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DELAYED ACTION CONTROL PROBLEMS

D. H. Chyung** and E. B. Lee***

**Department of Electrical Engineering, University of Iowa
Iowa City, Iowa, U.S.A.

***Center for Control Sciences, University of Minnesota
Minneapolis, Minnesota, U.S.A.

1. Introduction

In this paper we indicate modifications to be made in the theory of optimal control when the controlled system model has dependence on both the previous history of state type variables and control variables. Optimal control problems for models with time delays have been extensively studied and a number of recent publications[1-5,7-10] indicate a theory similar to that when the model has no such delays. More recently papers[1,10] have appeared in which the model considered contained terms which were dependent on the previous history of the control actions.

To indicate the modifications we consider as the basic model for the controlled system the linear functional differential equation

$$\dot{x}(t) = \int_{-\tau}^{0} A_0(t,s)x(t+s)ds + \sum_{i=0}^{\ell} A_i(t)x(t-h_i)$$
$$+ \int_{-\tau}^{0} B(t,s)u(t+s)ds + \sum_{i=0}^{\ell} B_i(t)u(t-h_i)$$

with continuous initial function $x(t) = \varphi(t)$ on $[t_0-\tau, t_0]$. Here

(i) $x(t)$ is an $(n \times 1)$ state vector

(ii) $u(t)$ is an $(m \times 1)$ measurable vector (control) function, and $u(t) \in \Omega$ for all t

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(iii) $\Omega \subset R^m$ (R^m is the m-dimensional real number space) is a compact convex controller restraint set

(iv) $\tau > 0$ is a constant

(v) $0 = h_o < h_1 < h_2 < \cdots < h_\ell \leq \tau$ are constants and ℓ is a positive integer

(vi) $A_o(t,s)$ defined for $t \geq t_o$, $-\infty < s < \infty$, is an $n \times n$ matrix continuous in t uniformly with respect to s, $-\tau \leq s \leq 0$. Each element of $A_o(t,s)$ is integrable with respect to s, $-\tau \leq s \leq 0$. $A_i(t)$, $i = 0,1,\ldots,\ell$, are $n \times n$ continuous matrices in $t \geq t_o$

(vii) $B(t,s)$ defined for $t \geq t_o$, $-\infty < s < \infty$, is an $n \times m$ matrix continuous in t uniformly with respect to s, $-\tau \leq s \leq 0$. Each element of $B(t,s)$ is integrable with respect to s, $-\tau \leq s \leq 0$. $B_i(t)$, $i = 0,1,\ldots,\ell$, are $n \times m$ continuous matrices in $t \geq t_o$.

For the sake of brevity let $A(t,s)$ be the $n \times n$ matrix defined for $t \geq t_o$, $-\infty < s < \infty$, such that

$$\int_{-\tau}^{0} \left\{d_s A(t,s)\right\} x(t+s) = \int_{-\tau}^{0} A_o(t,s)x(t+s)\,ds + \sum_{i=0}^{\ell} A_i(t)x(t-h_i)$$

Here the left hand side integral is in the sense of Lebesgue-Stieltjes. Then the equation of the system becomes

$$\dot{x}(t) = \int_{-\tau}^{0} \left\{d_s A(t,s)\right\} x(t+s) + \int_{-\tau}^{0} B(t,s)u(t+s)\,ds + \sum_{i=0}^{\ell} B_i(t)u(t-h_i) \qquad (1)$$

In addition to Eq. (1) consider the following two scalar differential equations

$$\dot{x}^{n+1}(t) = f^1(x(t),t) + h^1(u(t),t), \quad x^{n+1}(t_o) = 0 \qquad (2)$$

$$\dot{x}^{n+2}(t) = f^2(x(t),t) + h^2(u(t),t), \quad x^{n+2}(t_o) = 0 \qquad (3)$$

where $f^1(x,t)$ and $f^2(x,t)$ are C^1 real non-negative convex functions in $x \in R^n$ for all t and are piecewise continuous in t, and $h^1(u,t)$ and $h^2(u,t)$ are non-negative real continuous functions in $u \in R^m$ for each t and are piecewise continuous in t.

Let $x(t,\varphi)$ be the solution of the homogeneous equation

$$\dot{x}(t) = \int_{-\tau}^{0} \{d_s A(t,s)\} x(t+s)$$

with the initial function $x(t) = \varphi(t)$ on $[t_o-\tau, t_o]$, and let $Y(s,t)$ be the $n \times n$ matrix solution of

$$Y(s,t) + \int_{s}^{s+\Delta(s)} Y(\sigma,t)A(\sigma,s-\sigma)\,d\sigma = I$$

where

$$\Delta(s) = \begin{cases} \tau & \text{if} \quad t_o \leq s \leq t-\tau \\ t-s & \text{if} \quad t-\tau \leq s \leq t \end{cases}$$

and I is the $n \times n$ identity matrix. Then for any given measurable function $u(t) \subset \Omega$ on $t_o-\tau \leq s \leq t$ there exists an absolutely continuous solution (response) of Eq. (1), and it can be written as:

$$x(t) = x(t,\varphi) + \int_{t_o}^{t} Y(s,t)\left\{\int_{-\tau}^{0} B(s,\sigma)u(s+\sigma)\,d\sigma + \sum_{i=0}^{\ell} B_i(s)u(s-h_i)\right\}ds \tag{4}$$

In what follows the Eqs. (2) and (3) will be associated with the cost of control and a possible side contraint or with a multiple cost of control problem.

In the remaining sections of this paper we discuss first the geometric properties of the set of attainability which has proved so useful in the development of necessary and sufficient conditions for optimal control and in establishment of existence results. We then consider optimal control problems when the cost functional is convex with possible convex type side constraints. Systems with time varying delays are also considered.

2. Sets of Attainability

The results for optimal control are obtained later in this paper by studying the geometric properties of various sets of attainability. We now investigate these properties.

Define $n+1$ and $n+2$ dimensional vectors $\tilde{x}$ and $\hat{x}$ by adding x^{n+1} and x^{n+2}, defined by Eqs. (2) and (3) respectively.

to the n vector x, that is, $\tilde{x} = (x, x^{n+1})$, and $\hat{x} = (x, x^{n+1}, x^{n+2})$. Define the sets of attainability $K \subset R^n$, $\tilde{K} \subset R^{n+1}$ and $\hat{K} \subset R^{n+2}$ at $t = t_1$ of the system (1) together with Eqs. (2) and (3) to be the collections of all response endpoints $x(t_1)$, $\tilde{x}(t_1)$ and $\hat{x}(t_1)$ respectively for all admissible controllers u(t). A controller u(t) is called admissible if it is a measurable function and $u(t) \epsilon \Omega$ for all $t \epsilon \left[t_o - \tau, t_1\right]$. Also define the saturation sets $\tilde{K}_s$ and $\hat{K}_s$ of $\tilde{K}$ and $\hat{K}$ to be the sets of all points $\tilde{x} = (x, x^{n+1})$ in R^{n+1} and $\hat{x} = (x, x^{n+1}, x^{n+2})$ in R^{n+2} such that there exist points $\tilde{y} = (y, y^{n+1})$ and $\hat{y} = (y, y^{n+1}, y^{n+2})$ in $\tilde{K}$ and $\hat{K}$ respectively with $y = x$, $y^{n+1} \leq x^{n+1}$ and $y^{n+2} \leq x^{n+2}$.

Since $x^{n+1}(t)$ and $x^{n+2}(t)$ are non-negative for all t, $x^{n+1} \geq 0$ and $x^{n+2} \geq 0$ for all points $\tilde{x} \epsilon \tilde{K}$ and $\hat{x} \epsilon \hat{K}$. Also, from the definitions, $\tilde{K} \subset \tilde{K}_s$ and $\hat{K} \subset \hat{K}_s$. Furthermore $\tilde{K}$ is the orthogonal projection of $\hat{K}$ on the plane $x^{n+2} = 0$ in R^{n+2} and K is the orthogonal projection of $\tilde{K}$ on the plane $x^{n+1} = 0$ in R^{n+1}.

Throughout this paper it will be assumed that $\tilde{K}_s$ and $\hat{K}_s$ have nonempty interiors in R^{n+1} and R^{n+2} respectively. If they have empty interiors then similar results can be obtained in the subspaces spanned by $\tilde{K}_s$ and $\hat{K}_s$.

The following theorems summarize some of the more important properties of the sets of attainability.

<u>Theorem 1</u>. The set K is compact and convex in R^n.

<u>Theorem 2</u>. The set $\tilde{K}_s$ is convex and closed in R^{n+1}. The set $\tilde{K}$ is bounded in R^{n+1}.

<u>Theorem 3</u>. The set $\hat{K}_s$ is convex and closed in R^{n+2}. The set $\hat{K}$ is bounded in R^{n+2}.

The essential steps of the proofs can be found in reference 4 for the case without delays in the control action.

3. <u>Convex Cost Functionals</u>

Consider the system given by Eq. (1) together with Eq. (2). Let $G \subset R^n$ be a given closed convex target set and $t_1 > t_o$ be a given finite final time. Let g(x) be a C^1 convex function in $x \epsilon R^n$. The control problem is as follows: Find an admissible controller u(t) on $\left[t_o - \tau, t_1\right]$ that steers the corresponding response $x(t)$ from the given initial function $\varphi(t)$ on $\left[t_o-\tau, t_o\right]$ to the target set G at $t = t_1$ and minimizes the cost functional

$$C(u) = g(x(t_1)) + x^{n+1}(t_1)$$

$$= g(x(t_1)) + \int_{t_o}^{t_1} \left\{f^1(x(t),t) + h^1(u(t),t)\right\}dt \qquad (5)$$

If an admissible control function satisfies the above conditions then we call it an optimal controller. From theorems 1 and 2 one can obtain the following existence result.

__Theorem 4.__ If there exists an admissible controller which steers the corresponding response of (1) to G at $t = t_1$, then there exists an optimal controller.

Let $\widetilde{G} = G \times R^1 \subset R^{n+1}$. Then an admissible controller is an optimal controller if and only if the correcponding response endpoint $\widetilde{x}(t_1) = (x(t_1), x^{n+1}(t_1))$ is in $\widetilde{G}$ and the cost functional at that point $C(u) = g(x(t_1)) + x^{n+1}(t_1)$ is minimum for all points $\widetilde{x} = (x, x^{n+1})$ in $\widetilde{G} \cap \widetilde{K}$. It can be shown, from the convexity and closedness of $\widetilde{K}_s$ and G, that this point is a boundary point of $\widetilde{K}_s$, that is, $\widetilde{x}(t_1) \in \partial\widetilde{K}_s$.

Define an admissible controller $u^*(t)$ on $\left[t_o-\tau, t_1\right]$ with corresponding response $\widetilde{x}^*(t) = (x^*(t), x^{n+1*}(t))$ to be a maximal controller in R^{n+1} in case there exists a nontrivial (n+1) dimensional vector solution $\widetilde{\eta}(t) = (\eta(t), \eta^{n+1}(t))$ $(\eta \in R^n, \eta^{n+1} \in R^1)$ of the equation

$$\eta^{n+1}(t) = \eta_{n+1} = \text{constant} \leq 0$$

$$\eta(t) + \int_t^{t+\Delta(t)} \eta(s) A(s,t-s)\, ds + \int_{t_1}^{t} \eta_{n+1} \frac{\partial f^1}{\partial x}(x^*(s),s)\, ds = \text{constant} \qquad (6)$$

$$\Delta(t) = \begin{cases} \tau & \text{if} \quad t_o \leq t \leq t_1-\tau \\ t_1-t & \text{if} \quad t_1-\tau \leq t \leq t_1 \end{cases}$$

such that

$$\left\{\int_{t_o}^{t+\tau} \eta(s)B(s,t-s)ds\right\}u^*(t) = \max_{u\in\Omega}\left\{\int_{t_o}^{t+\tau} \eta(s)B(s,t-s)ds\right\}u$$

almost everywhere on $\left[t_o-\tau, t_o-h_\ell\right]$

$$\left\{\int_{t_o}^{t+\tau} \eta(s)B(s,t-s)ds + \sum_{i=k}^{\ell} \eta(t+h_i)B_i(t+h_i)\right\}u^*(t)$$

$$= \max_{u\in\Omega}\left\{\int_{t_o}^{t+\tau} \eta(s)B(s,t-s)ds + \sum_{i=k}^{\ell} \eta(t+h_i)B_i(t+h_i)\right\}u \tag{7}$$

almost everywhere on $\left[t_o-h_k, t_o-h_{k-1}\right]$, $k = 1,2,\ldots,\ell$

$$\left\{\int_{t}^{t+\tau} \eta(s)B(s,t-s)ds + \sum_{i=0}^{\ell} \eta(t+h_i)B_i(t+h_i)\right\}u^*(t) + \eta_{n+1}h^1(u^*(t),t)$$

$$= \max_{u\in\Omega}\left\{\int_{t}^{t+\tau} \eta(s)B(s,t-s)ds + \sum_{i=0}^{\ell} \eta(t+h_i)B_i(t+h_i)\right\}u + \eta_{n+1}h^1(u,t)$$

almost everywhere on $\left[t_o, t_o-\tau\right]$

$$\left\{\int_{t}^{t_1} \eta(s)B(s,t-s)ds + \sum_{i=0}^{\ell} \eta(t+h_i)B_i(t+h_i)\right\}u^*(t) + \eta_{n+1}h^1(u^*(t),t)$$

$$= \max_{u\in\Omega}\left\{\int_{t}^{t_1} \eta(s)B(s,t-s)ds + \sum_{i=0}^{\ell} \eta(t+h_i)B_i(t+h_i)\right\}u + \eta_{n+1}h^1(u,t)$$

almost everywhere on $\left[t_1-\tau, t_1-h_\ell\right]$

$$\left\{\int_{t}^{t_1} \eta(s)B(s,t-s)ds + \sum_{i=0}^{k-1} \eta(t+h_i)B_i(t+h_i)\right\}u^*(t) + \eta_{n+1}h^1(u^*(t),t)$$

$$= \max_{u\in\Omega}\left\{\int_{t}^{t_1} \eta(s)B(s,t-s)ds + \sum_{i=0}^{k-1} \eta(t+h_i)B_i(t+h_i)\right\}u + \eta_{n+1}h^1(u,t)$$

almost everywhere on $\left[t_1-h_k, t_1-h_{k-1}\right]$, $k = 1,2,\ldots,\ell$.

<u>Theorem 5</u>. An admissible controller $u(t)$ steers the response $\tilde{x}(t) = (x(t), x^{n+1}(t))$ to the boundary of $\tilde{K}_s$ at $t = t_1$, i.e.,

$\tilde{x}(t_1) \varepsilon \partial\tilde{K}_s \cap \tilde{K}$ if and only if it is a maximal controller in R^{n+1}. Furthermore in this case the corresponding $\tilde{\eta}(t_1)$ is an exterior normal vector to $\tilde{K}_s$ at $\tilde{x}(t_1)$.

Since an optimal controller steers the response to the boundary of $\tilde{K}_s$, by the above theorem, it must be a maximal controller. Also if a controller is maximal and certain end-point conditions are satisfied then by studying the geometry of the set $\tilde{K}_s$ one can conclude that it is an optimal controller. The following theorem summarizes the results.

Theorem 6. (A) Suppose $G = R^n$ (free endpoint problem). A controller u(t) on $\left[t_0-\tau, t_1\right]$ is an optimal controller if and only if it is a maximal controller in R^{n+1} and the corresponding solution $\tilde{\eta}(t_1) = (\eta(t_1), \eta_{n+1})$ of (6) satisfies the transversality condition $\eta(t_1) = -\eta_{n+1}$ grad $g(x(t_1))$, $\eta_{n+1} < 0$. (B) Suppose $g(x) = 0$, i.e., $C(u) = x^{n+1}(t_1)$. Then a control function u(t) is an optimal control function if and only if it is a maximal controller in R^{n+1} and either $x(t_1) \varepsilon G$, $\eta_{n+1} < 0$, $\eta(t_1) = 0$ or $x(t_1) \varepsilon \partial G$, $\eta_{n+1} \leq 0$, $\eta(t_1)$ interior normal to G at $x(t_1)$.

4. Multiple Cost Functionals

Often an optimal controller obtained in the proceeding section is not unique, that is, there are many optimal control functions. In this case it is rather natural to introduce a second cost functional C'(u) and then to choose among the optimal control functions the one that minimizes the second cost functional. We examine this problem in this section.

Consider the same problem as in section 2. Assume that there are more than one admissible control functions which minimize the given cost functional C(u). Define a second cost functional C'(u) by Eq. (3)

$$C'(u) = x^{n+2}(t_1) = \int_{t_0}^{t_1} \left\{f^2(x(t),t) + h^2(u(t),t)\right\}dt$$

The new optimal control problem is now to find an admissible control function u(t) such that

(i) it steers the response $x(t)$ from the initial function $\varphi(t)$ on $[t_o-\tau, t_o]$ to the target set G at $t = t_1$

(ii) it minimizes the cost functional $C(u)$ for all admissible control functions which satisfy (i)

(iii) it minimizes the second cost functional $C'(u)$ for all admissible control functions which satisfy (i) (ii).

If a control function satisfies the above conditions then it is called an optimal controller in this section.

Let $\hat{G} = G\times R^1\times R^1 \subset R^{n+2}$. Then an admissible control function $u(t)$ with corresponding response $\hat{x}(t) = (x(t), x^{n+1}(t), x^{n+2}(t))$ is an optimal controller if $C(u) = g(x(t_1)) + x^{n+1}(t_1)$ is minimum for all $\hat{x} = (x, x^{n+1}, x^{n+2})$ in $\hat{G}\cap\hat{R}$ and $C'(u) = x^{n+2}(t_1)$ is minimum for all $\hat{x}$ in $\hat{G}\,\hat{R}$ with $x^{n+1} = x^{n+1}(t_1)$. Furthermore if such a point exists in $\hat{R}\cap\hat{G}$ then there exists an optimal controller. By Theorems 1 and 3 the following existence theorem can be established:

<u>Theorem 7</u>. If there exists an admissible controller which steers the response to G at $t = t_1$, then there exists an optimal controller.

As in the previous section define an admissible control function $u^*(t)$ with response $\hat{x}^*(t) = (x^*(t), x^{n+1^*}(t), x^{n+2^*}(t))$ to be a maximal controller in R^{n+2} if and only if there exists n+2 dimensional nontrivial vector solution $\hat{\lambda}(t) = (\lambda(t), \lambda^{n+1}(t), \lambda^{n+2}(t))$, $\lambda \in R^n$, $\lambda^{n+1} \in R^1$, $\lambda^{n+2} \in R^1$, of

$$\lambda^{n+1}(t) = \lambda_{n+1} = \text{constant} \leq 0$$

$$\lambda^{n+2}(t) = \lambda_{n+2} = \text{constant} \leq 0$$

$$\lambda(t) + \int_t^{t+\Delta(t)} \Lambda(s, t-s)\,ds + \int_{t_1}^{t} \left\{\lambda_{n+1}\frac{\partial f^1}{\partial x}(x^*(s), s) + \lambda_{n+2}\frac{\partial f^2}{\partial x}(x^*(s), s)\right\}ds$$

$$= \text{constant}$$

$$\Delta(t) \begin{cases} \tau & \text{if} \quad t_o \leq t \leq t_1-\tau \\ t_1-t & \text{if} \quad t_1-\tau \leq t \leq t_1 \end{cases}$$

such that

$$\left\{\int_{t_o}^{t+\tau} \lambda(s)B(s,t-s)\,ds\right\}u^*(t) = \max_{u\in\Omega}\left\{\int_{t_o}^{t+\tau} \lambda(s)B(s,t-s)\,ds\right\}u$$

almost everywhere on $\left[t_o-\tau, t_o-h_\ell\right]$

$$\left\{\int_{t_o}^{t+\tau} \lambda(s)B(s,t-s)\,ds + \sum_{i=k}^{\ell} \lambda(t+h_i)B_i(t+h_i)\right\}u^*(t)$$

$$= \max_{u\in\Omega}\left\{\int_{t_o}^{t+\tau} \lambda(s)B(s,t-s)\,ds + \sum_{i=k}^{\ell} \lambda(t+h_i)B_i(t+h_i)\right\}u \tag{9}$$

almost everywhere on $\left[t_o-h_k, t_o-h_{k-1}\right]$, $k = 1,2,\ldots,\ell$.

$$\left\{\int_{t}^{t+\tau} \lambda(s)B(s,t-s)\,ds + \sum_{i=0}^{\ell} \lambda(t+h_i)B_i(t+h_i)\right\}u^*(t)+\lambda_{n+1}h^1(u^*(t),t)$$

$$+ \lambda_{n+2}h^2(u^*(t),t)$$

$$= \max_{u\in\Omega}\left\{\int_{t}^{t+\tau} \lambda(s)B(s,t-s)\,ds+\sum_{i=0}^{\ell} \lambda(t+h_i)B_i(t+h_i)\right\}u+\lambda_{n+1}h^1(u,t)+\lambda_{n+2}h^2(u,t)$$

almost everywhere on $\left[t_o, t_o-\tau\right]$

$$\left\{\int_{t}^{t_1} \lambda(s)B(s,t-s)\,ds+ \sum_{i=0}^{\ell} \lambda(t+h_i)B_i(t+h_i)\right\}u^*(t)+\lambda_{n+1}h^1(u^*(t),t)$$

$$+ \lambda_{n+2}h^2(u^*(t),t)$$

$$= \max_{u\in\Omega}\left\{\int_{t}^{t_1} \lambda(s)B(s,t-s)\,ds+ \sum_{i=0}^{\ell} \lambda(t+h_i)B_i(t+h_i)\right\}u+\lambda_{n+1}h^1(u,t)$$

$$+ \lambda_{n+2}h^2(u,t)$$

almost everywhere on $\left[t_1-\tau, t_1-h_\ell\right]$

$$\left\{\int_{t}^{t_1} \lambda(s)B(s,t-s)\,ds+ \sum_{i=0}^{k-1} \lambda(t+h_i)B_i(t+h_i)\right\}u^*(t)+\lambda_{n+1}h^1(u^*(t),t)$$

$$+ \lambda_{n+2}h^2(u^*(t),t)$$

$$= \max_{u\in\Omega}\left\{\int_{t}^{t_1} \lambda(s)B(s,t-s)\,ds+ \sum_{i=0}^{k-1} \lambda(t+h_i)B_i(t+h_i)\right\}u+\lambda_{n+1}h^1(u,t)$$

$$+ \lambda_{n+2}h^2(u,t)$$

almost everywhere on $\left[t_1-h_k, t_1-h_{k-1}\right]$, $k = 1,2,\ldots,\ell$.

