}^{n} \frac{f_1 g_i}{A_i s^2 + C_i s + B_i}\right\}$$

where g_i is the displacement in the ith mode at the accelerometer position chosen so that $g_i = 0$, but implying that for certain higher frequency modes $f_i g_i < 0$.

Conclusions

This paper has surveyed the theorems and construction methods of the Liapunov functional technique for partial differential equations. A number of linear problems have been tackled, and it has turned out here that the Liapunov method is not only simpler than the classical Laplace transform analysis, but can lead to new design principles - for example, the accelerometer positioning in flexible missiles explained in example 4. The Liapunov method is particularly valuable for linear non-uniform problems illustrated by example 3(ii).

It is in non-linear problems involving partial differential equations that the method promises to be of greatest value and indeed certain non-linear problems have been tackled - for example, non-linear hysteretic damping in [13] and full non-linear form of the Navier-Stokes equations of hydrodynamics in [14]. Many such problems involve the search for stability conditions on certain parameters, and the Liapunov method usually yields sufficient but not necessary conditions deduced from $\frac{dV}{dt}$. Sharp criteria include efficient use of the calculus of variations to establish inequalities between integrals.

There is a need to examine

(i) what problems, old and new, are amenable to Liapunov functional analysis,

(ii) develop new construction methods for functionals,

(iii) devise methods which lead to sharp stability criteria.

The method of Liapunov applied to partial differential equations is still in its infancy, but when one considers the progress in its application to ordinary differential equations in the last ten years one feels confident that much progress towards (i) (ii) and (iii) above will come in the next decade.

References

(1) P.K.C. Wang, 'Theory of Stability and Control for Distributed Parameter Systems (a Bibliography)'. Int. J. Control, 1968, Vol.7, no.2, 101-116.

(2) A.A. Movchan, 'On Liapunov's direct method in problems of Stability of Elastic Systems'. PMM Vol.23, No.3, 483-493, 1959.

(3) A.A. Movchan, 'Stability of Processes with respect to two Metrics'. PMM, Vol 24, No.6, 1506-1524, 1960.

(4) R.J. Knops and E.W. Wilkes, 'On Movchan's theorems for stability of continuous systems'. Int. J. Eng. Science, Vol.4, 303-329, 1966.

(5) J. Leray, 'Hyperbolic differential equations'. Lecture Notes, Institute for Advanced Study, Princeton, 1951-52.

(6) L. Gårding, Solution directe du problème de Cauchy pour les équations hyperboliques. Comptes rendus du colloque pour les equations aux dérivées partielles. Colloque International, CNRS, Nancy (1965), pp.71-90.

(7) G. Peyser, 'Energy integrals for the mixed problem in hyperbolic partial differential equations of higher order', J. Math and Mech., no.6 (1957).

(8) R. Courant and D. Hilbert, 'Methods of Mathematical Physics', Vol.II p.652-667, Interscience 1962.

(9) P.K.C. Wang, 'Control of distributed parameter systems' in Advances in Control Systems - Theory and Applications, Adacemic Press, N.Y. Vol.I, p.75-172, 1964, edited by C.T. Leondes.

(10) C. Hsu, 'Stability analysis of reactor systems via Liapunov's second method'. Trans. ASME, J. Basic Eng., Vol.89, Series D, p.307-10, 1967.

(11) R.E.D. Bishop, 'The vibration of rotating shafts', J. Mech. Eng. Sci., Vol.1, No.1, pp.50-65, 1959.

(12) P.C. Parks, 'Liapunov functionals for aeroelastic problems'. J.Franklin Inst. Vol.283, No.5, p.426-429, 1967.

(13) P.C. Parks, 'A stability criterion for panel flutter via the second method of Liapunov'. AIAA J., Vol.4, No.1, p.175-177, 1966.

(14) A.J. Pritchard, 'A study of two of the classical problems of hydrodynamic stability by the Liapunov method'. J. Inst. Math. Applic. Vol.4, pp.78-93, 1968.

(15) T.K. Sirazetdinov, 'On the theory of stability of processes with distributed parameters'. Prik Mat. Mekh., Vol.31, No.1, pp.37-48, 1967.

(16) G.R. Buis and W.G. Vogt, 'Liapunov functionals for a class of wave equations'. Electronics Letters, Vol.4, No.7, pp.128-130, April 1968.

(17) Hardy, G.H., Littlewood, J.E. and Pólya, G. 1959. Inequalities. Oxford University Press.

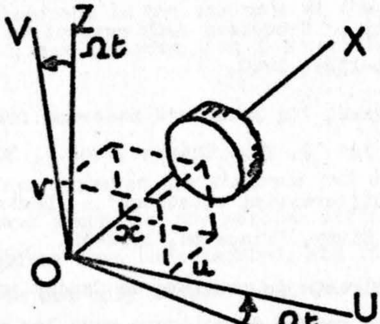


FIG. I. SHAFT.

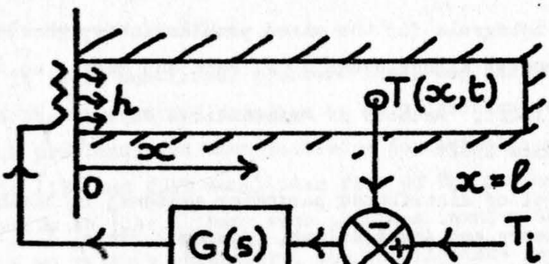


FIG. 2. HEATING OF BAR.

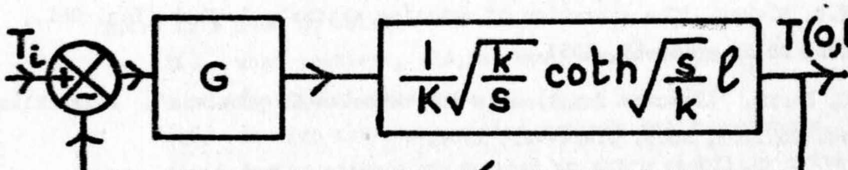


FIG.3. TRANSFER FUNCTION.

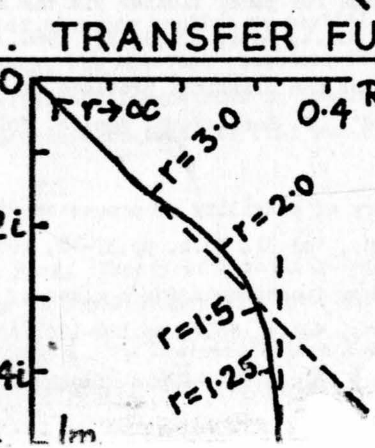


FIG 4

NYQUIST PLOT:

$$\frac{1}{\sqrt{s}}\coth\sqrt{s},$$

$$s = i\omega,$$

$$r = \sqrt{\omega}$$

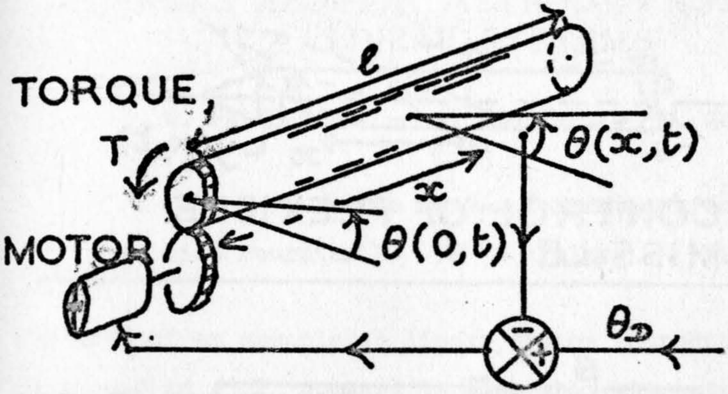


FIG. 5. ANGULAR CONTROL OF SHAFT.

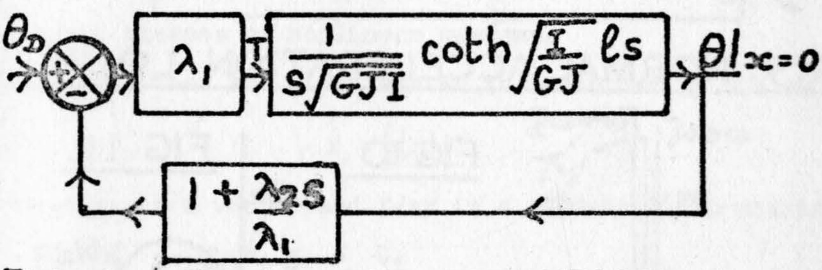


FIG. 6. TRANSFER FUNCTION.