Theorem 8. An admissible control function $u(t)$ steers the corresponding response $\hat{x}(t) = (x(t), x^{n+1}(t), x^{n+2}(t))$ to the boundary of $\hat{K}_s$ at $t = t_1$, that is, $\hat{x}(t_1) \epsilon \partial\hat{K}_s$ if and only if $u(t)$ is a maximal control function on $\left[t_o, t_1\right]$ in R^{n+2}. Furthermore in this case the corresponding $\hat{\lambda}(t_1)$ is an exterior normal vector to $\hat{K}_s$ at $\hat{x}(t_1)$ in R^{n+2}.

Theorem 9. (A) Suppose $G = R^n$. Then an optimal control function is a maximal controller in both R^{n+1} and R^{n+2} (i.e., the control satisfies Eqs. (7) and (9)) with endpoint conditions $\eta_{n+1} = \lambda_{n+1} < 0$, $\lambda_{n+2} \leq 0$, $\eta(t_1) = \lambda(t_1) = -\eta_{n+1}$ grad $g(x(t_1))$. Conversely if $u(t)$ is a maximal control function in both R^{n+1} and R^{n+2} with the above endpoint conditions and $\lambda_{n+2} \neq 0$, then it is an optimal controller. (B) Suppose $g(x) = 0$. Then an optimal control function is a maximal control function in both R^{n+1} and R^{n+2} with either $x(t_1) \epsilon G$, $\eta_{n+1} = \lambda_{n+1} < 0$, $\lambda_{n+2} < 0$, $\eta(t_1) = \lambda(t_1) = 0$, or $x(t_1) \epsilon \partial G$, $\eta_{n+1} = \lambda_{n+1} \leq 0$, $\lambda_{n+2} \leq 0$, $\eta(t_1) = \lambda(t_1)$ interior normal to G at $x(t_1)$. Conversely a maximal controller in both R^{n+1} and R^{n+2} with the above endpoint conditions and $\lambda_{n+1} \neq 0$, $\lambda_{n+2} \neq 0$, is an optimal control.

The condition that an optimal control function must be a maximal control function in R^{n+1} follows from section 2 because it minimizes the cost functional $C(u)$, and the requirement that it must be a maximal control function in R^{n+2} follows from the fact that the corresponding response endpoint $\hat{x}(t_1)$ must be a boundary point of $\hat{K}_s$. The sufficiency part can be established by examining the endpoint conditions together with the condition $\tilde{x}(t_1) \epsilon \partial\tilde{K}_s$, $\hat{x}(t_1) \epsilon \partial\hat{K}_s$ and $\tilde{K}_s$ and $\hat{K}_s$ convex in R^{n+1} and R^{n+2} respectively.

5. Side Constraints

Consider the optimal control problem of section 2. In some cases there are additional constraints imposed on the system. A limit on the available fuel for operation of a controlled system is an example. This problem can often be handled by introducing a side constraint of the form

$$x^{n+2}(t_1) = \int_{t_o}^{t_1} \left\{f^2(x(t),t) + h^2(u(t),t)\right\}dt \le d$$

for a given positive constant d.

The problem now is to find an admissible control function which steers the response to the target G at $t = t_1$, minimizes the cost functional $C(u) = g(x(t_1)) + x^{n+1}(t_1)$ and satisfies the side constraint $x^{n+2}(t_1) \le d$. This problem can be solved by examining the sets $\hat{R}$ and $\hat{R}_s$ in much the same manner as in section 3. If we let $\hat{G} = \left\{\hat{x} = (x,x^{n+1},x^{n+2}) \,|\, x\epsilon G, x^{n+1}\epsilon R^1, x^{n+2} \le d\right\}$ then it is obvious that the optimal response endpoint $\hat{x}(t_1) = (x(t_1),x^{n+1}(t_1),x^{n+2}(t_1))$ is a point in $\hat{R}\cap\hat{G}$ with minimum x^{n+1} for $\hat{x}$ in $\hat{R}\cap\hat{G}$. It can also be shown that the endpoint $\hat{x}(t_1)$ is a boundary point of $\hat{R}_s$, i.e., $\hat{x}(t_1)\epsilon\partial\hat{R}_s\cap\hat{R}$.

Thus the following results are easily obtained.

<u>Theorem 10</u>. If there exists an admissible control function which steers the response to G at $t = t_1$ and satisfies the side constraint $x^{n+2}(t_1) \le d$ then there exists an optimal controller.

<u>Theorem 11</u>. Suppose $G = R^n$. Then an optimal control function is a maximal controller in R^{n+2} with endpoint conditions either $\lambda_{n+1} \le 0$, $\lambda_{n+2} < 0$, $x^{n+2}(t_1) = d$ or $\lambda_{n+1} \le 0$, $\lambda_{n+2} = 0$, $x^{n+2}(t_1) \le d$. Conversely a maximal controller in R^{n+2} with the above endpoint conditions and $\lambda_{n+1} \ne 0$ is an optimal controller. If G is not the whole space R^n in the above theorem then the endpoint conditions should be modified as in Theorem 9(B).

6. <u>Remarks and Other Extensions</u>

Throughout this paper it was assumed that one can select the initial control segment u(t) on $\left[t_o-\tau,t_o\right]$. If the initial control function is given on the initial time interval, then the results in the paper are still applicable except that the given initial control function should be used instead of the maximal control function on the initial interval.

The right hand side of Eq. (1) is such that certain dependence on time varying delayed history is covered. However because of the assumed continuity of $A(t,s)$ with respect to t it is not readily apparent how terms of the form $x(\omega(t))$ could be handled. We therefore treat this special case by adding to the right hand side these terms to indicate what changes have to be made. It is sufficient to consider the equation

$$\dot{x}(t) = A(t)x(t) + B(t)x(\omega(t)) + C(t)u(t) + D(t)u(\zeta(t)). \quad (10)$$

with continuous initial function $x(t) = \varphi(t)$ on $\omega(t_o) \leq t \leq t_o$. It is assumed that the $n \times n$ matrices $A(t)$ and $B(t)$ and the $n \times m$ matrices $C(t)$ and $D(t)$ have integrable functions as their elements. To insure that certain inverse functions exist assume that

$$\omega(t) = t - \theta(t),\ \theta(t) \geq 0, \text{ in class } C^1, \text{ and } \alpha_1 \leq \dot{\theta} \leq \alpha_2 < 1$$

$$\zeta(t) = t - \gamma(t),\ \gamma(t) \geq 0, \text{ in class } C^1, \text{ and } \alpha_3 \leq \dot{\gamma} \leq \alpha_4 < 1$$

Then both $\omega(t)$ and $\zeta(t)$ are strictly increasing and $r = \omega^{-1}$, $\rho = \zeta^{-1}$ exist and are C^1.

Let $x(t,\varphi)$ be the solution of the homogeneous equation as in section 1 and let $y(s,t)$ $t_o \leq t$, $t_o \leq s \leq t$ be the $n \times n$ matrix solution of

$$\begin{aligned}\frac{\partial y}{\partial s}(s,t) &= -y(s,t)A(s),\ \omega(t) = t - \theta(t) \leq s \leq t \\ &= -y(s,t)A(s) - y(r(s),t)B(r(s))\dot{r}(s),\ t_o \leq s \leq \omega(t)\end{aligned}$$

with $y(t,t) = I$. Then given any measurable function $u(t) \subset \Omega$ the response of (10) can be obtained from

$$x(t) = x(t,\varphi) + \int_{t_o}^{t} y(s,t)\left\{C(s)u(s) + D(s)u(\zeta(s))\right\}ds.$$

Therefore certain time varying delay type terms can be included in the theory of section 2-4.

If Ω is not compact as was assumed throughout this paper it is necessary to put other restrictions on the control functions to be considered. For example, one can require that $u(t)$ be integrable and $\int_{t_o}^{t_1} ||u||_U^2\, dt < \infty$ so that $u(t)$ belongs to $L_2(t_o,t_1)$. The problem of optimal control with such restrictions was considered in reference 3 for the problem with time delays and the results are easily extended to the functional differential equation (1).

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OPTIMUM CONTROL OF LINEAR TIME-LAG PROCESSES WITH QUADRATIC PERFORMANCE INDEXES

by

Andrzej Manitius

Chair of Automatic Control
Technical University of Warsaw, Poland

1. Introduction

The aim of this paper is to give an analytic solution to to the optimum control problem for processes described by systems of linear differential-difference equations of retarded type. The performance index considered is a quadratic functional of the state and control variables. The problem is a generalized version of the so-called "linear state regulator problem", solved by R.E. Kalman and others in 1960-1964 [1,2,3]. Delays existing in process equations raise a number of difficulties, so that - to the author's knowledge - the problem considered has had no full and adequate analytic solution [5-10].

In this paper, a new solution based on the maximum principle and Fredholm integral equations is presented. Main results are given by lemmas and theorems. Less important proofs are omitted, but they may be found in [4].

Notation. Standard vector-matrix notation is used. Vectors will be designated by underlined lower case letters, matrices by underlined capitals, scalars by not underlined characters. Prime denotes matrix transposition, $\underline{0}$ means zero matrix, $\underline{I}$ - unit matrix. $\underline{R} > \underline{0}$ (or $\underline{R} \geq \underline{0}$) means that matrix $\underline{R}$ is positive definite (or semidefinite).

2. Statement of the Problem

2.1. We are dealing with control processes described by the equations

$$\dot{\underline{x}}(t) = \sum_{i=0}^{m} \underline{A}_i(t)\underline{x}(t - h_i) + \underline{B}(t)u(t) \qquad (2.1)$$

where: $\underline{x}(t)$ - n x 1 vector; $\underline{u}(t)$ - p x 1 vector; $0 = h_0 < h_1 < \ldots < h_m$ - constant delays (real numbers).

The above equation is a general form of the system of linear differential-difference equations of retarded type.

We shall call: $x_i(t)$ - state variables, $\underline{x}(t)$ - instantaneous state, or state, at time t.

<u>Definition</u>. A complete state at time τ of the process de-- scribed by eq. (2.1) is a system of n functions $\varphi_k(s)$, $s \in \in [\tau - h_{j_k}, \tau]$, $k = 1, \ldots, n$, where h_{j_k} is the maximum de- lay appearing in the argument of $x_k(t)$ in eq. (2.1).

The complete state is then a system of n initial functions for the eq. (2.1) at time τ, and is an element of an infinite dimensional space. In this paper we shall deal with continuous initial functions.

2.2. Two fixed values, t_o and t_f, $t_o < t_f$, specyfying the interval $[t_o, t_f]$ are given.

2.3. The complete state at time t_o is given.

2.4. The terminal state $\underline{x}(t_f)$ is not specified.

2.5. Admissible control $\underline{u}(t)$ is any vector-valued, piece- -wise continuous function with a finite number of discontinuities of the first kind, and satysfying the condition

$$\int_{t_o}^{t_f} \underline{u}'(t)\, \underline{u}(t)\, dt < \infty$$

No other constraints on the control $\underline{u}(t)$ are assumed.

2.6. Performance index has the form:

$$J(\underline{u}) = \frac{1}{2}\, \underline{x}'(t_f)\underline{Z}\underline{x}(t_f) + \frac{1}{2} \int_{t_o}^{t_f} \Big[\underline{x}'(t)\underline{Q}(t)\underline{x}(t) + + \underline{u}'(t)\underline{R}(t)\underline{u}(t) \Big] dt \tag{2.2}$$

where $\underline{Z}$, $\underline{Q}$, $\underline{R}$ - quadratic, real, symmetric matrices of dimensions $n \times n$, $n \times n$ and $p \times p$, respectively

$\underline{Q}(t)$, $\underline{R}(t)$ are continuous $\forall t \; [t_o, t_f]$

$\underline{Z} \geqslant \underline{O}$, $\underline{Q}(t) \geqslant \underline{O}$, $\underline{R}(t) > \underline{O}$ $\forall t \in [t_o, t_f]$

2.7. The problem is to find an adimssible control bringing the process from given complete initial state to unspecified terminal state along a trajectory that minimizes functional (2.2). More precisely, we wich to determine the optimum control $\underline{u}^o(t)$, assuming the knowledge of process history up to time $\tau \leqslant t$, i.e. assuming the knowledge of vector function $\underline{x}(\sigma)$, $\forall \sigma \in [t_o - h_m, \tau]$ *. The following special cases are

* Note see page 3.

included in this formulation:

(a) if $\tau = t_o$, then determining control $\underline{u}^o(t)\ \forall t \in [t_o, t_f]$ we find the so-called "open-loop" control;

(b) if $\tau = t\ \ \forall t \in [t_o, t_f]$, then at each time t we find control $\underline{u}^o(t)$ based on knowledge of the complete state at time t, i.e. we construct an operator A

$$\underline{u}^o(t) = A[\underline{x}(\sigma)] \qquad \sigma \in [t - h_m, t]$$

that implements the optimum control in "closed-loop" system;

(c) if $\tau = t + h$, then we determine the anticipated optimum control.

3. Maximum Principle and Fredholm Integral Equations

An existence theorem for the optimum solution was given in work of Chyung and Lee [11]. Basing upon the maximum principle for processes with delayed states [11-14], we obtain the following necessary conditions of optimality**:

(i) there exists a continuous vector-valued function $\underline{p}(t)$ satisfying the adjoint canonical equation:

$$\dot{\underline{p}}(t) = -\sum_{i=0}^{m} \underline{A}_i'(t + h_i)\underline{p}(t + h_i) - \underline{Q}(t)\underline{x}(t) \tag{3.1}$$

with terminal condition

$$\underline{p}(t) = \begin{cases} \underline{Z}\ \underline{x}(t) & \text{for} \quad t = t_f \\ \underline{O} & \text{for} \quad t > t_f \end{cases} \tag{3.2}$$

(ii) the optimum control $\underline{u}^o(t)$ is a hamiltonian-minimizing control; in the problem considered, hamiltonian's minimum is absolute and unique and is reached for

$$\underline{u}^*(t) = -\underline{R}^{-1}(t)\underline{B}'(t)\underline{p}(t) \tag{3.3}$$

As shown by Chyung and Lee [11], the above conditions are also sufficient in our case. Hence $\underline{u}^o = \underline{u}^*$.

Substituting (3.3) to (2.1) and denoting

$$\underline{H}(t) = \underline{B}(t)\underline{R}^{-1}(t)\underline{B}'(t) \tag{3.4}$$

we have

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* As we shall see the optimum solution obtained in the sequel will allow to reject the information about the function $x(s)$ for $s \in [t_o - h_m, t - h_m]$, at least for a deterministic problem, but this cannot be assumed at this point.

** We use here "minimum principle" in-lieu of "maximum principle".

$$\dot{\underline{x}}(t) = \sum_{i=0}^{m} \underline{A}_i(t)\underline{x}(t - h_i) - \underline{H}(t)\underline{p}(t) \tag{3.5}$$

Eq. (3.5) with initial condition at time t_0 defined by the complete initial state, together with (3.1) and (3.2) forms the system of canonical equations; their solution $\underline{p}(t)$ determines the optimum control $\underline{u}^o(t)$. Unfortunately, we have no formula for a general solution of such a system of differential equations, because eq. (3.5) is of retarded type, while eq. (3.1) is of advanced type. Hence, to find a relation between $\underline{p}(t)$ and $\underline{x}(t)$ is a difficult task.

To overcome this difficulty we can transform the system of canonical equations (in view of their linearity), into an equivalent system of linear integral equations. Assuming the knowledge of the complete state at time τ, and integrating eq. (3.5) in the forward direction of time, from τ to t, and equation (3.1 in the backward direction, from t_f to t, we have

$$\underline{x}(t) = \underline{X}(t, \tau)\underline{x}(\tau) + \sum_{i=1}^{m} \int_{\tau - h_i}^{\tau} \underline{X}(t, \alpha + h_i)\underline{A}_i(\alpha + h_i)\underline{x}(\alpha)d\alpha$$

$$- \int_{\tau}^{t} X(t, \sigma)H(\sigma)p(\sigma)d\sigma \quad \text{for } t \in [\tau, t_f] \tag{3.6}$$

$$\underline{p}(t) = \underline{X}'(t_f, t)\underline{Z}\underline{x}(t_f) + \int_{t}^{t_f} \underline{X}'(s, t)\, \underline{Q}(s)\underline{x}(s)ds, \; t \in [\tau, t_f] \tag{3.7}$$

where $\underline{X}(t, \tau)$ - the resolvent kernel of homogeneous equation (see Halanay [15] and Manitius [4]).

Substituting (3.6) into (3.7) and changing the order of integration (which is permitted due to continuity of $\underline{X}(t, \;)$, we obtain the following integral equation with unknown vector valued function $\underline{p}(t)$:

$$\underline{p}(t) = \underline{g}(t) + \lambda \int_{\tau}^{t_f} \underline{K}(t, \sigma)\underline{p}(\sigma)d\sigma \quad \text{for } t \in [\tau, t_f] \tag{3.8}$$

where $\lambda = -1$,

$$\bar{\underline{g}}(t) = \underline{\bar{M}}(t, \tau)\underline{\bar{x}}(\tau) + \sum_{i=1}^{m} \int_{\tau - h_i}^{\tau} \underline{M}(t, \alpha + h_i)\underline{A}(\alpha + h_i)\underline{x}(\alpha)d\alpha \tag{3.9}$$

$$\underline{M}(t,\sigma) \overset{df}{=} \underline{X}'(t_f,t)\underline{Z}\,\underline{X}(t_f,\sigma) + \int_{\omega}^{t_f} \underline{X}'(s,t)\underline{Q}(s)\underline{X}(s,\sigma)ds \qquad (3.10)$$

$$\omega = \max(t, \ )$$

$$\underline{N}(t,\sigma) = \underline{M}(t,\sigma)\,\underline{B}(\sigma) \qquad (3.11)$$

Eq. (3.8) represents a system of linear Fredholm integral equations of the second kind, with respect to unknown functions $p_i(t)$, $i = 1, \ldots, n$. Basic properties of the integral equation (3.8) are summarized below:

Lemma 1. The kernel $\underline{M}(t,\sigma)$ defined by relation (3.10) is continuous $\forall\, t,\sigma \in [\tau, t_f]$, $\tau \in [t_o, t_f]$ and is piece-wise continuously differentiable with respect to t and σ in the rectangle $[\tau, t_f; \tau, t_f]$, except for the line $t = \sigma$, $t = t_f - h_k$, $\sigma = t_f - h_k$, $k = 1, \ldots, m$.

Proof: see Manitius[4] section 6.3.

By lemma 1 and relation (3.11), the kernel $\underline{N}(t, \ )$ is continuous in the given range and is piece-wise continuously differentiable with respect to t $\forall\, t \in [\tau, t_f]$.

Lemma 2. The integral equation (3.8) and the system of canonical equations (3.1), (3.5), with its boundary conditions, are mutually equivalent, i.e. every continuous and piece-wise continuously differentiable vector-valued function $\underline{p}(t)$ satisfying the canonical system $\forall\, t \in [\tau, t_f]$, satisfies also the integral equation (3.8) and vice-versa.