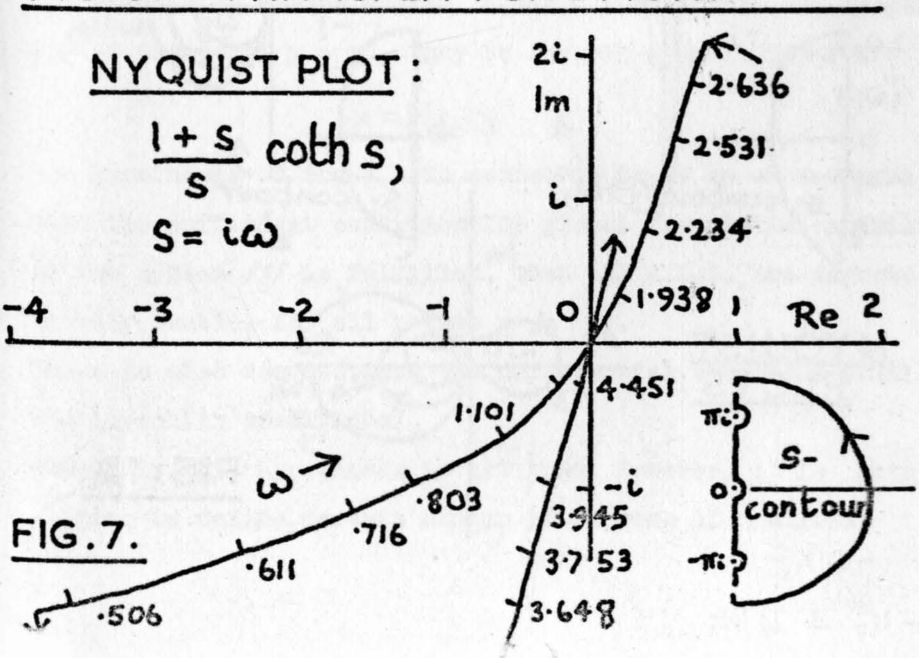


FIG. 7.

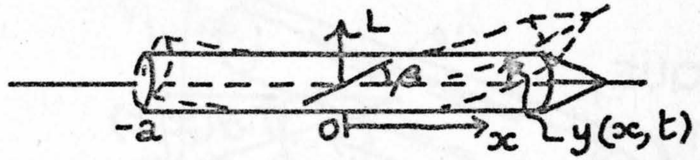


FIG. 8. CONTROL OF FLEXIBLE MISSILE.

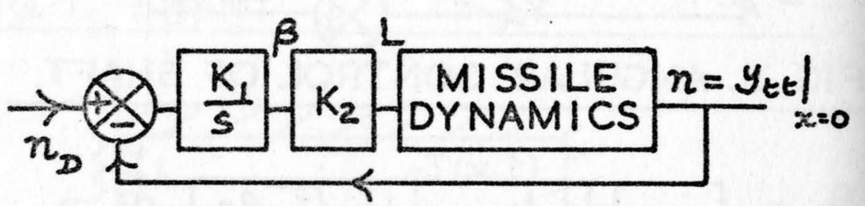


FIG. 9. NORMAL ACCELERATION LOOP.