Proof: see Manitius[4] sections 6.2 and 6.4.

Theorem 1. If: (a) assumptions of sections 2.1 and 2.2 on continuity of matrices $\underline{A}_i$, $\underline{B}$, $\underline{Q}$, $\underline{R}$, $\forall t \in [t_o,t_f]$, $i = 0, \ldots, m$, hold; (b) $\underline{Z} \geq \underline{0}$, $\underline{Q}(t) \geq \underline{0}$, $\underline{R}(t) > \underline{0}$, $\forall\, t \in [t_o,t_f]$, then $\lambda = -1$ is not an eigenvalue of kernel (3.11)

Proof. Since $\underline{R}(t) > \underline{0}$, $\forall\, t \in [t_o,t_f]$, it can be written as $\underline{R}(t) = \underline{T}(t)\underline{T}'(t)$, where $\underline{T}(t)$ - $p \times p$ nonsingular matrix. Next, by change of variable $\underline{p}(t)$ into $\underline{v}(t) = \underline{T}\,\underline{B}'\,\underline{p}(t)$ eq. (3.8) can be transformed into an integral equation with symmetric kernel

$$\underline{v}(t) = \underline{T}'(t)\,\underline{B}'(t)\,\underline{q}(t) + \lambda\int_{\tau}^{t_f} \underline{T}'(t)\underline{B}'(t)\underline{M}(t,s)\underline{B}(s)\underline{T}(s)\underline{v}(s)ds$$

$$\lambda = -1 \qquad (3.12)$$

(Symmetry follows from symmetry of the kernel $\underline{M}(t,\sigma)$).

In addition we have

$$\underline{u}^o(t) = -\underline{T}(t)\,\underline{v}(t) \qquad (3.13)$$

Now, we examine the kernel of the above equation. To this effect consider a nonnegative functional:

$$F(\underline{u}) = \underline{x}'(t_f)\underline{Z}\,\underline{x}(t_f) + \int_{\tau}^{t_f} \underline{x}'(t)\underline{Q}(t)\underline{x}(t)dt \qquad (3.14)$$

on a trajectory $\underline{x}(t)$ corresponding to zero complete state at time τ and any admissible control satisfying the relation

$$\underline{u}(t) = -\underline{T}(t)\,\underline{z}(t) \qquad (3.15)$$

where $\underline{z}(t)$ is any $p \times 1$ vector-valued function square integrable in the interval $[\tau, t_f]$. We have then:

$$F(\underline{u}) = \int_{\tau}^{t_f}\int_{\tau}^{t_f} \underline{z}'(s)\underline{T}'(s)\underline{B}'(s)\underline{M}(t,\sigma)\underline{B}(\sigma)\underline{T}(\sigma)\underline{z}(\sigma)ds\,d\sigma \geqslant 0 \qquad (3.16)$$

Last inequality shows that the kernel of eq. (3.12) is positive [16], so it can have only positive eigenvalues, and $\lambda = -1$ is not its eigenvalue. On the basis of second Fredholm theorem we can say immediately that for $\underline{g}(t) = \underline{0}\ \forall t \in [\tau, t_f]$ the equation (3.12) has unique solution $\underline{v}(t) \equiv \underline{0}$, which by (3.13) involves $\underline{u}^o(t) \equiv \underline{0}$ and - by canonical system - $\underline{p}(t) = \underline{0}$ $\forall t \in [\tau, t_f]$. It follows, that a homogeneous equation corresponding to eq. (3.8) has only trivial solution, hence $\lambda = -1$ is not an eigenvalue of the kernel $\underline{N}(t,\sigma)$, Q.E.D.

From the theorem 1 it follows now:

Corollary 1.1. From the first Fredholm theorem [16] it follows that for $\lambda = -1$ there exists a resolvent kernel of the eq. (3.8), i.e. $\underline{R}(t,\vartheta,\tau;\lambda)\big|_{\lambda=-1} = \underline{R}(t,\vartheta,\tau)$ satisfying integral equations

$$\underline{R}(t,\vartheta,\tau) = \underline{N}(t,\vartheta) + \int_{\tau}^{t_f} \underline{N}(t,\sigma)\underline{R}(\sigma,\vartheta,\tau)d\sigma =$$

$$= \underline{N}(t,\vartheta) + \int_{\tau}^{t_f} \underline{R}(t,\sigma,\tau)\underline{N}(\sigma,\vartheta)d\sigma \qquad (3.17)$$

where $= -1$.

Moreover, in view of continuity of kernel $\underline{N}(t,\vartheta)$ the kernel $\underline{R}(t,\vartheta,\tau)$ is continuous $\forall t,\vartheta \in [\tau, t_f]$.

Corollary 1.2. From the first Fredholm theorem again eq. (3.8) has for $\lambda = -1$ unique, continuous solution $\underline{p}(t)$ defined by

$$\underline{p}(t) = \underline{q}(t) + \lambda \int_{\tau}^{t_f} \underline{R}(t, s, \tau)\underline{q}(s)ds \Big|_{\lambda = -1} \tag{3.18}$$

Corollary 1.3. By substitution of (3.8) to (3.18) and denoting

$$\underline{P}(t, \vartheta, \tau) \overset{df}{=} \underline{M}(t, \vartheta) - \int_{\tau}^{t_f} \underline{R}(t, \sigma, \tau)\underline{M}(\sigma, \vartheta)d\sigma \tag{3.19}$$

we obtain

$$\underline{p}(t) = \underline{P}(t, \vartheta, \tau)\underline{x}(\tau) + \\ + \sum_{i=1}^{m} \int_{\tau - h_i}^{\tau} \underline{P}(t, \alpha + h_i, \tau)\underline{A}_i(\alpha + h_i)x(\alpha)d\alpha \tag{3.20}$$

The states, that the function $\underline{p}(t)$ for $t \geqslant \tau$, satisfying eq. (3.2), i.e. satisfying the canonical system with its boundary conditions is uniquely determined by some linear operator on process complete state at time τ. We have then a full analogy to the "state regulator problem" for linear processes without delays where the relation $\underline{p}(t) = \underline{P}(t)\underline{x}(t)$ holds [17].

Combining (3,3) and (3.20) we have immediately:

Theorem 2. (Solution to the optimum control problem). If a linear process with delays of state variables, represented by eq. (2.1), and a quadratic functional (2.2), both satisfy the assumptions listed in sec. 2.1, 2.7, then, for given complete state at time $\tau \in [t_0, t_f]$ the optimal control $u^0(t)$ exists is unique and for $t \geq \tau$ is uniquely determined by

$$\underline{u}^0(t) = \underline{K}(t, \tau, \tau)\underline{x}(\tau) + \\ + \sum_{i=1}^{m} \int_{\tau - h_i}^{\tau} \underline{K}(t, \alpha + h_i, \tau)\underline{A}_i(\alpha + h_i)\underline{x}(\alpha)d\alpha \tag{3.21}$$

where

$$\underline{K}(t, \theta, \tau) \overset{df}{=} - \underline{R}^{-1}(t)\underline{B}'(t)\underline{P}(t, \theta, \tau) \tag{3.22}$$

This theorem implies now:

Corollary 2.1. The optimum control $\underline{u}^o(t)$ for $t \geq \tau$ is linear with respect to the complete state at time τ, i.e. for fixed t each scalar control $u_i(t)$, $i = 1, \ldots, p$, is a linear functional of complete state at time $\tau \leq t$.

Corollary 2.2. The linear operator determining the optimum control is a sum of $m + 1$ operators; first of them is a matrix depending on t and τ, and the m remaining are integral operators. Moreover, all of $m + 1$ operators are determined by one matrix kernel $\underline{P}(t, \vartheta, \tau)$.

Corollary 2.3. To obtain an effective solution it is necessary to determine the matrix kernel $\underline{P}(t, \vartheta, \tau)$ at least in a two-dimensional domain in the space t, ϑ, τ, i.e. either for $\tau = t$, $\vartheta \in [t, t + h_m]$ or for $\tau = t_o$, $\vartheta \in [t_o, t_o + h_m]$, for closed loop or open loop control, respectively. This differs essentially from the case without delays, where we have $\tau = \vartheta = t$ for closed loop control.

The matrix kernel $\underline{P}(t, \vartheta, \tau)$ is therefore a key to solution of the problem, and will be called the resolvent kernel.

4. Properties of the Resolvent Kernel $\underline{P}(t, \vartheta, \tau)$

(i) The following relation between the resolvent kernels P and R is valid:

$$\underline{P}(t, \vartheta, \tau)\underline{H}(\vartheta) = \underline{R}(t, \vartheta, \tau) \tag{4.1}$$

(ii)(Lemma 3). The kernel $\underline{P}(t, \vartheta, \tau)$ satisfies integral equations:

$$\underline{P}(t, \vartheta, \tau) = \underline{M}(t, \vartheta) - \int_{\tau}^{t_f} \underline{P}(t, \sigma, \tau)\underline{H}(\sigma)\underline{M}(\sigma, \vartheta)d\sigma \tag{4.2}$$

$$\underline{P}(t, \vartheta, \tau) = \underline{M}(t, \vartheta) - \int_{\tau}^{t_f} \underline{M}(t, \sigma)\underline{H}(\sigma)\underline{P}(\sigma, \vartheta, \tau)d\sigma \tag{4.3}$$

(iii) The kernel $\underline{P}(t, \vartheta, \tau)$ is symmetric with respect to arguments t and ϑ, i.e.:

$$\underline{P}'(\vartheta, t, \tau) = \underline{P}(t, \vartheta, \tau) \tag{4.4}$$

Proof of (i) - (iii) is given in the Appendix.

(iv) The kernel $\underline{P}(t, \vartheta, \tau)$ is continuous with respect to its arguments in respective intervals $t, \vartheta \in [\tau, t_f]$, $\tau \in [t_o, t_f]$; moreover it is piece-wise continuosly differentiable with respect to τ , giving:

$$\frac{\partial}{\partial\tau}\underline{P}(t,\vartheta,\tau) = \underline{P}(t,\vartheta,\tau)\underline{H}(\tau)\underline{P}(\tau,\vartheta,\tau) \qquad (4.5)$$

Proof: see Manitius [4] section 7.4.

These properties imply that the kernel $\underline{P}(t,\vartheta,\tau)$ may be found as a solution of Fredholm integral eqs. (4.2) or (4.3). For the closed-loop control problem these equations become Volterra equations ($\tau = t$). Therefore a numerical solution to the problem can be obtained by standard numerical procedures for solving integral equations [18]. Some difficulties appear, however, in the computation of kernel $\underline{M}(t,\vartheta)$.

For open-loop control computation the integral eq. (3.12) together with the relation (3.13) can be used. The vector $\underline{v}(t)$ has the dimension of control vector $\underline{u}(t)$, which is often much lower than the dimension of the vector $\underline{x}(t)$. Thus, solving the integral eq. (3.12) seems to be an advantageous way compared to solving the canonical system.

In the case where $\underline{Q}(t) = \underline{0} \ \ \forall\, t \in [t_0,t_f]$ ("terminal control") the kernel $\underline{M}(t,\vartheta)$ and $\underline{N}(t,\vartheta)$ become separable kernels . This allows to find the resolvent kernel $P(t,\vartheta,\tau)$ explicitly:

$$\underline{P}(t,\vartheta,\tau) = \underline{X}'(t_f,t)\underline{Z}\left[\underline{I} + \underline{W}(t_f,\tau)\underline{Z}\right]^{-1} \underline{X}(t_f,\vartheta) \qquad (4.6)$$

where

$$\underline{W}(t_f,\tau) = \int_{\tau}^{t_f} \underline{X}(t_f,s)\underline{H}(s)\underline{X}'(t_f,s)ds \qquad (4.7)$$

and the matrix inverse in (4.6) always exists [4] .

From (4.6) we may draw an interesting conclusion on the nature of optimum control for $\underline{Q} \equiv \underline{0}$. To do it let us denote:

$$\hat{\underline{x}}(t_f|\tau) = \underline{X}(t_f,\tau)\underline{x}(\tau) + \sum_{i=1}^{m}\int_{\tau-h_i}^{\tau} \underline{X}(t_f, s + h_i)\underline{A}_i(s + h_i)\underline{x}(s)ds \qquad (4.8)$$

The vector $\hat{\underline{x}}(t_f|\tau)$ is equal (see (3.6)) to the terminal state corresponding to free motion of system (2.1), given complete state at time τ . Substituting now (4.6) to (3.22) and to (3.21) and denoting

$$\underline{G}(t,\tau;t_f) = \underline{R}^{-1}(t)\underline{B}'(t)\underline{X}'(t_f,t)\underline{Z}\left[\underline{I} + \underline{W}(t_f,\tau)\underline{Z}\right]^{-1} \qquad (4.9)$$

we obtain

$$\underline{u}^o(t) = -\underline{G}(t,\tau;t_f)\hat{\underline{x}}(t_f|\tau) \qquad (4.10)$$

Now we can state the following: In the case of terminal control ($\underline{Q} \equiv \underline{0}$) the optimum control at any time $t \in [\tau, t_f]$ consists in a linear counteraction (negative sign in eq. (4.10)) to the anticipated terminal state $\hat{\underline{x}}(t_f|\tau)$, which would result from free motion in the interval $[\tau, t_f]$, given complete state at time τ.

This generalizes several similar well known results [5,19].

5. Riccati Differential Equation for $\underline{P}(t, t, t)$

By making use of lemma 1 and property 4.(iv) stated previously the formulas for partial derivatives of $\underline{P}(t,\vartheta,\tau)$ with respect to t and ϑ can be derived from eqs. (4.1) and (4.3). Together with property 4(iii) it leads to the following differential equation ($m = 1$ for simplicity of notation):

$$\frac{d}{dt}\underline{P}(t,t,t) = -\underline{A}_0'(t)\underline{P}(t,t,t) - \underline{P}(t,t,t)\underline{A}_0(t) - Q(t)$$
$$+ \underline{P}(t,t,t)\underline{H}(t)\underline{P}(t,t,t) - \underline{A}_1'(t+h)\underline{P}(t+h,t,t) - \underline{P}(t,t+h,t)\underline{A}_1(t+h) \qquad (5.1)$$

with boundary conditions:

$$\underline{P}(t_f, t_f, t_f) = \underline{Z}$$
$$\underline{P}(t, \ , t) = 0 \quad \text{for } t > t_f \text{ or } \vartheta > t_f \qquad (5.2)$$

Proof: see Manitius [4].

Equation (5.1) is a matrix differential equation of Riccati type with respect to matrix $\underline{P}(t, t, t)$. Matrix $\underline{P}(t + h, t, t)$ and its transpose play the role of forcing terms and are identically zero for $t \in (t_f - h, t_f]$ (see (5.2)). For $\underline{A}_1 \equiv \underline{0}$ we have exactly the well known Riccati equation for processes without delays.

For $t \in (t_f - h, t_f]$ equation (5.1) can be solved exactly as it was in the case without delays. This, with an appropriate equation for $\frac{\partial}{\partial\vartheta}\underline{P}(t,\vartheta,t)$, can furnish the kernel $\underline{P}(t,\vartheta,t)$.

But for $t \leq t_f - h$ this procedure becomes extensively complicated due to the presence of nonzero $\underline{P}(t + h, t, t)$ term, described by another Riccati differential equation involving $\underline{P}(t+h, t+h, t)$, which in turns involves other terms appearing succesively as time t goes back from $t_f - h$, $t_f - 2h$, etc. Thus the total dimensionality of Riccati equation is growing with the number of delay intervals remaining to the terminal time (for more details see Manitius [4] sec 7.6). We can conclude therefore that the presented equation does not seem to indicate a practical way to solve the problem numerically. It demo strates, however, the dependence of the structure of kernel $\underline{P}(t, \vartheta, t)$ upon the control interval $[t, t_f]$ relative to process delay h.

6. Optimal Performance Functional

Kernel $\underline{P}(t, \vartheta, \tau)$ appears to be closely related to the value of performance functional evaluated along an optimal trajectory. In fact we have:

Theorem 3. Given assumptions 2.1 - 2.6 and any complete state at time t_o, the performance functional evaluated along an optimal trajectory is equal to:

$$J(\underline{u}^o) = \frac{1}{2} \sum_{j=0}^{m} \sum_{k=0}^{m} \int_{t_o-h_j}^{t_o} \int_{t_o-h_k}^{t_o} \underline{x}'(s)\underline{A}_j'(s+h_j) \times$$

$$\times \underline{P}(s+h_j, \vartheta + h_k, t_o)\, \underline{A}_k(\vartheta + h_k)\underline{x}(\vartheta)\, ds\, d\vartheta \qquad (6.1)$$

Proof. Substituting (3.6) into (2.2), changing the order of integration, and reducing terms by making use of optimality conditions supplied by integral eqs. (3.12) and (4.2) along with (3.13), we obtain eq. (6.1) after some elementary but lengthy calculations.

Comment. The right hand side of eq. (6.1) can be interpreted as a quadratic form of an infi ite number of variables which are the values of functions defining the initial complete state.

It is interesting to note also, that the method presented allows to obtain the minimal performance functional without the aid of dynamic programming concepts, which have been generally used for this purpose [5-10, 17].

7. Concluding Remarks

The results presented appear to be analogous to those obtained by Kalman for processes without delays [1-3]. The linearity of optimum control with respect to the complete state, the existence of matrix kernel $P(t, \vartheta, \tau)$, the differential equation of Riccati type and the form of optimum performance functional may be interpreted as the extensions of Kalman results. Moreover, an unified method for threating these problems upon the basis of maximum principle is given.

Several related topics yet remain unsolved however. These are for example: the existence of optimum stationary control law (for infinite control interval), optimum control for processes with multiple input-delays, and estimation of state-variables for processes with time delay and random perturbations.

8. Appendix

Proof of properties 4 (i) - (iii): Multiplying both sides of eq. (3.19) by the matrix $\underline{H}(\vartheta)$ and recalling eq. (3.11) we find:

$$\underline{P}(t,\vartheta,\tau)\underline{H}(\vartheta) = \underline{N}(t,\vartheta) - \int_{\tau}^{t_f} \underline{R}(t,\sigma,\tau)\underline{N}(\sigma,\vartheta)d\sigma \qquad (8.1)$$

Comparing this with (3.17) we obtain:

$$\underline{P}(t,\vartheta,\tau)\underline{H}(\vartheta) = \underline{R}(t,\vartheta,\tau) \qquad (8.2)$$

which proves property (i). By substitution of (8.2) into (3.19) we obtain the first integral equation for kernel $\underline{P}(t,\vartheta,\tau)$ (4.2). Transposing the arguments t and ϑ and recalling the symmetry property of $\underline{M}(t,\vartheta)$ and eq. (3.11), we have

$$\underline{P}'(\vartheta, t, \tau) = \underline{M}(t,\vartheta) - \int_{\tau}^{t_f} \underline{N}(t,\sigma)\underline{P}'(\vartheta,\sigma,\tau)d\sigma \qquad (8.3)$$

which is an integral equation with $\underline{N}(t,\sigma)$ as a kernel and $\underline{P}'(\vartheta, t, \tau)$ as an unknown function. By Fredholm first theorem the solution is:

$$\underline{P}'(\vartheta, t, \tau) = \underline{M}(t,\vartheta) - \int_{\tau}^{t_f} \underline{R}(t,\sigma,\tau)\underline{M}(\sigma,\vartheta)d\sigma \qquad (8.4)$$

Comparison of (8.4) with (3.19) and substitution of (8.2) into (8.4) completes the proof of property (iii) and lemma 3.

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ON BOUNDS OF PERFORMANCE MEASURE AND MIN-MAX CONTROLLERS IN TIME-LAG SYSTEMS*

A. J. Koivo**
Purdue University
Lafayette, Indiana

S. J. Kahne
University of Minnesota
Minneapolis, Minnesota

H. N. Koivo**

Introduction

Many aerospace and industrial applications have mathematical models specified by differential-difference equations of retarded type[1]. Such systems are commonly called time-lag systems.

In general, our mathematical models only approximate the dynamics of physical systems. The mathematical description is usually written for the nominal values of parameters for the purpose of design. Often some tolerances (bounds) for the nominal values can be determined or estimated. In studying the behavior of the system, the error in the system description can then be considered as an internal disturbance whose bounds are known. A system may also be subject to external bounded disturbances; for example, wind gusts acting on a space vehicle. It appears appropriate, therefore, to investigate the influence of internal and/or external bounded disturbances within the same framework. Such a study is an application of the optimum control theory[2]. Bounded disturbances cause variations in the value of the chosen performance measure. Its upper and lower bounds can be computed from necessary conditions obtained from the maximum principle. Then the range of the performance index (PI) can serve as a useful measure in evaluating the behavior of the system.

In many practical applications, the task of the designer is then to minimize the effect of the disturbances on the PI. Such a problem can be viewed as a game between nature (disturbance) and the designer. Since the bounds of the disturbances can be approximated in many cases, the designer can compute the worst effect of nature on the PI. In order to minimize this worst effect, the designer can determine the optimal input to the system or an optimal feedback controller so as to minimize the worst value of the PI. In some problems it may be desirable to force the maximum

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** Formerly Koivuniemi.

deviation of the PI (or the difference between the upper and lower bounds) to a minimum.

The design of systems described by ordinary differential equations has been investigated from a theoretic viewpoint[3,5,6]. For problems in which the PI is minimized and maximized over function spaces specified by admissible controls and disturbances, the determination of the optimal solution often appears quite complicated to be practical, at least at the present. For problems in which the PI is maximized over a function space and minimized over a parametric space, the determination of the optimal solution by means of computational methods has been proposed[4]. Investigations of problems in which the min-max of the PI is performed over parametric spaces have also been reported[7,8,9]. This paper studies the influence of deterministic disturbances on the PI in time-lag systems. Also, the design of min-max controllers for time-lag systems is investigated. Particularly, a good measure of the influence of bounded disturbances is determined by computing the upper and lower bounds of the PI. The former corresponds to antagonistic disturbances and the latter to cooperative ones. By taking a conservative viewpoint, the designer may want to minimize the effect of the antagonistic disturbances by introducing a min-max controller in the system. This paper presents the design of such min-max controllers for time-lag systems.

Mathematical Description of the System

The state transition in a dynamical system is governed by a set of differential-difference equations

$$\dot{x}^o = f[t, x^o(t), x^o(t-\tau(t)), u(t), \alpha(t), 0], \qquad (\dot{\ }) = d/dt \tag{1}$$

where $x^o(t)$, an n-vector in E^n, is the state of the system at time t. $x^o(t-\tau(t))$ is the delayed state at time $t-\tau(t)$, where $\tau(t)$ is a non-negative C^1 function of time satisfying $d_1 \leq \dot{\tau}(t) \leq d_2 < 1$ $(d_1, d_2 \in R^1)$ all time t; then $t_f - \tau(t_f) \geq t_o - \tau(t_o)$, where t_o is the initial time and t_f is the final time of the process. $u(t)$, an m-vector in an admissible set U, is a known input to the system. $\alpha(t)$ is a parameter vector of dimension s. $f[\cdot]$ is a vector-valued function which belongs to C^2. The superscript signifies the nominal values of the variables. The initial function for system (1) is

$$x^o(t) = g(t) \quad \text{for} \quad t_o - \tau(t_o) \leq t \leq t_o \tag{2}$$

where $g(t)$ is a continuous vector-valued function of t.

The performance of system (1) is measured by $J[x^o, u, 0, \alpha]$, which is a continuous functional in $x^o(t)$, $u(t)$, $w(t)$, and $\alpha(t)$, and the superscript indicates the absence of disturbances.

Suppose that system (1) now moves under the influence of the input $u(t) \in U$ and a bounded disturbance $w(t) \in W$. The evolution of the states with time is now governed by

$$\dot{x} = f[t, x(t), x(t-\tau(t)), u(t), \alpha(t), w(t)] \tag{3}$$

where f, u(t) and $\tau(t)$ are the same as in equation (1); $\alpha(t)$ signifies the nominal value of the parameter vector; $w(t)$, an r-vector, is a disturbance at time t in an admissible set W of bounded disturbances. The state vector at time t is $x(t)$; $x(t) \equiv x^o(t)$ if $w(t) \equiv 0$, $\forall$ t. The behavior of system (3) is measured by a well-defined PI, $J[x, u, w, \alpha]$.

It is noted that $w(t) \in W$ represents both internal and/or external bounded disturbances. The description for the system subject to internal disturbances is obtained from equation (1) by replacing $\alpha(t)$ by $\alpha(t) + w(t)$, where $w(t)$ is admissible. Such disturbances may be generated by the parameters in plant and/or control matrix. External disturbances can often be included in the system description. For example, the designer may approximate or determine from the speed profiles of the wind the upper bound of forces acting as bounded disturbances on a large flexible booster.

Determination of Bounds for Performance Measure

Disturbances acting on the system can be static or dynamic in nature. Disturbances caused by inaccurate information about the initial function (or initial condition) or about the initial time of the process are termed static. On the other hand, disturbances appearing explicitly in the dynamical equations of the system are dynamic[2]. If the constraints on admissible disturbances are known, the bounds of the PI can be obtained. The determination of the upper and lower bounds of the PI are studied in this section first for the static case and then for the dynamic case.

(A) Static Case

The approach for computing the upper and lower bounds of the PI in the presence of a static disturbance in time-lag systems is illustrated by an example.

Example 1:

The state transition in a time-lag system evolves in time according to

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -3 & -1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1(t-0.5)\\ x_2(t-0.5)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t) \quad t \in [0,5] \tag{4}$$

or $\dot{x} = Ax(t) + Bx(t-0.5) + \Gamma u(t)$ with obvious definitions. The initial function of the system is governed by

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix}\cos(t-t_o-0.5)\\ \sin(t-t_o-0.5)\end{bmatrix}, \quad t \in [-0.5,0] \tag{5}$$

The nominal value of t_o is known to be 1.0 with an accuracy of $\Delta t_o = 0.5$.

The problem is first to compute the unrestricted optimal control $u^*(t)$ so as to minimize

$$J[x,u,0,t_o] = 50\,\|x(5)\|^2 + \frac{1}{2}\int_0^5 (x_1^2 + x_2^2 + u^2)\,dt \tag{6}$$

Then the upper and lower bounds of the PI about the nominal value of the PI is to be determined.

Necessary conditions for optimality can be written by means of the maximum principle for time-lag systems[10] (see equations (6)-(10)). The optimal solution can then be computed, for example, by a second-order variational method[12]. It yields the nominal value of the PI: $J[x^*, u^*, 0, 1.0] \approx 0.2335$. Then the bounds of the PI in the presence of an opposing and co-operative disturbance Δt_o can be computed in a straightforward manner. The results are:

$$J[x^*, u^*, \Delta t_{o_M}, 0.5] \approx 0.7053 \quad ; \quad J[x^*, u^*, \Delta t_{o_m}, 1.5] \approx 0.1232$$

Thus, the bounds of the optimal value of the PI have been established when the system is subject to a bounded static disturbance (Δt_o).

(B) Dynamic Case

The nominal trajectory $x^o(t)$ and the corresponding nominal value of the PI $J[x^o, u, 0, \alpha]$ are obtained from equation (1) (disturbance absent). Then the bounds for the performance measure are determined by computing trajectories and the corresponding values of the PI from equation (2) (with antagonistic disturbance $w_M(t)$ present, and then with co-operative disturbance $w_m(t)$ present).

Necessary conditions for the determination of the range of the PI are written from the maximum principle for time lag systems[11]. Suppose the PI is expressed in the Mayer form:

$$J[x,u,\alpha,w] = G[w(t_f),t_f] \tag{7}$$

where $G[\cdot]$ is smooth non-negative scalar function of $x(t_f)$ and t_f (the former free and the latter fixed for convenience). Define the H-function of the system by

$$H[t,x,u,\alpha,w] = p'(t)f[t,x(t),x(t-\tau(t)),u(t),\alpha(t),w(t)] \tag{8}$$

where $p(t) = \mathrm{col}[p_1(t),\ldots,p_n(t)]$, and the superscript indicates the transposition. Necessary conditions for the extremum are as follows:

$$\dot{x} = f[t,x(t),x(t-\tau(t)),u(t),\alpha(t),w(t)],\quad t \in [t_o,t_f] \tag{3}$$

$$-\dot{p}(t') = p'(t')\left(\frac{\partial f[\cdot]}{\partial x(t)}\right)_{t=t'} + p'(\tau_1(t'))\left(\frac{\partial f[\cdot]}{\partial x(t-\tau(t))}\right)_{t=\tau_1(t')} \frac{d\tau_1(t')}{dt'} \tag{9}$$

where $t = \tau_1(t')$ is obtained by inverting $t-\tau(t) = t'$; $t' \in [t_o,t_f-\tau(t))$

$$-\dot{p}(t) = p'(t)\left(\frac{\partial f[\cdot]}{\partial x(t)}\right),\quad t \in [t_f-\tau(t_f),t_f] \tag{10}$$

$$\max_{w(t)\in W} H[t,x,u,\alpha,u] \text{ determines } w_m^*(t) \text{ (co-operative)} \tag{11$'$}$$

$$\min_{w(t)\in W} H[t,x,u,\alpha,u] \text{ determines } w_M^*(t) \text{ (antagonistic)} \tag{11$''$}$$

$$x(t_o-) = x(t_o+) \tag{12$'$}$$

$$x(t) = g(t)\quad,\quad t \in [t_o-\tau(t_o),t_o] \tag{12$''$}$$

$$p[t_f-\tau(t_f)-] = p[t_f-\tau(t_f)+] \tag{13$'$}$$

$$p(t_f) = \frac{\partial G[\cdot]}{\partial x(t_f)} \tag{13$''$}$$

where $\partial f[\cdot]/\partial x$ signifies the Jacobian matrix of $f[\cdot]$ relative to x (with proper arguments); similarly, $\partial G[\cdot]/\partial x$. The subscript in equation (9) indicates that t is replaced by $\tau_1(t')$; in case of constant τ, $\tau_1 = t+\tau$ and $f[\cdot]$ becomes $f[t+\tau,x(t+\tau),x(t),u(t+\tau),\alpha(t+\tau),w(t+\tau)]$. $w_m^*(t)$ denotes the admissible disturbance that minimizes $G[\cdot]$, whereas $w_M^*(t)$ is the admissible disturbance that maximizes $G[\cdot]$. Equations (12$''$) and (13$''$) specify the boundary conditions for the state $x(t)$ and the costate $p(t)$. Equations (12$'$) and (13$'$) express the continuity conditions for $x(t)$ at time t_o and for $p(t)$ at time $t_f-\tau(t_f)$. (The - or + signs in the arguments indicate the limiting value when the time is approached from the left or from the right, respectively.)

If the admissible set W for the disturbances is closed and bounded, e.g., $W = \{w(t);\ |w(t)| \le 1,\ \forall\ t\}$, and if $w(t)$ appears only linearly in $H[\cdot]$, then equation (11′) implies that

$$w_m^*(t) = \mathrm{sgn}\left[p'(t)\ \frac{\partial f[\cdot]}{\partial w(t)}\right] \tag{14}$$

where $\partial f[\cdot]/\partial w$ is the Jacobian of $f[\cdot]$ relative to $w(t)$; $\mathrm{sgn}\ Z = \mathrm{col}[\mathrm{sgn}\ z_1, \ldots, \mathrm{sgn}\ z_r]$ and $\mathrm{sgn}\ [z_i]$, $i=1,\ldots,r$ is a scalar function which is equal to 1 if $z_i > 0$ and -1 if $z_i < 0$, and undetermined if $z_i = 0$. For $w_M^*(t)$, equation (14) is taken with the opposite sign.

Equations (3) and (9)-(13′) furnish necessary conditions for determining the least upper and greatest lower bound of the PI. The solution to the split boundary value problem must, in general, be obtained by means of a digital computer.

Computing the Solutions

The upper bound and the lower bound for the PI can now be computed by determining candidates for the solution of the boundary value problem posed by the necessary conditions (equation (3) and (9)-(13′)). Algorithms for solving optimum control problems in time-lag systems have been proposed.[11,12] Both algorithms treat only unconstrained cases. Difficulties on the convergence of the algorithms are likely to arise when the inputs (disturbances) are bounded.

Since the disturbances in this study are usually bounded, a gradient method in the function space based on first order variations is applied. The disturbances are changed in the direction of the gradient of PI so that the PI will converge towards the optimal value. Whenever the values exceed the constraint set, appropriate boundary values are used. A good exposition of the method has been presented[13].

It is noted that the determination of the range of the PI requires solutions to two independent two-point boundary value problems.

Example 2:

The state-transition in a time-lag system is governed by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2-w(t) & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-0.5) \\ x_2(t-0.5) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{15}$$

or $\dot{x}(t) = A(w)x(t) + Bx(t-0.5) + \Gamma u(t)$ with obvious definitions. $w(t)$ represents an internal disturbance due to the variation of a parameter; it is constrained by $|w(t)| \le 0.5,\ \forall\ t$. $x(t) = \mathrm{col}[x_1(t), x_2(t)]$ and $u(t)$

is a unit step function $1(t)$ applied at $t = 0$. The initial function for the system is $x_1(t) = 1.0$ and $x_2(t) = 0$ for $-0.5 \leq t \leq 0$. To determine the bounds for the performance measure, the influence of $w(t)$ on the PI is studied:

$$J[x(t_f),1(t),w,-2] = \frac{1}{2}[x_1^2(t_f) - \frac{1}{3}]^2 + \frac{1}{2} x_2^2 (t_f) \tag{16}$$

where the terminal time is specified by $t_f = 6$ sec.

The nominal value of the PI is to be computed when $w(t) \equiv 0$, $0 \leq t \leq 6$. Also, the range of the PI is to be determined by minimizing $J[\cdot]$ and then maximizing $J[\cdot]$ relative to admissible disturbances.

Necessary conditions for a solution become now

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2+w(t) \\ -1 & 1 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1(t+0.5) \\ p_2(t+0.5) \end{bmatrix}, \quad 0 \leq t \leq 5.5 \tag{17'}$$

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2+w(t) \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \quad \text{for } 5.5 \leq t \leq 6.0 \tag{17''}$$

The boundary conditions are

$$p_1(t_f) = x_1(t_f); \; p_2(t_f) = x_2(t_f); \; p(5.5-) = p(5.5+) \tag{18}$$

The antagonistic disturbance is specified by

$$w_M(t) = 0.5 \operatorname{sgn}[p_2(t)x_1(t)] \tag{19}$$

and the co-operative disturbance by

$$w_m^*(t) = -0.5 \operatorname{sgn}[p_2(t)\, x_1(t)] \tag{20}$$

The nominal value ($w(t) \equiv 0$) of the PI is $J[x(t_f),1(t),0,-2] \approx 0.3484$. The application of the gradient method yields the upper bound $J[x(t_f),1(t), w_M^*,-2] \approx 0.5233$ and the lower bound $J[x(t_f),1(t),w_m^*,-2] \approx 0.1984$. Hence, the range of the PI is

$$0.1984 \lesssim J[x(t_f),1(t),w,-2] \lesssim 0.5233$$

Design of Min-Max Controllers

The application of necessary conditions for optimality of time-lag systems results in an open-loop optimal control which is dependent upon time and initial function. From the viewpoint of the engineers, it is desirable to obtain an optimal feedback control which is a function of the current state variables and possibly of time. However, a realizable optimal feedback control in time-lag systems can be determined only in very special cases (it is possible even for the general case of linear plant-quadratic criterion). Moreover, its implementation may not even be feasible.

A possible approach to the problem is then to introduce a controller in the system with known input-output relation. This controller operates on the available state and/or delayed state varisbles to generate a control that results in an approximately optimal system. It is accomplished by determining the parameters of the controller in an optimal manner. Since the optimal values of the parameters depend upon the initial conditions, one possibility is to determine the best parameter's so as to minimize the maximum degradation (relative to admissible initial conditions) of the PI from the optimal value[7,8,15]. Such a min-max controller is often called a specific optimal controller[8]. Another possibility is to use a performance index that is independent of the initial conditions. For example, one can assume a probability distribution for the initial conditions and then use the expected value of the PI[14].

The min-max controller to be used here for time-lag systems is essentially a specific optimal controller; its structure is prespecified within adjustable parameters. Since the use of the min-max controller results in a deterioration of the PI, and since the best parameters depend upon the parameters specifying the initial function, the goal in the design is to maximize the degradation of the PI (from the nominal value) relative to the parameters of the initial function and minimize the same degradation relative to the feedback coefficients of the controller. An additional requirement on the initial functions is that they have to be continuous.

To formulate the problem, let W signify the class of bounded antagonistic disturbances (static and dynamic), and Ω is the class of co-operative controls. The problem is then to determine

$$\min_{\Omega} \max_{W} DJ$$

where DJ denotes the performance measure chosen. It is noted that the class W may consist of initial conditions (functions), initial (final) time, and/or external (internal) disturbances, and that Ω may include the control input, parameters of the feedback controller, and/or initial (final) time. The choice of DJ is determined by the goals of the design. It can be, for example, the degradation of the PI from the nominal value due to the use of a suboptimal feedback controller in the system.

The procedure to design suboptimal feedback controllers for time-lag systems on the basis of min-max criterion is illustrated by examples.

Example 3:

Suppose that system (4) of Example 1 employs a suboptimal feedback controller specified by

$$u_s[t, x(t-0.5)] = + k\, x_1(t-0.5) \tag{21}$$

where k is the feedback gain. The best value k^* of k in $\Omega = \{k; 0 \le k \le 1\}$ is to be determined so that

$$\underset{k \in \Omega}{\text{Min}} \ \underset{t_o}{\text{Max}} \left\{ 50\left[\|x(t)\|^2 - \|x^*(5)\|^2\right] + \frac{1}{2}\int_0^5 \left[\|x(t)\|^2 - \|x^*(t)\|^2 + u_s^2 - u^{*2}\right] \right\} dt \tag{22}$$

is attained. The superscript in expression (22) refers to the ideally optimal solution obtained by the application of necessary conditions to system (4) and x(t) is obtained when the suboptimal controller specified by equation (21) is employed. When expression (22) is used for DJ, the designer attempts to "match" the value of the PI associated with the suboptimal system with that of the PI obtained for the ideally optimal system.

The behavior of the solution is illustrated in Figure 1 by displaying DJ vs. t_o using k as a parameter. The application of a standard gradient method to the problem yields:

$$k^* \approx 0.60 \ , \quad t_o^* \approx 2.44 \ , \quad DJ \approx 0.1168$$

where k^* and t_o^* provide the min-max solution. The degradation, DJ, of the PI from the ideally optimal value of the PI is approximately 14.4%.

Example 4:

The system studied in Example 2 is to be so designed that the worst influence of the internal disturbance (parameter variation) on the PI specified by equation (16) is minimized. It is accomplished by determining the constant coefficient α_1 of $x_1(t)$ in equation (15) so as to attain

$$\underset{\alpha_1 \in \Omega}{\text{Min}} \quad \underset{w \in W/\alpha_1}{\text{Max}} \quad \frac{1}{2}\left\{ \left[x_1(t_f) - \frac{1}{1+\alpha_1}\right]^2 + x_2^2(t_f)\right\} \tag{23}$$

The solution to the problem can be determined by the approximate gradient method[2,8]. For every specified value of α_1, the worst disturbance $w^*(t) \in W$ is computed. Then the value of α_1 is changed to $\alpha_1 + \Delta\alpha_1$ so as to decrease the maximum of the PI. It can be accomplished by means of writing the first-order differential for DJ (the expression in the braces):

$$\Delta DJ[\alpha_1, w^*; \Delta\alpha] \approx \left[\left(x_1(t_f) - \frac{1}{1+\alpha_1}\right)\left(\frac{\partial x_1(t_f)}{\partial \alpha_1} + \frac{1}{(1+\alpha_1)^2}\right) + x_2(t_f)\frac{\partial x_2(t_f)}{\partial \alpha_1}\right]\Delta\alpha_1 \quad (24)$$

where $x_1(t_f)$ and $x_2(t_f)$ are evaluated for α_1 and the worst disturbance $w^*(t) \in W$ and $\Delta\alpha_1$ signifies a change in α_1. The value of $\Delta\alpha_1$ is determined so that $\Delta DJ[\cdot]$ becomes negative. It is noted that $\partial x_1(t_f)/\partial\alpha_1$ and $\partial x_2(t_f)/\partial\alpha_1$ can be computed from the perturbation equations.

The solution is furnished by

$$\alpha_1^* \approx 4.02 \qquad DJ \approx 0.2631$$

It is emphasized that the solution is local. Other solutions may be found by starting from a different initial value. In many practical problems the parameter α_1 is constrained to lie in a closed and bounded set. In such cases, global solutions can be determined.

Conclusions

When a system is subject to bounded disturbances, the value of the PI deviates from the nominal one. The upper and lower bounds of the PI for time-lag systems can be determined by the application of the optimum control theory. Such bounds establish a good measure of the influence of possible disturbances acting on the system. Since the optimal feedback control can rarely be determined for time-lag systems, an approximately optimal controller that operates on the current state and/or delayed state variables offers a practical solution. Such controllers can be designed on the basis of min-max criterion. A procedure for such a design is presented. The presentation is illustrated by examples.