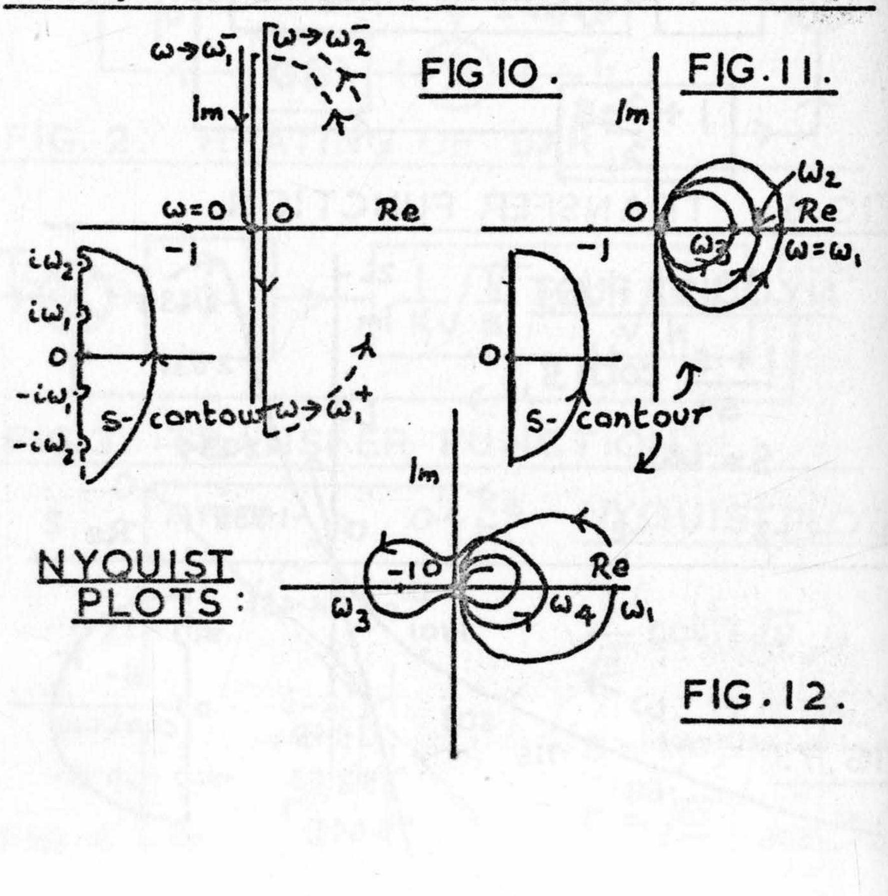


FIG. 12.

REMARKS ABOUT A METHOD OF ASSOCIATED LINEAR SYSTEMS

O. PALUSINSKI - A. LAURANS - M. GAUVRIT

Research Center for Automation

Cité Fournier - 78 - Velizy - Villacoublay

The idea of an associated linear system introduced in 3 and developed in 1,2 appears to be highly attractive. This is in the analysis of nonlinear systems stability, due to the simplicity of a stability criterion which can be formulated for certain classes of nonlinear systems.

We recall that the given system is described by the equation

$$\dot{\underline{x}} = f(\underline{x})\,\underline{x} \qquad /1/$$

where $\underline{x}$ is a vector and f/x/ is a quadratic nxn matrix

f/x/ ≠ o when $\underline{x} \neq 0$.

The associated linear system /A.L.S./ relevent to the system /1/ at the point $\underline{x} = \underline{x}_0$ may be described as follows:

$$\dot{\underline{x}} = f(\underline{x}_0)\,\underline{x}$$

The hypothesis of the A.L.S. method is based on an assumption that the sufficient condition for global asymptotic stability of the system /1/ is fulfilled, when all A.L.S. are asympto - tically stable, for all points $\underline{x} = \underline{x}_0$.

There is also assumed that the differential equations fulfill the Lipschitz conditions.

Generally this hypothesis is not true. However, it is inte - resting to define certain number of classes of nonlinear systems, whose stability may be investigated using this method. This subject has already been discussed in a number of papers.

In this paper, the authors give results with regard to certain classes of control systems containing a linear plant and nonlinear controller, and indicate how to apply a linear stabilization method to certain nonlinear systems. Next, they present a new, fairly general class of systems containing several nonlinearities.

1. CLASS OF ONE - DIMENSIONAL SYSTEMS

The results presented below apply to the class of control systems containing a linear plant. The last is defined by the differential equation L/p/s = ku, where p is the derivative operator, L/p/ - n-th degree polynomial of p, u - out put of the controller with nonhysteresis characteristic contained in the first and third quadrant /see Fig.1 and 2/.

This may be described: u = f/x/x.

1.1. The case when L/p/ = p /M/p//; where M/p/ is a polynomial of p, of /n-1/-th degree /velocity-controlled system/.

Equation of the system may be written as:

$$x^{(n)} + a_1 x^{(n-1)} + \dots\dots\dots + a_{n-1} x^{(1)} + k f(x) x = 0$$

Let $\underline{x}$ be a vector of components x, $x^{/1/}$, ... $x^{/n-1/}$, δ be a linear combination of components of $\underline{x}$.