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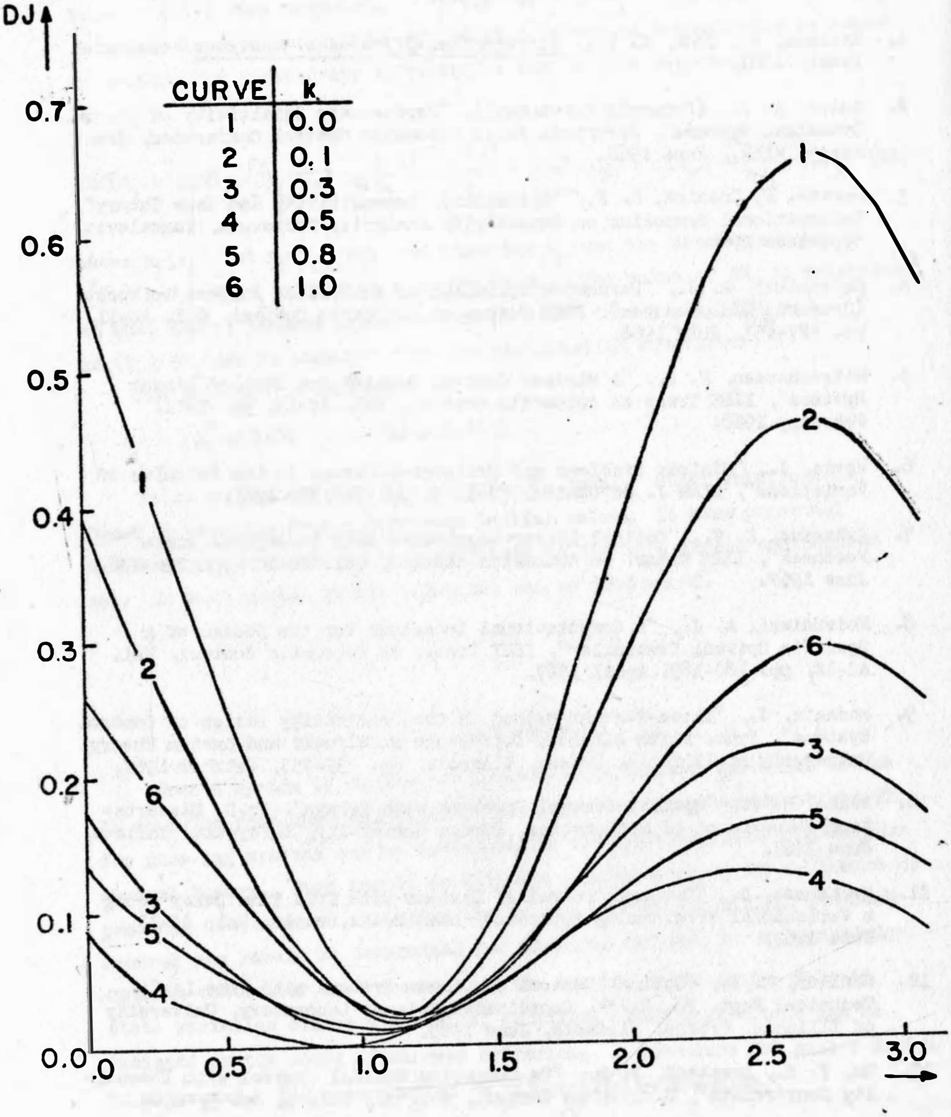

FIG. 1.

DETERIORATION DJ VS t_0 USING k AS A PARAMETER

OPTIMAL DISCRETE CONTROL OF TIME LAG SYSTEMS

James F. Banas
Staff Member - Technical*
Sandia Corporation
Albuquerque, New Mexico
87106
U.S.A.

André G. Vacroux
Assistant Professor
Dept. of Electrical Engineering
Illinois Institute of Technology
Chicago, Illinois 60616
U.S.A.

INTRODUCTION

In recent years, there has been a growing interest in the area of system optimization. Necessary conditions for optimality, the Maximum Principle, were derived by Pontryagin[1] for systems described by differential equations with piecewise continuous control. This work was extended by Kharatishvilli[2] to the optimal control of systems described by differential-difference equations containing a constant delay.

Ragg[3] used the calculus of variations to find necessary conditions for the optimal piecewise continuous control of time lag systems with time-varying delays. The delay was considered to be a function of time, state, and the time derivative of the state but an error in the derivation[4] restricted the range of applicability of his results.

The synthesis of the optimal control often necessitates a means of carrying on many accurate computations in a short time. The digital computer, while providing such a means, gives output in the form of discrete values. Jordan and Polak[5] found local necessary conditions for the optimality of systems described by differential equations with piecewise constant control. A maximum principle was formulated by Becker and Lopresti[6] for the optimal piecewise constant control of linear time lag systems containing a single constant delay in state. In another paper[7], the authors extended the above results to include time lag systems with multiple constant delays in state and control.

In this paper, consideration is given to the optimal piecewise constant control of time lag systems with time-varying delays. A Hamiltonian type functional and a costate system are defined which include the time dependence of the delay in state. Local necessary conditions are found for the optimal control of systems which are not necessarily linear and a maximum principle is found for linear systems with a quadratic index of performance. Finally, an analytic example is given for a particular form of the time-varying delay.

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TIME-VARYING DELAY IN STATE

Formulation of the Problem

State System Equation. Consider a system described by the differential-difference equation

$$\tilde{\dot{x}}(t) = \tilde{f}\big(\tilde{x}(t), \tilde{x}_\theta(t), u(t), t\big), \quad t_o \le t \le t_K, \tag{1}$$

with the initial condition

$$\tilde{x}(t) = \tilde{\varphi}(t), \quad t_o - \theta(t_o) \le t \le t_o, \tag{2}$$

where $\tilde{x} \in E^n$ is a column vector with components $x_1, \cdots, x_n$ representing the state of the system, $u \in U$ (a given subset of E^r) is a column vector with components $u_1, \cdots, u_r$ representing the system controller, and $\tilde{f}$ is a vector function with each component $f_1, \cdots, f_n$ of class C^1 over $E^n \times U$.

The delay $\theta(t)$ is a given function of time satisfying (the reason for these restrictions will become clear later)

$$\theta(t) \ge 0 \text{ and } 0 \le \dot{\theta}(t) < 1 \text{ for } t_o \le t \le t_K,$$

and the delayed state is represented by $\tilde{x}_\theta(t) \equiv \tilde{x}\big(t-\theta(t)\big)$. The initial function $\tilde{\varphi}(t)$ is specified for the state for $t_o - \theta(t_o) \le t \le t_o$ so that a unique solution to the system (1) is obtained. The process runs from a given fixed initial time t_o to a given fixed final time t_K.

Admissible Controls. Let U be a given subset of E^r with the properties[5]: (i) for every $v \in U$ there exists at least one $\delta v \in E^r$, $\delta v \ne 0$, and a constant $\beta(v, \delta v) > 0$ such that $(v + \varepsilon\, \delta v) \in U$ for all $0 \le \varepsilon \le \beta(v, \delta v)$ (ii) the set of all δv for a given $v \in U$ satisfying (i) is a convex cone Λ.

Let the sampling instants $t_1, \cdots, t_{K-1}$ be given fixed instants of time such that $t_o < t_1 < \cdots < t_{K-1} < t_K$ for some fixed positive integer K. Let V be the set of all vectors $\nu = (v_o, \cdots, v_{K-1}) \in E^{rK}$ such that $v_i \in U$, $i = 0, \cdots, K-1$. Then for any $\nu \in V$ let the piecewise constant controller $u(t;\nu)$ be defined by

$$u(t;\nu) = v_{k-1}, \quad t_{k-1} \le t < t_k, \quad k = 1, \cdots, K$$
$$u(t_K;\nu) = v_{K-1},$$

which is denoted by $u(t)$, $t_o \le t \le t_K$. All such piecewise constant controllers will be referred to as admissible.

Terminal State. On the interval $t_o \le t \le t_K$ the system (1) has a solution $\tilde{x}(t;\tilde{\varphi},t_o,u)$, denoted by $\tilde{x}(t)$ and referred to as the trajectory of the system, which corresponds to the controller $u(t)$. The final state $\tilde{x}(t_K)$ is constrained to lie in some given set S. For simplicity the set S is chosen here to be E^n, that is, the endpoint is free.

The choice of the terminal constraint set S does not alter the basic results of this work; the specific form of S determines a transversality condition[5].

Index of Performance. To each admissible controller $u(t)$, $t_o \leq t \leq t_K$, with the corresponding trajectory $\tilde{x}(t)$, $t_o \leq t \leq t_K$, is assigned a "cost" according to the index of performance $J[u]$ with

$$J[u] = \int_{t_o}^{t_K} f_o\big(\tilde{x}(t), u(t), t\big)\, dt \tag{4}$$

where the scalar function f_o is continuous and continuously differentiable with respect to $\tilde{x}$ and u over $E^n \times U$.

Define a new variable x_o which satisfies the differential equation

$$\dot{x}_o(t) = f_o\big(\tilde{x}(t), u(t), t\big), \quad t_o \leq t \leq t_K, \tag{5}$$

with the initial condition

$$x_o(t_o) = 0. \tag{6}$$

Then it is clear that $x_o(t_K;u) = J[u]$ represents the "cost" of transition between the initial state $\tilde{x}(t_o)$ and the final state $\tilde{x}(t_K)$.

Statement of the Problem. For the system (1) with initial function (2) and the index of performance (4), it is required to find among all admissible controllers $u(t;\nu)$, $\nu \in V$, $t_o \leq t \leq t_K$, which yield $\tilde{x}(t_K) \in S$, a control $u^*(t) = u(t;\nu^*)$, $\nu^* \in V$, $t_o \leq t \leq t_K$, such that $x_o(t_K;u^*) \leq x_o(t_K;u)$. Such a control $u^*(t)$, $t_o \leq t \leq t_K$, will be called an optimal control and the corresponding trajectory $\tilde{x}^*(t) = \tilde{x}(t;\tilde{\varphi},t_o,u^*)$ will be called an optimal trajectory.

Necessary Conditions for an Optimal Control

Augmented System Equation. For notational convenience, the cost variable x_o and the state variables $x_1,\cdots,x_n$ are combined into the column vector x with components $x_o,x_1,\cdots,x_n$; similarly the column vector function f is defined to have components $f_o,f_1,\cdots,f_n$. The augmented system is described by the differential-difference equation

$$\dot{x}(t) = f\big(\tilde{x}(t), \tilde{x}_\theta(t), u(t), t\big), \quad t_o \leq t \leq t_K, \tag{7}$$

with the initial conditions

$$x(t) = \varphi(t), \quad t_o - \theta(t_o) \leq t \leq t_o \tag{8}$$

where the initial function φ is a column vector with components $0, \varphi_1,\cdots,\varphi_n$.

Time Lead Function. The advance function $\rho(\cdot)$ is introduced to take into account the functional dependence of the delay $\theta(\cdot)$ on time. If

$$s = t - \theta(t), \; t_o \le t \le t_K \tag{9a}$$

can be solved for t, then $\rho(s)$ is defined by

$$t = s + \rho(s). \tag{9b}$$

For example, in the case of a linear time-varying delay $\theta(t) = at + b$ where a and b are constants, it follows that

$$t = \frac{s+b}{1-a}, \quad \rho(s) = \frac{as+b}{1-a}. \tag{10}$$

Note for the case of a constant delay (a = 0) the equations (10) become

$$t = s + b, \quad \rho(s) = b, \tag{11}$$

that is, the advance is equal to the delay.

Costate System Equation. Let the costate p, a column vector with components $p_o, p_1, \cdots, p_n$, be the solution to the differential-difference equation

$$\dot{p}(t) = - \left[\frac{\partial f}{\partial x}(t)\right]^T p(t) - \left[\frac{\partial f}{\partial x_\theta}(t+\rho(t))\right]^T \frac{p_\rho(t)}{1 - \dot{\theta}(t+\rho(t))}, \; t_o \le t \le t_K - \theta(t_K)$$

$$\dot{p}(t) = - \left[\frac{\partial f}{\partial x}(t)\right]^T p(t), \quad t_K - \theta(t_K) \le t \le t_K, \tag{12}$$

where $p_\rho(t) \equiv p(t+\rho(t))$, the $(n+1) \times (n+1)$ matrices $\frac{\partial f}{\partial x}(t)$, $\frac{\partial f}{\partial x_\theta}(t+\rho(t))$ have j, k^{th} elements $\frac{\partial f_j}{\partial x_k}$, $\frac{\partial f_j}{\partial x_{\theta k}}$ and are evaluated for the control and trajectory at the times t, $t+\rho(t)$ respectively. The superscript T denotes the matrix transpose.

The Hamiltonian. For the arbitrary times t' and t'' $(t_o \le t' \le t'' \le t_K)$ define a Hamiltonian type functional for the system (7) by

$$H(x,x_\theta,p,p_\rho,u,t',t'') = p^T(t'')\Delta x(t'',t') + \int_{t'-\theta(t')}^{t''} \frac{p_\rho^T(s)}{1-\dot{\theta}(s+\rho(s))} \left[\frac{\partial f}{\partial x_\theta}(s+\rho(s))\right] \Delta x(s,t')ds \tag{13}$$

where $\Delta x(s,t')$, the difference in the state at time s and time t', can be written

$$\Delta x(s,t') = x(s) - x(t') = \int_{t'}^{s} f(\bar{x}(\alpha),\bar{x}_\theta(\alpha),u(\alpha),\alpha)d\alpha.$$

Also, for reasons to be seen later, make the definition

$$\left[\frac{\partial f}{\partial x_\theta}(s+\rho(s))\right] \triangleq 0 \text{ for } s+\rho(s) \ge t_K \tag{14}$$

Theorem 1: General Necessary Conditions. If $u^*(t) = u(t;v^*)$, $v^* = (v_o^*, \cdots, v_{K-1}^*) \in V$ is an optimal control with corresponding optimal trajectory $x^*(t) = x(t;\bar{\varphi},t_o,u^*)$ for $t_o \le t \le t_K$, then there exists a nonzero function $p^*(t)$, $t_o \le t \le t_K$, satisfying the costate equation (12) along the optimal control and optimal trajectory, such that

(i) the Hamiltonian (13) evaluated for successive sampling instants is either stationary or a local maximum with respect to the control along the optimal trajectory $x^*(t)$, $t_o \leq t \leq t_K$, and the costate trajectory $p^*(t)$, $t_o \leq t \leq t_K$;

(ii) $p_o^*(t_K) \leq 0$;

(iii) the transversality condition $p_i^*(t_K) = 0$, $i = 1, \cdots, n$ is satisfied.

As discussed earlier, the endpoint condition on the state determines the transversality condition (iii). Then if

(1) the terminal state is a fixed point, there is no constraint on $\bar{p}^*(t_K)$,

(2) the terminal state is required to lie on a manifold S, $\bar{p}^*(t_K)$ must obey $\bar{p}^{*T}(t_K)\,\alpha = 0$ where α is any vector in the plane tangent to S at $\bar{x}^*(t_K)$.

The derivations of these conditions are similar to Jordan and Polak[5].

Perturbations in the Control and Trajectory. Assume the existence of an optimal control $u^*(t) = u(t;\nu^*)$, $\nu^* = (v_o^*,\cdots,v_{K-1}^*) \in V$, $t_o \leq t \leq t_K$, and corresponding optimal trajectory $x^*(t) = x(t;\bar{\varphi},t_o,u^*)$, $t_o \leq t \leq t_K$. Consider the variation in the control $\delta\nu = (\delta v_o,\cdots,\delta v_{K-1})$ where $\delta\nu \in \Lambda(v_i^*)$, $i = 0,\cdots,K-1$, then for an $\epsilon > 0$ sufficiently small, $(\nu^* + \epsilon\delta\nu) \in V$. Let $u(t;\epsilon)$ denote the control $u(t;\nu^* + \epsilon\delta\nu)$ and $x(t;\epsilon)$ denote the trajectory $x(t;\bar{\varphi},t_o,u(t;\epsilon))$. The variation in the trajectory $\delta x(t;\epsilon)$ is defined by $\delta x(t;\epsilon) = x(t;\epsilon) - x^*(t)$ for $t_o \leq t \leq t_K$. Consider the particular perturbation of the optimal control such that $\delta v_i = 0, i = 0,\cdots, K-1$, $i \neq k-1$. It follows that

$$\left(\frac{\partial u}{\partial \epsilon}(t;\epsilon)\right)_{\epsilon=0} = \begin{cases} \delta v_{k-1}, & t_{k-1} \leq t < t_k \\ 0\ , & \text{otherwise} \end{cases} \tag{15}$$

and if $x(t;\epsilon)$ is expanded in a Taylor series about $\epsilon = 0$, the first term $y(t) = \left(\frac{\partial x}{\partial \epsilon}(t;\epsilon)\right)_{\epsilon=0}$ satisfies the linear differential-difference equation

$$\dot{y}(t) = \left[\frac{\partial f}{\partial x}(t)\right]_o y(t) + \left[\frac{\partial f}{\partial x_\theta}(t)\right]_o y(t-\theta(t)) + \\ + \left[\frac{\partial f}{\partial u}(t)\right]_o \left(\frac{\partial u}{\partial \epsilon}(t;\epsilon)\right)_{\epsilon=0}, \quad t_o \leq t \leq t_K \tag{16}$$

with the initial condition $y(t) = 0$, $t_o - \theta(t_o) \leq t \leq t_o$. The notation $|\ |_o$ is used to denote those matrices which are evaluated for the optimal control along the optimal trajectory.

It can be shown[8] that an equation of the form (16) has the solution

$$y(t) = \int_{t_o}^{t} Y(s,t)\left|\frac{\partial f}{\partial u}(s)\right|_{\bullet} \left(\frac{\partial u}{\partial \varepsilon}(s;\varepsilon)\right)_{\varepsilon=0} ds, \quad t_o \leq t \leq t_K \tag{17}$$

where the (n+1) x (n+1) matrix $Y(s,t)$ satisfies the adjoint equation

$$\frac{\partial}{\partial s}Y(s,t) = -Y(s,t)\left|\frac{\partial f}{\partial x}(s)\right|_{\bullet} - \frac{Y(s+\rho(s),t)}{1-\dot\theta(s+\rho(s))}\left|\frac{\partial f}{\partial x_\theta}(s+\rho(s))\right|_{\bullet}, \quad t_o \leq s \leq t - \theta(t)$$

$$\frac{\partial}{\partial s}Y(s,t) = -Y(s,t)\left|\frac{\partial f}{\partial x}(s)\right|_{\bullet}, \quad t - \theta(t) \leq s \leq t \tag{18}$$

with the initial condition $Y(t,t) = I$, the (n+1) x (n+1) identity matrix. Substituting the variation in control (15) into the solution for $y(t)$ of equation (17)

$$y(t_K) = \int_{t_{k-1}}^{t_k} Y(s,t_K)\left|\frac{\partial f}{\partial u}(s)\right|_{\bullet} \delta v_{k-1}\, ds, \tag{19}$$

and the variation in the optimal trajectory at time t_K becomes

$$\delta x(t_K;\varepsilon) = \varepsilon y(t_K) + O(\varepsilon^2) \tag{20}$$

where $O(\varepsilon^2)$ is a column vector with components $O_o(\varepsilon^2),\cdots,O_n(\varepsilon^2)$ satisfying $\lim\limits_{\varepsilon\to 0} \frac{O_i(\varepsilon^2)}{\varepsilon} = 0, i = 0,\cdots,n.$

The Terminal Cone. Let C be the set of all vectors $x^*(t_K) + \varepsilon y(t_K)$ where $y(t_K)$ is given by (19). Since $y(t_K)$ is linear in the control perturbation, it is clear that C is a convex cone with vertex at the terminal point $x^*(t_K)$.