Applying the Lurye-type Liapunov function:

$$V(\underline{x}) = \underline{x}^T B \underline{x} + \lambda \int_0^{\delta} f(u) u \, du \qquad /2/$$

/where: B is a positive definite matrix, the elements of which depend on the coefficients of M/p/, λ is a positive number/, it has been proved [5,7] that the method of A.L.S. was valid for n = 1, 2, 3.

Accordingly to [6] it can be shown that such a Liapunov function does not give the stability conditions for n $\geqslant$ 5. It seems to be possible to obtain these for n = 4.

It is worth while to mention that for n = 3 the method of A.L.S. is valid for differential equations of the following type:

$$x''' + a_1 x'' + f_2(x'')x' + a_3 x = 0 \qquad \text{/see ref } 7/$$

$$x''' + f_2(x')x'' + a_2 x' + a_3 x = 0 \qquad \text{/see ref } 4/$$

The above can be shown by using the Liapunov function of the type mentioned above.

It is interesting to study as an example the problem of stabilizing a third-order system which belong to the class described above.

In the case the nonlinear characteristic may be arbitrary, provided it is of a nonhysteresis type, included in the first and third quadrant.

The phase lead element may be introduced for stabilizing the system, similarly as for linear systems. This is shown in fig. 3.

Thus the equation of the system is:

$$x''' + a_1 x'' + [a_2 + \alpha K f(x + \alpha x')]x' + K f(x + \alpha x')x = 0$$

The problem arises how to find a sufficient condition for stability of the system whenever f/u/u is restricted to the 1-st and 3-rd quadrant and of a nonhysteresis type.

For this let us consider the Liapunov function

$$V = \underline{x}^T B \underline{x} + 2a_2 k\eta \int_0^{\delta} f(z) z\,dz$$

where $\underline{x} = \begin{bmatrix} x \\ x' \\ x'' \end{bmatrix}$ and $B = \begin{bmatrix} a^2 & a_1 a_2 & a_2 \\ a_1^2 a_2 & a_1^2\eta + a_2(\alpha\eta a_1 - 1) & a_1\eta \\ a_2 & a_1\eta & \alpha\eta a_1 \end{bmatrix}$

η is a fixed parameter.

The derivative $\dot{V}$ along the trajectory is:

$$\frac{dV}{dt} = x^T A x \quad \text{with} \quad A = \begin{bmatrix} -2a_2 k f(\delta) - \alpha a_2 k f(\delta) & & 0 \\ -\alpha a_2 k f(\delta) - 2a_1 a_2(\eta - 1) & & 0 \\ 0 & 0 & -2a_1\eta(\alpha a_1 - 1) \end{bmatrix}$$

So, the following conditions may be obtained:

$$a_1 > 0 \qquad a_2 > 0$$

$$\alpha > \frac{1}{a_1} \qquad 0 < k f(\delta) \leq E - \frac{4a_1(\eta - 1)}{\alpha^2}$$

where E is a real, positive, arbitrarily large number, since η is an arbitrary number satysfying $\eta > 1$.

It is worth while to note that the same condition $\alpha > 1/a_1$ is received when applying the Hurwitz criterion to the linear system, obtained by substitution of $\int f/z/ = k'$, if it is required that this system is to be globally asymptotically stable for each value of k'.

1.2. Case of $L/p/ = \prod_{i=1}^{n} (p+\alpha_i)$ with $\alpha_i > 0$. Equation of the system may be written in the form:

$$x^{(n)} + a_1 x^{(n+1)} + \cdots\cdots + a_{n-1} x^{(1)} + [a_n + K f(x)]\, x = 0$$

Using the Popov's theorem [9], it may be proved that the method of A.L.S. is adequate for $n \leq 5$.

In the particular case when all α_1 were equal, it was also shown that the method is valid for any value of n.

It is also possible to prove that the control systems with a conservative plants described by:

$$x^{(n)} + a_1 f x^{(n-1)} + a_2 x^{(n-2)} + a_3 f x^{(n-2)} + \cdots\cdots + = 0$$

where $f = f(x, x^{(1)}, \ldots\ldots, x^{(n-1)}, t)$

fulfill the A.L.S. hypothesis [8].