By hypothesis the control $u^*(t)$, $t_o \leq t \leq t_K$, is optimal and must give the minimum cost, therefore all points in C (formed by the use of perturbed controls) have a cost component $x_o(t_K)$ which is greater than the optimal cost $x_o^*(t_K)$. Thus C must be disjoint of the halfspace $G = \{x | x_o < x_o^*(t_K)\}$. The plane $F = \{x | x_o = x_o^*(t_K)\}$ must separate the cone C from the halfspace G. A vector in G that is transverse to F is a with components $a_o, 0, \cdots, 0$ where $a_o \leq 0$. Then the inequality $a^T y(t_K) \leq 0$ must hold where $y(t_K)$ is given by (19). Let $p_o^*(t_K) = a_o \leq 0$ and $p_i^*(t_K) = 0, i = 1,\cdots,n$, then

$$p^{*T}(t_K)y(t_K) \leq 0. \tag{21}$$

A Relation Between the Costate System and Variations in the Trajectory. Since (21) gives a relation between the costate and trajectory at the final time, it is desired to relate this inequality to other instants of time. To achieve this end, consider the quantity Δ_k where

$$\Delta_K = \left\{ p^{*T}(t_K)y(t_K) + \int_{t_K-\theta(t_K)}^{t_K} \frac{p_0{*}^T(s)}{1-\dot\theta(s+\rho(s))}\left[\frac{\partial f}{\partial x_\theta}(s+\rho(s))\right]_* y(s)\, ds \right\} - \left\{ p^{*T}(t_k)y(t_k) + \int_{t_k-\theta(t_k)}^{t_k} \frac{p_0{*}^T(s)}{1-\dot\theta(s+\rho(s))}\left[\frac{\partial f}{\partial x_\theta}(s+\rho(s))\right]_* y(s)\, ds \right\} \tag{22}$$

which can be written in the form

$$\Delta_k = \int_{t_k}^{t_K} \frac{d}{ds}\left\{ p^{*T}(s)y(s) + \int_{s-\theta(s)}^{s} \frac{p_0{*}^T(\gamma)}{1-\dot\theta(\gamma+\rho(\gamma))}\left[\frac{\partial f}{\partial x_\theta}(\gamma+\rho(\gamma))\right]_* y(\gamma)\, d\gamma \right\} ds. \tag{23}$$

Carrying out the indicated differentiation in (23), substituting the differential-difference equations (12) and (16) for $p^*(t)$ and $y(t)$ respectively, and noting that the variation in control is zero over the interval of integration

$$\Delta_K = \int_{t_K-\theta(t_K)}^{t_K} \frac{p_0{*}^T(s)}{1-\dot\theta(s+\rho(s))}\left[\frac{\partial f}{\partial x_\theta}(s+\rho(s))\right]_* y(s)\, ds. \tag{24}$$

It follows as a consequence of the definition (14) that Δ_K given by equation (24) is equal to zero if it can be shown that

$$s + \rho(s) \geq t_K \tag{25a}$$

for

$$s \geq t_K - \theta(t_K). \tag{25b}$$

By adding $\rho(s)$ to both sides of (25b), it is evident that $s + \rho(s) \geq t_K - \theta(t_K) + \rho(s)$. Assuming from physical considerations that

$$\theta(t) \geq 0,\ t_0 \leq t \leq t_K, \tag{26}$$

inequality (25a) will be satisfied for the condition of (25b) if

$$\theta(t_K) \leq \rho(s). \tag{27}$$

Evaluating at time $t = t_K$ the equations (9a) and (9b) which define the advance function, $\rho(t_K-\theta(t_K)) = \theta(t_K)$, then in order for (27) to be satisfied, it is sufficient that

$$\frac{d\rho}{ds}(s) \geq 0. \tag{28}$$

Differentiating (9a) and (9b) with respect to t and s respectively yields

$$\frac{1}{1-\dot\theta(t)} = 1 + \frac{d\rho}{ds}(s) \tag{29}$$

and (28) will hold if

$$0 \leq \dot\theta(t) < 1,\ t_0 \leq t \leq t_K. \tag{30}$$

Summarizing, in order that $s + \rho(s) \geq t_K$ for $s \geq t_K - \theta(t_K)$ it is sufficient to require $\theta(t) \geq 0$ and $0 \leq \dot{\theta}(t) < 1$ for $t_o \leq t \leq t_K$. This is the reason for originally imposing these conditions on the time-varying delay. Thus the expression for Δ_k in equation (24) is equal to zero. If the above considerations are applied to equation (22),

$$p^{*T}(t_K)y(t_K) = p^{*T}(t_k)y(t_k) + \int_{t_k-\theta(t_k)}^{t_k} \frac{p_\rho^{*T}(s)}{1-\dot{\theta}(s+\rho(s))}\left[\frac{\partial f}{\partial x_\theta}(s+\rho(s))\right]_* y(s)ds. \quad (31)$$

<u>Derivation of the Necessary Conditions</u>. Since the right hand side of equation (31) has the same basic form as the Hamiltonian (13), it is desired to relate the variation in the trajectory $y(s)$ to the difference in states $\Delta x(s,t_{k-1})$. Now $y(s)$ can be written

$$y(s) = \frac{\partial x}{\partial \epsilon}(s;\epsilon)\bigg|_{\epsilon=0} \left[= \left[\frac{\partial x}{\partial u}(s)\right]_* \frac{\partial u}{\partial \epsilon}(s;\epsilon)\bigg|_{\epsilon=0}\right]. \quad (32)$$

And since $x^*(t_{k-1})$ is independent of u,

$$\frac{\partial x^*}{\partial u}(t_{k-1}) = 0. \quad (33)$$

It is clear that

$$x(s) = \begin{cases} x^*(t_{k-1}) + \Delta x^*(s,t_{k-1}), & s < t_{k-1} \\ x^*(t_{k-1}) + \Delta x(s,t_{k-1}), & t_{k-1} \leq s \leq t_K \end{cases} \quad (34)$$

because the control and trajectory are assumed optimal for $s < t_{k-1}$. Then

$$y(s) = \begin{cases} \frac{\partial}{\partial u}\Delta x(s,t_{k-1})\Big|_{u=u^*} \delta v_{k-1}, & t_{k-1} \leq s < t_k \\ 0 & , \text{ otherwise.} \end{cases} \quad (35)$$

The first line of (35) will be used in what follows; if the first argument ever becomes smaller than the second argument of Δx, the partial derivative is zero as indicated in the second line of (35). Equation (35) is combined with (31) and the terminal inequality (21) to yield

$$p^{*T}(t_k)\frac{\partial}{\partial u}\Delta x(t_k,t_{k-1})\Big|_{u=u^*}\delta v_{k-1} + $$

$$+ \int_{t_k-\theta(t_k)}^{t_k} \frac{p_\rho^{*T}(s)}{1-\dot{\theta}(s+\rho(s))}\left[\frac{\partial f}{\partial x_\theta}(s+\rho(s))\right]_* \frac{\partial}{\partial u}\Delta x(s,t_{k-1})\Big|_{u=u^*}\delta v_{k-1}\, ds \leq 0. \quad (36)$$

Applying the definition of the Hamiltonian (13), inequality (36) becomes

$$\frac{\partial}{\partial u}H(x^*,x_\theta^*,p^*,p_\rho^*,u^*,t_{k-1},t_k)\delta v_{k-1} \leq 0. \quad (37)$$

The integer k was not restricted to any specific $k = 1,\cdots,K$, then (37) must hold for each $k = 1,\cdots,K$. This is a precise statement of the necessary condition that along the optimal trajectory and costate trajectory, the Hamiltonian, evaluated for successive sampling instants, is either stationary or a local maximum with respect to the optimal control.

A Maximum Principle for Linear Systems with a Quadratic Index of Performance.

System and Cost Equations. Consider a particular form of the system (1) which is described by a linear nonstationary differential-difference equation

$$\dot{\tilde{x}}(t) = A_o(t)\tilde{x}(t) + A_1(t)\tilde{x}\big(t-\theta(t)\big) + B(t)u(t), \quad t_o \leq t \leq t_K, \tag{38}$$

with the initial function $\tilde{x}(t) = \tilde{\phi}(t)$ for $t_o - \theta(t_o) \leq t \leq t_o$ and the matrices A_o, A_1, B with dimensions $n \times n$, $n \times n$, and $n \times r$ respectively. The time-varying delay $\theta(t)$ is assumed to satisfy $\theta(t) \geq 0$ and $0 \leq \dot{\theta}(t) < 1$ for $t_o \leq t \leq t_K$.

Consider also the particular form of the cost function (5) which is described by a nonstationary differential equation

$$\dot{x}_o(t) = \tilde{x}^T(t)P(t)\tilde{x}(t) + u^T(t)Q(t)u(t) \tag{39}$$

with the initial condition $x_o(t_o) = 0$ and the positive semidefinite matrix P and positive definite matrix Q with dimensions $n \times n$ and $r \times r$ respectively.

Theorem 2: A Maximum Principle. Consider the problem stated earlier in this paper for the linear system (38) and the quadratic cost function (39). If $u^*(t)$, $t_o \leq t \leq t_K$, is an optimal control with the corresponding optimal trajectory $x^*(t)$, $t_o \leq t \leq t_K$, then there exists a nonzero function $p^*(t)$, $t_o \leq t \leq t_K$, satisfying the costate equation (12) such that

(i) the Hamiltonian (13) evaluated for successive sampling instants is a global maximum with respect to the optimal control on the optimal trajectory $x^*(t)$ and costate trajectory $p^*(t)$;

(ii) $p_o{}^*(t_K) \leq 0$;

(iii) the transversality condition $p_i{}^*(t_K) = 0$, $i = 1, \cdots, n$ is satisfied.

Necessary conditions (ii) and (iii) follow immediately as a consequence of Theorem 1. The candidates for the optimal control are those controls which render the Hamiltonian either a local maximum or stationary. If it can be shown that the Hamiltonian is downward convex with respect to the control, then there can be only one local maximum or stationary point. It is therefore required to show for any $0 \leq \lambda \leq 1$ that the following inequality is satisfied

$$H\left(\lambda u^{(1)} + (1-\lambda)u^{(2)}\right) \geq \lambda\, H\left(u^{(1)}\right) + (1-\lambda)H\left(u^{(2)}\right) \tag{40}$$

where $u^{(1)}$ and $u^{(2)}$ are any admissible controls. From the properties of a linear system

$$x\left(\lambda u^{(1)} + (1-\lambda)u^{(2)}\right) = \lambda x\left(u^{(1)}\right) + (1-\lambda)x\left(u^{(2)}\right) \tag{41}$$

and by an argument similar to Chyung[9]

$$x_o\left(\lambda u^{(1)} + (1-\lambda)u^{(2)}\right) \leq \lambda x_o\left(u^{(1)}\right) + (1-\lambda)x_o\left(u^{(2)}\right) \tag{42}$$

Substituting these relations (41) and (42) into the definition of the Hamiltonian, inequality (40) follows.

The convexity of the Hamiltonian requires that the optimal control must yield

$$H(x^*, x_\theta^*, p^*, p_\rho^*, u^*, t_{k-1}, t_k) \geq H(x^*, x_\theta^*, p^*, p_\rho^*, u, t_{k-1}, t_k) \tag{43}$$

for each $k = 1, \cdots, K$ and all admissible controls u. This is a precise statement of the necessary condition that the Hamiltonian is a global maximum with respect to the optimal control. It is clear that if the control set U is unbounded, the global maximum must occur at the stationary point in which case

$$\frac{\partial}{\partial u} H(x^*, x_\theta^*, p^*, p_\rho^*, u^*, t_{k-1}, t_k) = 0 \tag{44}$$

for each $k = 1, \cdots, K$.

TIME-VARYING DELAY IN STATE AND CONTROL

Although the theorems developed have been concerned with systems containing a time-varying delay in state alone, the results can easily be extended to also include a time-varying delay in control. The system is described by a differential-difference equation of the form

$$\dot{\bar{x}}(t) = \bar{f}(\bar{x}(t), \bar{x}_\theta(t), u(t), u_\tau(t), t)$$

where the delayed control is given by $u_\tau(t) = u(t-\tau(t))$. An advance function $\omega(\cdot)$ is defined for the delay in control $\tau(\cdot)$ by

$$t - \tau(t) = s, \quad s + \omega(s) = t, \quad t_o \leq t \leq t_K.$$

For a system which is not necessarily linear, the general condition (i) of Theorem 1 becomes: the Hamiltonian is either a local maximum or stationary "with respect to the control" along the optimal trajectories, i.e.,

$$\left\{\frac{\partial}{\partial u} H^*(t_{k-1}, t_k) + \frac{\partial}{\partial u_\tau} H^*\left(t_{k-1} + \omega(t_{k-1}), t_k + \omega(t_k)\right)\right\} \delta v_{k-1} \leq 0$$

for $k = 1, \cdots, m$,

$$\left\{\frac{\partial}{\partial u} H^*(t_{k-1},t_k) + \frac{\partial}{\partial u_\tau} H^*\left(t_{k-1} + \omega(t_{k-1}),\ t_K\right)\right\}\delta v_{k-1} \leq 0$$

for $k = m + 1$, and

$$\left\{\frac{\partial}{\partial u} H^*(t_{k-1},t_k)\right\}\delta v_{k-1} \leq 0$$

for $k = m + 2,\cdots,K$

where m is the integer $1 \leq m \leq K$ such that

$$t_m + \omega(t_m) < t_K \text{ but } t_{m+1} + \omega(t_{m+1}) \geq t_K.$$

For a linear system described by

$$\dot{\bar{x}}(t) = A_o(t)\bar{x}(t) + A_1(t)\bar{x}_\theta(t) + B_o(t)u(t) + B_1(t)u_\tau(t)$$

and a quadratic index of performance, the maximum principle, necessary condition (i) of Theorem 2 becomes:

the Hamiltonian is a global maximum "with respect to the control" along the optimal trajectories, that is,

$$H\left(u^*,u_\tau^*,t_{k-1},t_k\right) + H\left(u^*,u_\tau^*,t_{k-1} + \omega(t_{k-1}),\ t_k + \omega(t_k)\right) \geq$$

$$\geq H(u,u_\tau^*,t_{k-1},t_k) + H\left(u^*,u_\tau,t_{k-1} + \omega(t_{k-1}),\ t_k + \omega(t_k)\right)$$

for $k = 1,\cdots,m$,

$$H(u^*,u_\tau^*,t_{k-1},t_k) + H\left(u^*,u_\tau^*,t_{k-1} + \omega(t_{k-1}),t_K\right) \geq$$

$$\geq H(u,u_\tau^*,t_{k-1},t_k) + H\left(u^*,u_\tau,t_{k-1} + \omega(t_{k-1}),t_K\right)$$

for $k = m + 1$, and

$$H(u^*,u_\tau^*,t_{k-1},t_k) \geq H(u,u_\tau^*,t_{k-1},t_k)$$

for $k = m + 2,\cdots,K$

where m is the integer $1 \leq m \leq K$ such that $t_m + \omega(t_m) < t_K$ but

$$t_{m+1} + \omega(t_{m+1}) \geq t_K.$$

EXAMPLE

A simple analytic example is now presented to illustrate the foregoing theory. It is a fixed endpoint problem in which the system is described by a linear differential-difference equation with a time-varying delay. The delay $\theta(t)$ is chosen to have a linear dependence on time, t. It is required to find the appropriately bounded piecewise constant control which minimizes the control energy needed to drive the state from the origin to the fixed endpoint.

Statement of the Problem

Consider the system

$$\dot{x}(t) = -x(t) - x\left(t-\theta(t)\right) + u(t)$$

between the initial time $t_o = 0$ and the terminal time $t_2 = 2$ with the initial condition $x(t) = 0$, $-\frac{1}{2} \leq t \leq 0$. The time-varying delay is given by

$$\theta(t) = \tfrac{1}{2}(t+1),$$

and the index of performance is chosen to be

$$\text{I.P.} = \int_0^2 \tfrac{1}{2} u^2(t)\, dt.$$

For this problem an admissible piecewise constant control $u(t)$, $0 \leq t \leq 2$, with the sampling instant $t_1 = 1$, is to have the form

$$u(t) = \begin{cases} v_o, & 0 \leq t < 1 \\ v_1, & 1 \leq t \leq 2 \end{cases}$$

where the control amplitudes must satisfy $|v_o| \leq 1$ and $|v_1| \leq 1$.

Find the admissible control which brings the system to the terminal state

$$x(2) = R$$

with a minimum index of performance.

<u>Solution</u>

The system is integrated forward in time to find the trajectory

$$x(t) = (1-e^{-t})v_o, \quad 0 \leq t \leq 1,$$

$$= \left(2e^{-\frac{1}{2}(t-1)} - e^{-t} - 1\right)v_o + \left(1-e^{-(t-1)}\right)v_1, \quad 1 \leq t \leq 2,$$

and therefore the terminal state constraint is expressed by

$$x(2) = R = \left(2e^{-\frac{1}{2}} - e^{-2} - 1\right)v_o + \left(1-e^{-1}\right)v_1.$$

The cost equation, $\dot{x}_o(t) = \frac{1}{2}u^2(t)$, has the solution

$$x_o(t) = \tfrac{1}{2}tv_o^2, \quad 0 \leq t \leq 1,$$

$$= \tfrac{1}{2}v_o^2 + \tfrac{1}{2}(2-t)v_1^2, \quad 1 \leq t \leq 2.$$

The costates $p_o^*(t)$ and $p^*(t)$ for $0 \leq t \leq 2$ must satisfy (12) with the advance function given by $\rho(s) = s + 1$, then

$$\dot{p}_o^*(t) = 0, \quad 0 \leq t \leq 2$$

$$\dot{p}^*(t) = p^*(t) + 2p^*(2t+1), \quad 0 \leq t \leq \tfrac{1}{2},$$

$$= p^*(t), \quad \tfrac{1}{2} \leq t \leq 2$$

which have the solutions

$$p_o^*(t) = \text{constant}, \quad 0 \leq t \leq 2$$

$$p^*(t) = \gamma^* e^{(t-2)}\left(1+2e^{(t+1)} - 2e^{3/2}\right), \quad 0 \leq t \leq \tfrac{1}{2},$$

$$= \gamma^* e^{(t-2)}, \quad \tfrac{1}{2} \leq t \leq 2$$

where the constant γ^* represents the terminal costate $p^*(2)$, Using the definition of the Hamiltonian

$$H(0,1) = p_o(1)\Delta x_o(1,0) + p(1)\Delta x(1,0) - \int_0^{\frac{1}{2}} \frac{p_o(s)}{(1-\frac{s}{2})}\Delta x(s,0)ds$$

$$= p_o(1)\tfrac{1}{2}v_o^2 + p(1)(1-e^{-1})v_o - \int_0^{\frac{1}{2}} 2p(2s+1)(1-e^{-s})v_o ds.$$

Applying the maximum principle of Theorem 2, the optimal control amplitude v_o^* must satisfy

$$-\tfrac{1}{2}v_o^{*2} + p^*(1)(1-e^{-1})v_o^* - \int_0^{\frac{1}{2}} 2p^*(2s+1)(1-e^{-s})v_o^* ds \geq$$
$$\geq -\tfrac{1}{2}v_o^2 + p^*(1)(1-e^{-1})v_o - \int_0^{\frac{1}{2}} 2p^*(2s+1)(1-e^{-s})v_o\, ds$$

which reduces to

$$\left(v_o^*-(2e^{-\frac{1}{2}}-e^{-2}-1)\gamma^*\right)^2 \leq \left(v_o-(2e^{-\frac{1}{2}}-e^{-2}-1)\gamma^*\right)^2$$

for all v_o such that $|v_o| \leq 1$.

Similar calculations find that the optimal control amplitude v_1^* must satisfy

$$\left(v_1^*-(1-e^{-1})\gamma^*\right)^2 \leq \left(v_1-(1-e^{-1})\gamma^*\right)^2$$

for all v_1 such that $|v_1| \leq 1$.

For convenience, let $C_o = 2e^{-\frac{1}{2}}-e^{-2}-1$ and $C_1 = 1-e^{-1}$. Since the constant γ^* is not specified by a transversality condition, the following cases must be considered:

(a) $|C_o\gamma^*| \leq 1$, $|C_1\gamma^*| \leq 1$, (b) $|C_o\gamma^*| \leq 1$, $C_1\gamma^* > 1$,
(c) $|C_o\gamma^*| \leq 1$, $C_1\gamma^* < -1$, (d) $C_o\gamma^* > 1$, $|C_1\gamma^*| \leq 1$,
(e) $C_o\gamma^* < -1$, $|C_1\gamma^*| \leq 1$, (f) $C_o\gamma^* > 1$, $C_1\gamma^* > 1$,
(g) $C_o\gamma^* < -1$, $C_1\gamma^* < -1$

When case (a) occurs, choose $v_o^* = C_o\gamma^*$ and $v_1^* = C_1\gamma^*$. By the endpoint condition with these controls, $\gamma^* = R/(C_o^2 + C_1^2)$ and to determine for which values of R these amplitudes are valid, solve the inequalities $\left|\frac{RC_o}{C_o^2+C_1^2}\right| \leq 1$ and $\left|\frac{RC_1}{C_o^2+C_1^2}\right| \leq 1$. Considering also cases (b) through (g) the following optimal control amplitudes result

$$v_o^* = \begin{cases} \dfrac{R-C_1}{C_o}, & \dfrac{C_o^2+C_1^2}{C_1} \leq R \leq (C_o+C_1), & +1 \\[2ex] \dfrac{RC_o}{C_o^2+C_1^2}, & -\dfrac{C_o^2+C_1^2}{C_1} \leq R \leq \dfrac{C_o^2+C_1^2}{C_1}, & \dfrac{RC_1}{C_o^2+C_1^2} \\[2ex] \dfrac{R+C_1}{C_o}, & -(C_o+C_1) \leq R \leq -\dfrac{C_o^2+C_1^2}{C_1}, & -1 \end{cases} = v_1^*$$

Note that there is no admissible control which can meet the endpoint condition for $R > (C_o+C_1)$ and $R < -(C_o+C_1)$, hence there is no optimal solution.