The proof is based on a total energy formula: $V = \dot{q}^T P \dot{q} + q^T R q$ as a Liapunov function for a system represented by the following matrix expression:

$$P\ddot{q} + f D\dot{q} + Rq = 0$$

where P, D, R, are quadratic matrices, and q is a vector. For example if n = 2 m ; q is m - dimensional vector of elements $x, x^{/2/}, \ldots x^{/n-2/}$.

2. CLASS OF SECOND ORDER, MATRIX SYSTEMS

Let us study the stability of the systems defined by a matrix equation

$$P\ddot{q} + D\dot{q} + Rq = 0 \qquad /3/$$

where $q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix}$ and P as well as R are quadratic, symmetric matrices of dimension m, with constant off-diagonal elements, while the diagonal elements P_{ii} and r_{ii} may be functions of certain components of the vectors q and $\dot{q}$, i.e.:

$$P_{ii} = P_{ii}(\dot{q}_i) \qquad /4/$$

$$r_{ii} = r_{ii}(q_i) \qquad /5/$$

The elements d_{ij} of a matrix D can also nonlinear functions of the vectors q and $\dot{q}$ components :

$$d_{ij} = d_{ij}(q, \dot{q}) \qquad /6/$$

It is also assumed that the Lipschitz conditions are fulfilled for /3/.

<u>Theorem</u>

If the matrices P, R, D are positive definite, the system of equations /3/ has got a globally asymptotically stable equilibrium point at the origin.

Therefore it belongs to the class of systems for which the A.L.S. method is valid.

<u>Proof</u>

The proof utilizes Liapunov's second method, with the Liapunov function having an analogous form to the total energy of the system.

$$V = q^T A q + \dot{q}^T B \dot{q} \qquad /7/$$

where A and B are quadratic m x m matrices with respective elements:

$$a_{ij} = \begin{cases} q_i^{-2}\int_0^{q_i} r_{ii}(u)\,u\,du & \text{for } i=j \\ \frac{1}{2} r_{ij} & \text{for } i \neq j \end{cases} \qquad /8/$$

$$b_{ij} = \begin{cases} \dot{q}_i^{-2}\int_0^{\dot{q}_i} P_{ij}(u)\,u\,du & \text{for } i=j \\ \frac{1}{2} P_{ij} & \text{for } i \neq j \end{cases} \qquad /9/$$

Let $A^{/i/}$ and $R^{/i/}$ be the principal minors of i-th order, associated to the matrices A and R respectively.
Using results proved in the Appendix, we obtain:

$$\det A^{(i)} = \frac{1}{q_1^2 q_2^2 \ldots q_i^2}\int_0^{q_1}\int_0^{q_2}\int_0^{q_i} \det R^{(i)} u_1 u_2 \ldots u_i \, du_1 du_2 \ldots du_i \qquad /10/$$

Similar relations can be obtained for the minors $B^{/i/}$ and $P^{/i/}$ of the matrices B and P, by replacing q_i by $\dot{q}_i$ in /10/.
According to the Sylvester's criterion it follows that if the matrices R and P are positive definite, the function V is also positive definite.
Let us find a derivative of the function V along the system's trajectories. From the mathematical manipulations developed in Appendix, follows that:

$$\dot{V} = \underline{q}^T R \dot{\underline{q}} + \dot{\underline{q}}^T P \ddot{\underline{q}}$$

or, considering /3/ :

$$\dot{V} = -\dot{\underline{q}}^T D \dot{\underline{q}}$$

Hence, if the matrix D is positive semidefinite the function $\dot{V}$ is negative semidefinite. This proves that the origin is stable.

In fact, using the Barbashin & Krasovskii theorem it is easy to prove that the system is asymptotically stable if D is a positive definite matrix, since no solution of the system /3/ does satisfy $\mathring{q} \equiv 0$, except of the origin.
Evidently $\dot{q} \equiv 0$ implies $Rq = 0$, hence $q = 0$ because R is nonregular matrix.

Application: see p. 4 and 4 bis, § 1, 2

Note:

In the case when each matrix P, D, R is a scalar, the following differential equation may be obtained:

$$f_1(x')x'' + f_2(x, x')x' + f_3(x)x = 0$$

where: f_1, f_2, f_3 are the arbitrary functions fulfilling the Lipschitz conditions.