It can be seen from this example that the solutions to optimal control problems involving even simple forms for the time-varying delay are difficult to obtain. More complex examples would require the assistance of a digital or analog computer to carry out the optimization procedure.

CONCLUSION

The optimal piecewise constant control of time lag systems with time-varying delays in state and control has been considered. The definition of an advance function was used to develop local necessary conditions for systems which are not necessarily linear and a maximum principle for linear systems with a quadratic index of performance.

The problem of synthesizing the optimal control for time lag systems is generally very difficult since there usually results a two-point boundary value problem involving differential-difference equations. The techniques which have been developed for systems without time lags and time lag systems with a constant delay should be investigated for possible application to the optimization of systems with a time-varying delay.

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К ТЕОРИИ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ПРИ ОГРАНИЧЕННЫХ ФАЗОВЫХ КООРДИНАТАХ ОБЪЕКТА

А.Б.Куржанский, Ю.С.Осипов
Свердловское отделение математического института им.В.А.Стеклова
Свердловск

С С С Р

§ 1. Введение. В докладе на основе развитой в [1,2] теории рассматриваются задачи управления линейными объектами с ограниченными фазовыми координатами, а также некоторые родственные задачи с бесконечномерными особенностями. Задачи данного класса в весьма общей постановке и с различных позиций рассматривались, например, в [3-9]. Для линейного случая необходимые условия оптимальности получены из общих функциональных соображений в [10], однако, свойства множителей Лагранжа (являющихся здесь элементами соответствующих функциональных пространств) и детальная структура решений не обсуждались.

Весьма важная особенность рассматриваемых задач состоит в том, что даже при известных множителях Лагранжа процесс вычисления оптимальных управлений из условия принципа максимума требует на некоторых интервалах времени (тех, где траектория объекта идет по заданному ограничению) привлечения дополнительных соображений. Это объясняется тем, что условие принципа максимума на упомянутых промежутках вырождается.

Применение подхода [1,2] в настоящем докладе имеет целью изучить для рассматриваемых задач некоторые свойства множителей Лагранжа, указать способ исследования этих задач путем аппроксимации или сведения их к соответствующим конечномерным задачам и наметить путь к вычислению оптимальных управлений на всем промежутке времени движения.

§ 2. Задача об управлении при ограниченных фазовых координатах.

1. Постановка задачи. Задан линейный управляемый объект, описываемый дифференциальным уравнением

$$\frac{dx}{dt} = A(t)x + B(t)u + \omega(t) \tag{2.1}$$

Здесь x - n-мерный вектор фазовых координат, u - r-мерный вектор управления, $A(t)$, $B(t)$ - непрерывные матрицы соответствующих размерностей, $\omega(t)$ - известная n - векторная функция, означающая внешнее возмущение. Предполагается, что заданы начальное состояние $x(t_\alpha) = x_\alpha$, выпуклое терминальное многообразие $\mathcal{M}$, ограничения

$$u \in \mathcal{U}, \quad x \in \mathcal{X}(t) \tag{2.2}$$

на мгновенные значения величин $x(t)$ и $u(t)$. Множества $\mathcal{U}$ и $\mathcal{X}$ - выпуклые и замкнутые, содержащие нулевые элементы в качестве внутренних точек. Множество $\mathcal{X}(t)$ (как функция времени) предполагается полунепрерывным сверху по включению.

Задача 2.1. Среди управлений $u(t)$, переводящих систему (2.1) из состояния $x(t_\alpha) = x_\alpha$ на многообразие $\mathcal{M}$ ($x_\beta = x(t_\beta) \in \mathcal{M}$) найти оптимальное по быстродействию управление $u^\circ(t)$, обеспечивающее условие $t_\beta - t_\alpha = min$.

2. Условия разрешимости. Принцип максимума. Пусть $\{t_i\}$ - множество точек вида $t_\alpha + p(N)(t_\beta - t_\alpha)\cdot 2^{-N}$, где $p \le 2^N$ ($N = 1, \ldots, m, \ldots$). Это множество - всюду плотное в $[t_\alpha, t_\beta]$. Задача 2.1 может быть сведена к счетной проблеме моментов

$$\begin{aligned} \int_{t_\alpha}^{t_\beta} h_k(t_\beta, \tau)u(\tau)d\tau &= c_{k\beta} + x(t_\beta) \\ \int_{t_\alpha}^{t_\beta} h_k(t_i, \tau)u(\tau)d\tau &= c_{k_i} + z_{k_i} \end{aligned} \tag{2.3}$$

Здесь

$$x(t_\beta) \in \mathcal{M}; \; z^{(i)} = \{z_{1_i}, \ldots, z_{n_i}\} \in \mathcal{X}(t_i); \; u(t) \in \mathcal{U}, \; t_\alpha \le t \le t_\beta \tag{2.4}$$

величины $h_k(t, \tau)$ суть k-ые компоненты n-вектора

$$h(t,\tau) = B'(t)S(\tau,t)$$
$$\left(\frac{d}{d\tau}S(\tau,t) = -S(\tau,t)A(\tau)\ ,\ S(t,t) = E\right) \tag{2.5}$$

причем $h_k(t,\tau) \equiv 0$, если $\tau \geq t$. (В (2.5), как и везде в дальнейшем, знак $'$ означает транспонирование). Числа c_{k_i} определяются как k -ые компоненты вектора

$$c(t_i) = \int_{t_\alpha}^{t_i} S(\tau,t_i)\omega(\tau)d\tau + S(t_\alpha,t_i)x_\alpha\ ,\ c_\beta = c(t_\beta) \tag{2.6}$$

Введем обозначения

$$\rho[h] = \int_{t_\alpha}^{t_\beta} \gamma_1[h(t)]dt\ ,\ \gamma_1[l] = \max_u l'u\ ,\ u \in \mathcal{U}\ ;$$
$$\gamma_2[l] = \max_x l'x\ ,\ x \in \mathcal{M};\ \gamma_t[l] = \max_x l'x\ ,\ x \in \mathcal{X}(t) \tag{2.7}$$

Пусть t_β фиксировано. Необходимые и достаточные условия разрешимости проблемы (2.3), (2.4) известны в функциональном анализе [12]. Однако, (и это наиболее существенно в данном пункте) использование специфического для данной задачи свойства функций $h(t,\tau)$ -их непрерывности- позволяет условия (2.3), (2.4) представить в виде соотношения

$$\rho[\lambda' h(t_\beta,t) + \int_t^{t_\beta} l'(\tau)h(\tau,t)d\tau] - \lambda' c -$$
$$- \int_{t_\alpha}^{t_\beta} l'(t)c(t)dt + \int_{t_\alpha}^{t_\beta} \gamma_t[l(t)]dt + \gamma_2[\lambda] \geq 0 \tag{2.8}$$

которое должно выполняться при любом n - векторе $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ и любой суммируемой с квадратом n - векторной функции $l(t) = \{l_1(t), \ldots, l_n(t)\}$.

Условия (2.8) могут быть записаны в иной, но эквивалентной форме, вытекающей из (2.5)-(2.8): для разрешимости проблемы (2.3), (2.4) при конечном фиксированном t_β необходимо и достаточно, чтобы при любом n -векторе λ и любой n -векторной функции $l(t)$, стесненных условием

$$\lambda'\lambda + \int_{t_\alpha}^{t_\beta} l'(t)l(t)dt \leq 1\ ,$$

$$\inf_{\lambda,l}\Big\{\max_{u\in U}\int_{t_d}^{t_p} u'(t)B'(t)s(t)dt + s'(t_d)x_d + \tag{2.9}$$

$$+\int_{t_d}^{t_p}\omega(t)s(t)dt + \int_{t_d}^{t_p}\gamma_t[l(t)]dt + \gamma_2[s(t_p)]\Big\} \geq 0$$

Здесь $s(t)$ есть решение сопряженной системы

$$ds/dt = -A'(t)s(t) + l(t)\ ,\ s(t_p) = s_p = \lambda \tag{2.10}$$

Условие (2.9) является аналогом известной в теории выпуклого программирования задачи о седловой точке функции Лагранжа [13].

Временем оптимального быстродействия будет наименьшее из чисел $t_p - t_d$, при которых соотношение (2.9) выполняется со знаком равенства. Существенно при этом подчеркнуть, что $\inf$ в (2.9) при $t_p = t_p^0$ достигается на ненулевом элементе $(s_p^0, l^0(t))$.

Из сказанного вытекает, что оптимальное управление u^0 необходимо удовлетворяет принципу максимума

$$\max_{u\in U}\int_{t_d}^{t_p} u'(t)B'(t)s^0(t)dt = \int_{t_d}^{t_p} u^{0\prime}(t)B'(t)s(t)dt \tag{2.11}$$

или иначе

$$\max_{u\in U} u'(t)B'(t)s^0(t) = u^{0\prime}(t)B'(t)s(t) \tag{2.12}$$

где (в терминологии [2]) $s^0(t)$ есть минимальное движение системы (2.10), а именно то, которое доставляет минимум выражению

$$\Psi[s_p, l] = \int_{t_d}^{t_p}\gamma_1[B'(t)s(t)]dt + s'(t_d)x_d + \tag{2.13}$$

$$+\int_{t_d}^{t_p} s'(t)\omega(t)dt + \int_{t_d}^{t_p}\gamma_t[l(t)]dt + \gamma_2[s(t_p)] \geq 0$$

при

$$\|s_p\|^2 + \int_{t_\alpha}^{t_\beta} \|l(t)\|^2 dt \leq 1 \qquad (2.14)$$

Символ $\|x\|$ означает евклидову норму вектора x .

Функционал $\Psi(s_p, l)$ – неотрицательный, выпуклый.

Примечание. Пусть многообразие $\mathcal{M}$ является гиперплоскостью $x_k(t_p)=x_k$, $k=1,\dots,i$. Тогда множество $\mathcal{Y}$ всех точек $x = \{x_1,\dots,x_i\}$, удовлетворяющих (2.13) при любых s_p, $l(t)$, стесненных (2.14), образует область достижимости (по первым i координатам) системы (2.1) из состояния $x(t_\alpha)$ к моменту $t=t_p$ при ограничениях (2.2).

§3. Некоторые свойства решений.

1. Минимальная функция. Пусть найдено решение $h^\circ = B's^\circ$ задачи (2.13), (2.14). Обозначим через $x^\circ(t)$ оптимальную траекторию системы (2.1). Из свойств решений проблемы (2.13) (2.14) вытекает, что $l^\circ(t) \equiv 0$ там и только там, где траектория $x^\circ(t)$ не выходит на фазовое ограничение. Если компаненты h_i функции $h^\circ(t)$ могут обращаться в нуль лишь на множестве нулевой меры, то $u^\circ(t)$ определяется условием (2.12). Существенна, однако, иная ситуация. Ограждая себя от разбора наиболее нерегулярных случаев, предположим, что $h_i^\circ(t)$ могут обращаться в нуль лишь на конечном числе m замкнутых отрезков $[\tau_{k_1}, \tau_{k_2}]$ ($k=1,\dots,m$) оси t , суммарная мера которых меньше, чем $t_p^\circ - t_\alpha$. Предположим также, что система (2.1) вполне управляема [14] в усиленном смысле [2] и что ее траектории, удовлетворяющие (2.12) при $h_i^\circ \neq 0$, могут пересекаться с границей $\mathcal{X}(t)$ лишь на множестве меры нуль. Тогда функции h° обращаются в нуль при выходе системы (2.1) на ограничение и только лишь тогда. Таким образом, моменты выхода $x^\circ(t)$ на границу множества $\mathcal{X}(t)$ и схода с нее совпадают здесь с множеством $\{\tau_{k_i}\}$. Заметим, что точки τ_{k_i} определяются автоматически из условия (2.13), т.е. без обычного для вариационного исчисления обращения к условиям типа Вейерштрасса-Эрдмана.

2. О вычислении минимальной функции. Итак, для нахождения множителей s_p°, $l^\circ(t)$ надлежит решить вариационную задачу (2.13), (2.14) минимизации функционала Ψ при вы-

пуклых же ограничениях (2.14). Решение конкретной задачи (2.13), (2.14) можно, однако, упростить. Поясняя сказанное, предположим, что система (2.1) - стационарная, вполне управляемая скалярным воздействием и траектория ее выходит на границу фазового ограничения лишь один раз, на отрезке $[\tau_1, \tau_2]$; предположим, что ограничена лишь одна координата: $|x_1(t)| \leq f(t)$. Имеем

$$h^\circ(t) = b' s^\circ(t) = s_\beta^{\circ\prime} h(t_\beta, t) + \int_t^{t_\beta} \ell_1^\circ(\tau) h_1(\tau, t)\, d\tau \equiv 0 \qquad (3.1)$$

Воспользуемся теперь еще одним (более сильным) свойством функции $h(\tau, t)$, в данном случае ее дифференцируемостью. Последовательно дифференцируя (3.1) и выполняя некоторые преобразования получаем, что $\ell_1^\circ(\tau)$ удовлетворяет однородному дифференциальному уравнению $(n-1)$ порядка

$$b'\Big\{A'^{n-1}\ell^\circ + \sum_{j=1}^{n-1}\Big[(-1)^j A'^{n-(j+1)} \frac{d^j \ell^\circ}{d\tau^j} + d_j \sum_{i=0}^{j-1} (-1)^{i+1} A'^{j-(i+1)} \frac{d^i \ell^\circ}{d\tau^i}\Big]\Big\} = 0 \qquad (3.2)$$

где $\ell^\circ(\tau) = \{\ell_1^\circ(\tau), 0, \ldots, 0\}$ - n - векторная функция, d_j ($j = 0, \ldots, n-1$) - коэффициенты разложения вектора $b'A'^n$ по базису $\{b'A'^k;\ k = 0, \ldots, n-1\}$: $b'A'^n = \sum_{k=0}^{n-1} d_k b'A'^k$.

Условие непрерывности функции $h^\circ(t)$ и ее производных справа, в точке τ_1 дают $n-1$ краевое условие, которые образуют систему алгебраических уравнений, связывающих τ_1 и τ_2 с s_β. Решая уравнение (3.2) с учетом краевых условий, получим $\ell_1^\circ(t) = f'(t; \tau_1, \tau_2) s_\beta^\circ$, где $f(t; \tau)$ - n - векторная функция, нелинейно зависящая от τ_1 и τ_2. Подставляя ℓ_1° в (2.13) получим, что при фиксированных τ_1, τ_2 $\psi[s_\beta^\circ, \ell^\circ] = \varphi(s_\beta; \tau_1, \tau_2)$ является выпуклой функцией от s_β. Ограничение (2.14) преобразуется теперь к виду $v(s_\beta, \tau_1, \tau_2) \leq 1$, причем для фиксированных τ_1, τ_2 соотношение $v(s_\beta, \tau_1, \tau_2) = c$ есть уравнение поверхности эллипсоида. Таким образом, решение задачи (2.13) можно получить минимизируя при каждой фиксированной паре функцию $\varphi(s_\beta, \tau_1, \tau_2)$ по s_β ($v(s_\beta, \tau_1, \tau_2) \leq 1$), а затем определяя ту пару $\tau_1^\circ, \tau_2^\circ$, для которой нелинейная функция $\varphi(s_\beta(\tau_1, \tau_2), \tau_1, \tau_2)$ принимает наименьшее значение. Если оптимальная траектория выходит на ограничение на m участках $[\tau_{k_1}, \tau_{k_2}]$ ($k=1,\dots,m$), то $l_1^\circ(t)$ удовлетворяет уравнению (3.2) на каждом из них. Тогда функция $\psi(s_\beta, \tau_{1_1}, \tau_{1_2}, \dots, \tau_{n_1}, \tau_{n_2}, \dots, \tau_{m_1}, \tau_{m_2})$ – выпуклая по s_β и нелинейная по τ_{k_i}. Пусть известна оценка сверху числа m. Вычисление $l_1^\circ(t)$ следует тогда вести, полагая вначале $m=1$, затем $m=2$ и т.д., вплоть до условия $\psi(s_\beta^\circ, \tau_{1_1}^\circ, \tau_{1_2}^\circ, \dots, \tau_{m_1}^\circ, \tau_{m_2}^\circ)=0$

Уравнение, аналогичное (3.2) (но, естественно, более громоздкое) может быть выписано и в общем случае задачи (2.1). Однако матрицы $A(t)$; $B(t)$ должны быть нужное число раз дифференцируемы.

3. Аппроксимация решений. Отметим одно важное свойство функционала $\Psi(s_\beta, l)$ и минимальной функции $h^\circ(t)$ [15]. Именно, потребуем, чтобы ограничение (2.2) на координаты выполнялось не везде, а лишь в дискретные моменты времени t_{i_N} ($i_N = 1, \dots, 2^N$). В результате вместо (2.3), (2.4), получим конечномерную проблему моментов, разрешимую в том и только том случае, если неравенство

$$\min_{s_N(t)} \Psi(s_{\beta N}, l_N(t)) \geq 0 \tag{3.3}$$

выполняется на всех решениях $s_N(t)$ сопряженной системы

$$ds_N/dt = -A'(t)s_N + l_N(t),$$

$$s_{\beta N} = s_N(t_\beta), \quad l_N(t) = \sum_{i_N=1}^{2^N} l(t_{i_N})\delta(t - t_{i_N})$$

стесненных условием $s'_{\beta N} s'_{\beta N} + 2^{-N} \sum_{i_N=1}^{2^N} l'(t_{i_N}) l(t_{i_N}) = 1$
Функции $s_N(t)$ предполагаются непрерывными справа.

Оптимальное управление u_N° в дискретной задаче удовлетворяет принципу максимума

$$\max_{u \in U} \int_{t_\alpha}^{t^\circ_{\beta N}} u'_N(t)B'(t)s^\circ_N(t)dt = \int_{t_\alpha}^{t^\circ_{\beta N}} u^{\circ\prime}_N(t)B'(t)s^\circ_N(t)dt$$

на минимальной функции $h^\circ_N(t) = B'(t)\cdot s^\circ_N(t)$ условия (3.3). Последовательность $s^\circ_N(t)$ может быть выбрана так, чтобы выполнялись предельные соотношения:

$$t^\circ_{\beta N} \to t^\circ_\beta, \quad \Psi[s^\circ_{\beta N}, l^\circ_N(t)] \to \Psi[s^\circ_\beta, l^\circ(t)]; \tag{3.4}$$

$s^\circ_N(t) \to s^\circ(t)$, $x^\circ_N(t) \to x^\circ(t)$ равномерно по $t \in [t_\alpha, t^\circ_\beta - \varepsilon]$

для любого $\varepsilon > 0$

$$u^\circ_N(t) \dashrightarrow u^\circ(t) \quad \text{(в смысле слабой сходимости)} \tag{3.5}$$

§4. Вычисление оптимальных управлений. Определение функции $u^\circ(t)$ из принципа максимума на участках $[\tau_{k_1}, \tau_{k_2}]$, где $h^\circ_i(t) \equiv 0$, требует привлечения дополнительных соображений. Для нахождения $u^\circ(t)$ на этих участках может быть использовано соотношение (3.5), т.к. функции $h^\circ_{iN}(t) = 0$ лишь на множестве нулевой меры. Подчеркнем здесь, что для оптимальности $u^\circ(t)$ достаточно, чтобы u° получалось путем предельного перехода (3.5).

Управления u°_N, вообще говоря, разрывны, с возрастающим (при $N \to \infty$) числом точек разрыва, и сходятся к u° лишь в слабом смысле. Вычисление предела (3.5) в общем случае затруднительно. Процесс вычисления u° можно, однако, регуляризировать, введя в рассмотрение функции

$$u^\circ_\delta(t) = \frac{1}{\delta} \lim_{N \to \infty} \int_0^\delta u^\circ_N(t+\vartheta)\, d\vartheta \tag{4.1}$$

Функции $u^\circ_\delta(t)$ - непрерывные и сходятся (при $\delta \to 0$) равномерно к оптимальному управлению $u^\circ(t)$.