Using the preceeding theorem, the following sufficient stability conditions may be developed:

$$f_1(x) > 0, \quad f_2(x, \dot{x}) > 0, \quad f_3(x) > 0$$

It is worth while to note that it is impossible to generalize the A.L.S. method for systems with f_1 or f_3 being functions of x and $\dot{x}$.

This is shown in the example below.

Let us consider a system described by the differential ewuation

$$\ddot{x} + \dot{x} + \left[1 + (x - \dot{x})^2\right]x = 0 \qquad /11/$$

The functions f_1, f_2 and f_3 are strictly positive, but the system is still not globally stable. It can be proved that there exists an unstable limit cycle. It is intuitively obvious that

the system tends to diverge if its initial conditions are placed sufficiently far from the origin.
This may be seen from equation /11/ rewritten in the following form:

$$\ddot{x}+\dot{x}\left(1+x\dot{x}-2x^2\right)+\left(1+x^2\right)x=0 \qquad /12/$$

because the damping may strongly negative.
The above shows that the hypotheses about the form of the coefficients p_{ii} and r_{ii} of system /3/ is necessary if the A.L.S. method is to be used.

Conclusion

The application of the total energy concept for second order, matrix systems enables to generalize for higher order systems of particular type the interesting properties concerning the stability. However, it seems to be difficult to expect to find a number of other sufficiently general classes of systems that can be analysed by the method of associated linear system.

APPENDIX

I. Proof of relation /10/

The principal minor of i-th order of the matrix B is in the form:

$$\begin{bmatrix} \frac{1}{q_1^2}\int_0^{q_1} r_{11}(u)u\,du & \frac{1}{2}r_{12} & \dots & \frac{1}{2}r_{1i} \\ \frac{1}{2}r_{12} & \frac{1}{q_2^2}\int_0^{q_2} r_{22}(u)u\,du & \dots & \frac{1}{2}r_{2i} \\ \frac{1}{2}r_{1i} & & & \frac{1}{q_i^2}\int_0^{q_i} r_{ii}(u)u\,du \end{bmatrix}$$

Let us find the corresponding determinant

$$\det B^{(i)} = \frac{1}{q_1^2 q_2^2 \cdots q_i^2}\sum(-1)^K \int_0^{q_1} r_{1k_1}(u_1)u_1\,du_1 \int_0^{q_2} r_{2k_2}(u_2)u_2\,du_2 \cdots \int_0^{q_i} r_{ik_i}(u_i)u_i\,du_i$$

where $\sum$ denotes a sum taken over all permutations of the sequence k_1, k_2, k_i, K denotes a number of inversions in the above sequence.

We note that Rp_{k_p} is constant for $p \neq k_p$

The above relation can be rewritten as:

$$\det B^{(i)} = \frac{1}{q_1^2 q_2^2 \cdots q_i^2}\sum(-1)^K \int_0^{q_1}\int_0^{q_2}\cdots\int_0^{q_i} r_{1k_1}(u_1)r_{2k_2}(u_2)\dots r_{ik_i}(u_i)u_1u_2\dots u_i$$

$$du_1du_2\dots du_i = \frac{1}{q_1^2 q_2^2 \cdots q_i^2}\int_0^{q_1}\int_0^{q_2}\cdots\int_0^{q_i}\Big[\sum(-1)^K r_{1k_1}(u_1)r_{2k_2}(u_2)\dots r_{ik_i}$$

$$(u_i)u_1u_2\dots\dots\dots u_i\,du_1\,du_2\dots\dots du_i\Big]$$

which proves relation /10/.

II. <u>We have to prove the relation</u>

$$\frac{d}{dt}\left[q^T B q\right] = q^T R \dot{q}$$

Let us calculate a derivative of the quadratic form $q^T B q$

$$\frac{d}{dt}\left[q^T B q\right] = \dot{q}^T B q + q^T \dot{B} q + q^T B \dot{q} = 2 q^T B \dot{q} + q^T \dot{B} q$$

the elements b_{ij} of $\dot{B}$ can be easily obtained:

$$\dot{b}_{ij} = \begin{cases} -2\dfrac{\dot{q}_i}{q_i^3}\displaystyle\int_0^{q_i} r_{ii}(u)\,u\,du + \dfrac{\dot{q}_i}{q_i}\, r_{ii}(q_i) & \text{for } i=j \end{cases}$$

The form $q^T \dot{B} q$ can be therefore rewritten as $q^T C \dot{q}$,

where $\underline{C}$ is a matrix of elements C_{ij} defined by:

$$C_{ij} = \begin{cases} -\dfrac{2}{q_i^2}\displaystyle\int_0^{q_i} r_{ii}(u)\,u\,du + r_{ii}(q_i) & \text{for } i=j \\ 0 & \text{for } i\neq j \end{cases}$$

Hence, the following relation results:

$$\frac{d}{dt}\left[q^T B q\right] = q^T\left[2B + C\right]\dot{q} = q^T R \dot{q}$$

- - - - - - - - - - - - - - - - - - - -

REFERENCES

1. J.C.Gille. S.Węgrzyn: Une condition suffisante de stabilité pour asservissement non-linéaires Bull.Acad.Polon.Sci.; Sér. sci. techn. 9 /1961/ p.59.

2. L.R.Borgesvieira: Theory of associated systems for study of the stability in the large, Sao-Paulo 1962 pp.82 Ed. University of Sao-Paulo.

3. M.A.Aizerman: Ob odnoi problemie kasajuscejsia ustojcivosti "v bolsom" dinamiceskich sistiem. Usp.Mat.Nauk. T.IV, vyp. 4, 1949.

4. J.P.Lasalle: Stability by Liapunov's Direct Method with Applications, pp.134,Academic Press. N.York 1961.

5. A.R.Bergen, I.J.Wiliams: Verification of Aizerman's Conjecture for a Class of Third-Order Systems, JRE, AC-7. April 1962, N° 3.

6. M.A.Aizerman, F.R.Gantmacher: Absolute Stability of Regulator Systems, pp.172, Holden Day, London 1964.

7. S.Sedziwy: Asymptotic properties of solutions of non-linear differential equations of the higher order. Zesz.Nauk. V.J.; Prace Mat. Kraków 1966.

8. O.Palusiński: O sterowaniu obiektami konserwatywnymi. Prace IV KKA, I.I. pp. 305-320. Kraków 1967.

9. N.M.Trukhan: Ob odnokontournykh sistiemakh absolutno ustojcivykh v gourvitsevom ugle. pp. 5-8. Avtomatika i Telemekhanika N° 11 /1966/.

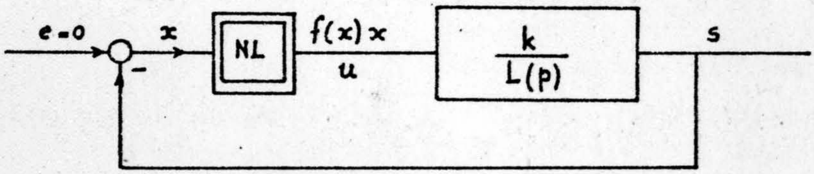
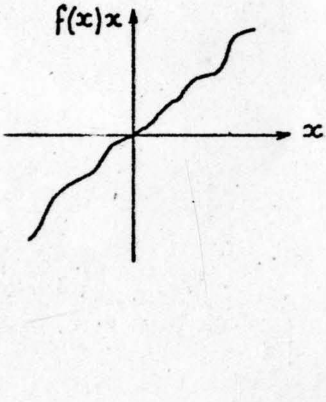


Fig. 1

Fig. 2



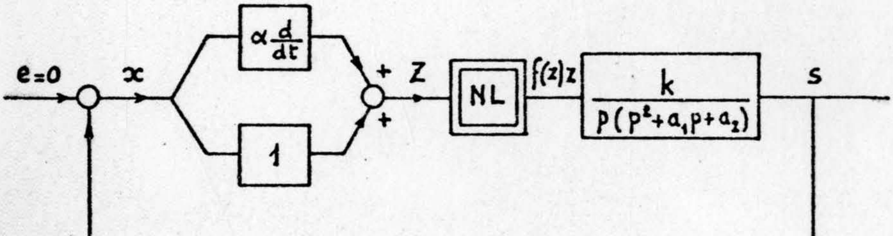


Fig. 3