Процедура (4.1) основана на непосредственном использовании предельного перехода из §3. Функцию $u^\circ(t)$ желательно находить, минуя этот переход. Пусть уже известны $\mathfrak{z}^\circ_\beta$, ℓ° и $\ell^\circ(t) \not\equiv 0$ на множестве $\Omega = \bigcup_k [\tau_{k_1}, \tau_{k_2}]$. Разобьем Ω на конечное число непересекающихся отрезков вида $[\tau', \tau'+\varepsilon)$, $\tau' \in \Omega$. Рассмотрим задачу 2.1, когда ограничение (2.2) на координаты имеет место всюду, кроме отрезка $[\tau', \tau'+\varepsilon)$. Пусть $[u^\circ(t)]_\varepsilon$ -решение такой задачи, и

$$u^\circ_\varepsilon(t) = \frac{1}{\varepsilon} \int_{\tau'}^{\tau'+\varepsilon} [u^\circ(t)]_\varepsilon\, dt, \quad \tau' \le t \le \tau'+\varepsilon$$

Пусть $u^\varepsilon(t)$ - управление, определенное при $\tau' \le t < \tau'+\varepsilon$ из принципа максимума

$$u^{\varepsilon\prime}(t)\, h^\varepsilon(t) = \max_{u \in U} u'(t)\, h^\varepsilon(t)$$

где $h^\varepsilon(t) = B'(t)\mathfrak{z}^\circ_\varepsilon(t)$, и $\mathfrak{z}^\circ_\varepsilon(t)$ определяется из условия минимизации по λ^1, λ^2 функционала Ψ из (2.13) при ограничении

$$d\mathfrak{z}_\varepsilon/dt = -A'(t)\mathfrak{z}_\varepsilon + \ell^*_\varepsilon(t) + \lambda'\delta(t-\tau') +$$
$$+ \lambda^2\delta(t-\tau'-\varepsilon)\,,\ \mathfrak{z}_{\varepsilon p} = \mathfrak{z}^\circ_p\,;$$

$$\ell^*_\varepsilon(t) = \begin{cases} \ell^\circ(t) & ,\ t \bar{\in} [\tau', \tau'+\varepsilon) \\ 0 & ,\ t \in [\tau', \tau'+\varepsilon) \end{cases}$$

$$(\lambda^1)^2 + (\lambda^2)^2 = \int_{\tau'}^{\tau'+\varepsilon} \ell^{\circ\prime}(t)\ell^\circ(t)dt$$

Примем обозначение $u^{\varepsilon o}(t) = \varepsilon^{-1}\cdot\int_0^\varepsilon u^\varepsilon(\tau'+\vartheta)d\vartheta$.
Справедливы условия $\|u^{\varepsilon o}(\tau') - u^\circ_\varepsilon(\tau')\| \to 0, \|u^\circ_\varepsilon(\tau') - u^\circ(\tau')\| \to 0, (\varepsilon\to 0)$
Таким образом, функция $u^{\varepsilon o}$ может служить аппроксимацией оптимального управления u°. При известных $\mathfrak{z}^\circ_p, \ell^\circ(t)$ вычисление $u^{\varepsilon o}$ требует на каждом шаге длины ε минимизации функции двух переменных.

§ 5. Пример.

Рассмотрим модельный пример, на котором, однако, хорошо прослеживаются все основные операции, составляющие суть изложенного подхода (и который можно, разумеется, решить из простых физических соображений). Система (2.1) имеет вид

$$dx_1/dt = x_2\ ,\quad dx_2/dt = -x_1 + u$$

Требуется построить управление $u^\circ(t)$, переводящее ее за наименьшее возможное время t°_p из $x(0) = \{0, 0\}$ в $x(t_p) = \{2, 0\}$ при условиях (2.2) вида

$$|u(t)| \le 2\ ,\quad |x_2(t)| \le 1$$

Соотношения (2.13), (2.14) принимают форму

$$\min_{\lambda,\ell}\left\{\int_0^{t_p}\left|\lambda_1\sin(t_p-t)+\lambda_2\cos(t_p-t) + \int_t^{t_p}\ell(\tau)\cos(\tau-t)d\tau\right|dt - \right.$$
$$\left. -\lambda_1\cdot 2 + \int_0^{t_p}|\ell(t)|dt \ge 0 \right. \quad \text{при} \quad \lambda_1^2+\lambda_2^2 + \int_0^{t_p}\ell^2(t)dt \le k, \tag{5.1}$$

$$k = 4(\sqrt{15}+\sqrt{3}-3)/15\ .$$

Решение (5.1) доставляют величины $\lambda^\circ_1 = \frac{1}{2}, \lambda^\circ_2 = -1/2\sqrt{15}$; $\ell^\circ = 0$ при $0 \le t < \tau_1$, $\tau_2 \le t \le t^\circ_p$; $\ell^\circ = -2/\sqrt{15}$ при $\tau_1 \le t < \tau_2$; $\tau_1 = \frac{\pi}{6}$,

$\tau_2=\pi/6+\sqrt{15}+\sqrt{3}-4$, $t^\circ_\beta=\tau_2+\arcsin(1/4)$; $h^\circ(t)\equiv 0$ при $\tau_1\le t\le\tau_2$ (Заметим, что уравнение (3.2), с помощью которого была найдена функция $l^\circ(t)$, имеет здесь простой вид: $dl/dt=0$, а краевое условие для него – линейное).

Применяя процедуру § 4 вычисления $u^\circ(t)$, получаем $u^\circ(t)=2$ при $0\le t<\tau_1$; $u^\circ(t)=2-\sqrt{3}-\frac{\pi}{6}-t$ при $\tau_1\le t<\tau_2$; $u^\circ(t)=-2$ при $\tau_2\le t\le t^\circ_\beta$.

§ 6. О задачах управления для систем с запаздыванием.

Пусть управляемый объект содержит элемент запаздывания[16] и описывается уравнением

$$\frac{dx}{dt}=Ax+Gx(t-\tau)+Bu\ ,\ \tau\ge 0 \tag{6.1}$$

с постоянными коэффициентами .

Задача 6.1. Задан промежуток времени $t_\alpha\le t\le t_\beta$ начальное состояние объекта $x_{t_\alpha}(\vartheta)$, $(-\tau\le\vartheta\le 0)$ и ограничения (2.2) на управление и координаты. Требуется выбрать управляющее воздействие u° так, чтобы

$$\varepsilon=\rho[x_{t_\beta}(\vartheta)]+\|x_\beta\|=\min_u=\varepsilon^\circ$$

Здесь символы $\rho[x(\vartheta)]$, $\|x\|$ означают соответственно некоторые нормы в пространстве n -векторных функций и в конечномерном евклидовом пространстве.

Изложенный выше в § 2 подход к решению задачи 2.1 можно применить (разумеется, после соответствующей модификации) и для исследования задачи 6.1. В результате получаем, что искомое управление $u^\circ(t)$ необходимо удовлетворяет принципу максимума

$$\int_{t_\alpha}^{t_\beta} h^\circ(t)u^\circ(t)dt=\max_{u\in U}\int_{t_\alpha}^{t_\beta} h^\circ(t)u(t)\,dt \tag{6.2}$$

где $h^\circ(t)=B'(t)s^\circ(t)$ и $s^\circ(t)$ есть решение сопряженной системы

$$\begin{aligned} &ds/dt=-A's-G's(t+\tau)+l(t)\ ,\quad t_\alpha\le t\le t_\beta-\tau\ ;\\ &ds/dt=-A's+p(t)\ ,\quad t_\beta-\tau\le t\le t_\beta\ ;\\ &s(t_\beta)=s_\beta\ ,\quad s(t_\alpha)=s_\alpha\ , \end{aligned} \tag{6.3}$$

обеспечивающее минимум

$$\Psi(s^\circ_\beta,p^\circ,l^\circ)=\min_{s_\beta,p,l}\Psi(s_\beta,p,l)=0 \tag{6.4}$$

неотрицательному, выпуклому функционалу

$$\Psi(s_\beta, p, l) = \int_{t_\alpha}^{t_\beta} \gamma_1 [B'(t) s(t)] dt + \int_{t_\alpha}^{t_\beta-\tau} \gamma_t [l(t)] dt +$$

$$+ k_1 \rho^* [p(t)] + k_2 \cdot \| s_\beta \|^* + s_\alpha' \cdot x(t_\alpha) + \qquad (6.5)$$

$$+ \int_{t_\alpha-\tau}^{t_\alpha} x'(t) G' s(t+\tau) dt \; ; \; k_1 \geq 0 \; , \; k_2 \geq 0$$

Здесь $\| s \|^*$ и $\rho^*[p(t)]$ суть нормы, сопряженные к $\| x \|$, $\rho[x(\vartheta)]$.

Значение ε° определяется как сумма чисел $k_1^\circ + k_2^\circ$, при которых выполняется (6.4).

Дальнейший ход решения повторяет в общих чертах соответствующую процедуру для задачи 2.1. Отметим только, что дискретную аппроксимацию задачи 6.1 удобно получать, заменяя систему 6.1 конечномерной системой

$$dy^\circ/dt = Ay^\circ + Gy^{(m)} + Bu$$

$$dy^{(i)}/dt = \frac{m}{\tau}\left(y^{(i-1)} - y^{(i)}\right), \; (i = 1, \ldots, m) \qquad (6.6)$$

по схеме работы [17]. Последнее объясняется, в частности, тем, что при $m \to \infty$ и $t - t_\alpha \geq 2\tau$ решение однородного уравнения 6.1 можно приблизить решениями $y^\circ(t), \ldots, y^{(m)}(t)$ однородного уравнения 6.6 равномерно на всем отрезке $[t-\tau, t]$ [18].

§ 7. Минимизация "чебышевского" функционала.

Изложенный выше подход может быть применен и к задаче о минимуме максимального отклонения [19].

Задача 7.1. При заданных краевых условиях $x(t_\alpha) = x_\alpha$, $x(t_\beta) = x_\beta$ и ограничении $|u_k(t)| \leq C$ найти управление u°, чтобы

$$\max_t |x_1(t)| = \min_u \qquad (7.1)$$

Решением задачи будут множители Лагранжа s_β°, $l^\circ(t) = \{l_1^\circ(t), 0, \ldots 0\}$, оптимальное управление u° и постоянная

$k^\circ = \max_t |x_1^\circ(t)|$, доставляющие седловую точку функции

$$\min_{s_\beta, l} \left\{ \max_{u\in U} \int_{t_\alpha}^{t_\beta} u'(t)B'(t)s(t)dt + s_\alpha' x_\alpha - s_\beta' x_\beta + \right.$$

$$\left. + k^\circ \int_{t_\alpha}^{t_\beta} |l_1(t)| dt \right\} = 0$$

где $s(t)$ есть решение (сопряженной системы) (2.10), при условии (2.14).

Аналогично решается и задача о минимизации промаха[20], когда в условиях задачи 7.1 вместо (7.1) выбран критерий

$$\max_t \| x(t) \| = \min_u = k^\circ$$

а конец траектории $x(t)$ при $t = t_\beta$ свободен и подчинен лишь неравенству $\| x_\beta \| \leq k^\circ$.

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НАХОЖДЕНИЕ НАЧАЛЬНЫХ ЗНАЧЕНИЙ ВСПОМАГАТЕЛЬНЫХ ПЕРЕМЕННЫХ ОДНОГО КЛАССА ЛИНЕЙНЫХ СИСТЕМ ПРИ ИХ ОПТИМАЛЬНОМ ПО БЫСТРОДЕЙСТВИЮ УПРАВЛЕНИИ

И.П.Цветанов
Институт по техническа кибернетика
София-13, Н.Р.Болгария

§ 1. Введение

При использовании принципа максимума для определения оптимального по быстродействию управления объектами, движение которых описывается следующими дифференциальными уравнениями в векторном виде [1]:

$$\dot{\bar{x}} = A\bar{x} + B\bar{u} \tag{1.1}$$

вспомагательные переменные $\psi_1(t), \psi_2(t), \ldots, \psi_n(t)$ вводятся при помощи векторного уравнения:

$$\dot{\bar{\psi}} = -A'\bar{\psi} \tag{1.2}$$

а условие экстремальности управления можно записать следующим образом:

$$\bar{\psi}(t)\, B\bar{u}(t) = \max_{\bar{u} \in U} \bar{\psi}(t) B\bar{u}(t) \tag{1.3}$$

В (1.1) $\bar{x}(t)$ является вектором фазового пространства X_n, $\bar{u}(t)$-вектором, принадлежащим области $U = \{\bar{u} : -1 \leq u_i(t) \leq 1,\ i = 1, 2, \ldots, r\}$ пространства допустимых управлений, $A = (a_{ij})$ -матрицей порядка $n \times n$ с реальными неположительными собственными значениями [*1], $B = (B_{ij})$ -матрицей порядка $n \times r$ а A' в (1.2) - транспонированной матрицей A.

Вектор $\bar{\psi}(t)$ определен при помощи (1.2) с точностью до n констант $\psi_1^\circ, \psi_2^\circ, \ldots, \psi_n^\circ$ (координаты начального значения $\bar{\psi}(0)$). Из этого следует что равенства (1.3) определяют обесконечное множество экстремальных управлений $U_{ext} \subset U$.

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*1 Это условие несущественное

Пусть в (1.1) накладывается дополнительное ограничение - $\vec{U}(t) \in U_{ext}$. Это означает, что фиксированному начальному состоянию $\vec{x}^o = (x_1^o, x_2^o, \ldots, x_n^o)$ из области управляемости Ω при помощи (1.1) сопоставляется ~~#~~бесконечное множество $\Phi_{ext}[\vec{x}^o]$ траектории, однозначно определенных элементами U_{ext}. Движение по этим траекториям, согласно достаточному условию оптимальности по быстродействию (1.3)[1], осуществляется с минимальными потерями времени и поэтому целесообразно назвать элементы $\Phi_{ext}[\vec{x}^o]$ экстремальными процессами.

Из теорем существования и единствености оптимального управления[1,2] следует, что существует единственный экстремальный процесс, который проходит через начало координат. К сожалению, однако, эти теоремы не дают ответ на вопрос, как построить вектор $\vec{\Psi}(0)$, который определяет $\vec{U}_{опт}(t)$. Это приводит к необходимости искать решения следующей задачи:

При произвольном $\vec{x}^o \in \Omega$ найти векторы $\vec{\Psi}(0)$, такие, что если они являются начальными значениями (1.2) то равенство (1.3) определяет $\vec{U}_{ext}(t)$, при котором соответствующий экстремальный процесс проходит через начало координат, т.е. $\vec{U}_{ext}(t) = \vec{U}_{опт}(t)$.

Это одна из основных, но все еще нерешенных проблем теории оптимального управления. В литературе известны различные приближенные методы, численного решения этой задачи, которые, однако, не дают точной зависимости между $\vec{x}^o$ и $\vec{\Psi}^o$ [3,4,8-13]

В настоящей работе дается полное решение рассматриваемой задачи ~~в~~ для одного более узкого, но важном для практики классе линейных объектов:

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \ldots & -\frac{a_{n-1}}{a_n} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \frac{b}{a_n} \end{bmatrix}$$

с одномерной областью $U = \{ |u(t)| \le 1 \}$

В сущности это класс всех управляемых объектов, закон движения которых записывается с помощью следующего линейного дифференциального уравнения с постоянными коэффициентами:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = bu(t) \tag{1.4}$$

§ 2. Разделение n-мерного пространства $\mathcal{T}^0$ на подобласти, содержащие те начальные $\bar{\Psi}(0)$, соответствующие экстремальные управления которых имеют одинаковое число точек разрыва (переключения)

Рассматриваемые объекты имеют одно управление: $-1 \le u(t) \le 1$ и экстремальные равенства (1.3) можно записать следующим образом:

$$u_{ext}(t) = \operatorname{sign} \Psi_n(t) \tag{2.1}$$

В зависимости, от того, есть ли у матрицы А нулевое собственное значение, или нет, в решениях появляется известное различие. Однако оно не внесет ничего нового в рассуждения и поэтому рассмотрим только случай когда:

Собственные значения матрицы А представляют реальные числа $\lambda_1 < \lambda_2 < \cdots < \lambda_m < 0$ со соответствующей кратностью $K_1, K_2, \ldots, K_m \ (K_1 + K_2 + \cdots + K_m = n)$ [**1]

$$u_{ext} = \operatorname{sign}\left\{-\sum_{i=1}^{m} \sum_{q=1}^{K_i} C_q^i \left[\sum_{r=1}^{q-1} \frac{t^r}{r!}\right] e^{-\lambda_i t}\right\} \tag{2.2}$$

Константы $C_q^i \left(\begin{matrix} i = 1, 2, \ldots, m \\ q = 1, 2, \ldots, K_i \end{matrix} \right)$ определяются однозначно [**2] из системы

$$\Psi_k^0 = \sum_{i=1}^{m} \sum_{q=1}^{K_i} C_q^i \left\{ \sum_{j=1}^{k-1} \frac{a_j}{a_n} \left[\sum_{\ell=0}^{q-1} \binom{k+\ell-j-1}{k-j-1} \frac{1}{\lambda_i^{\ell}} \right] \frac{1}{\lambda_i^{k-j}} \right\} \tag{2.3}$$

$$k = 1, 2, \ldots, n-1$$

$$\Psi_n^0 = -\sum_{i=1}^{m} \sum_{q=1}^{K_i} C_q^i$$

Если Δ_1 обозначает определитель коэффициентов переменных

**1 Если λ – собственное значение матрицы А то $-\lambda$ – собственное значение матрицы – А'

**2 Определитель коэффициентов переменных C_q^i не равен нулю

$C_q^i\left(\begin{smallmatrix} i=1,2,\dots,m \\ q=1,2,\dots,K_i \end{smallmatrix}\right)$ в (2.3), а d_{qs}^{1i} - минор , соответствующий элементу в Δ_1 , находящийся в s -ой строке и $K_1+K_2+\dots+K_{i-1}+q+s$ столбце, то

$$C_q^i=\frac{1}{\Delta_1}\sum_{s=1}^{n}(-1)^{K_1+K_2+\dots+K_{i-1}+q+s}d_{qs}^{1i}\Psi_s^0,\quad i=1,2,\dots,m;\ q=1,2,\dots,K_i$$

Функция $\operatorname{sign}\Psi_n(t)$ меняет свой знак только при тех значениях t которые представляют корни нечетной кратности в уравнении

$$\Psi_n(t)=0 \qquad (2.5)$$

Уравнение (2.5) определяет однозначно любому начальному вектору $\bar\Psi(0)$ счетное множество $T[\bar\Psi(0)]$ тех значений t, которые являются его корнями нечетной кратности. Так как (2.5) связано с реальным процессом, принимаем $t\geq 0$.

Из Леммы 2 [xx3] следует, что множество $T[\bar\Psi(0)]$ конечное и число его элементов $N_{T[\bar\Psi(0)]}$ не превышает $n-1$ [xx4]. Этот важный результат дает возможность разделить n -мерное векторное пространство начальных векторов ~~из (2.2)~~ T^0 на n взаимно непресекающихся подобластей. KT_ℓ^0 $(\ell=0,1,2,\dots,n-1)$ относятся все векторы $\bar\Psi(0)$, такие, что $T[\bar\Psi(0)]$ содержит $N_{T[\bar\Psi(0)]}=\ell$ элементов.

<u>Теорема 1</u>: Множества T_ℓ^0 являются ℓ -параметрическими системами подмножеств T^0 , каждое из которых есть объединение из конечного числа линейных подпространств. При этом в эти подпространства не входят их сечения с ℓ хиперплоскостями. Если $M[n;\ell]$ обозначает число всех наборов ℓ - нечетных неотрицательных чисел, сумма которых не превышает $n-1$ то объединение, соответствующее любому набору значений ℓ - параметров однозначно определено и содержит $M[n;\ell]$ подмно-

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[xx3] Лемма: Пусть $\lambda_1,\lambda_2,\dots,\lambda_m$ - действительные попарно различные числа, а $f_1(t),f_2(t),\dots,f_m(t)$ многочлены с действительными коэффициентами, имеющими степени $K_1,K_2,\dots,K_m$ соответственно Тогда функция

$$f_1(t)e^{\lambda_1 t}+f_2(t)e^{\lambda_2 t}+\dots+f_m(t)e^{\lambda_m t} \qquad (2.6)$$

имеет не более чем $K_1+K_2+\dots+K_m+m-1$ действительных корней.

[xx4] Из равенства (2.5) функцию $\Psi_n(t)$ можно записать в виде (2.6).

жеств.

Доказательство:

Совокупность T_e^o состоит из всех векторов $\Psi(o)$ для которых уравнение:

$$\sum_{i=1}^{m}\sum_{q=1}^{K_i}\left[\sum_{s=1}^{n}(-1)^{K_1+K_2+\ldots+K_{i-1}+s+q} d_{qs}^{ii}\Psi_s^o\right]\left[\sum_{r=0}^{q-1}\frac{t^r}{r!}\right]e^{-\lambda_i t}=0 \qquad (2.7)$$

имеет e различных корней с нечетной кратностью.

Пусть $t_1, t_2, \ldots, t_e$ суть произвольные неотрицательные значения t, удовлетворяющие неравенствам $0<t_1<t_2<\ldots<t_e$ и в тоже время являющиеся корнями уравнения (2.7) с нечетными кратностями, соответственно $r_1, r_2, \ldots, r_e$ т.е. если обозначим через $f_{qi}(t)=\left[\sum_{r=0}^{q-1}\frac{t^r}{r!}\right]e^{-\lambda_i t}$, а через $f_{qi}^{(\nu)}(t)$ ν- производную, числа t_j должны удовлетворять условиям:

$$\left.\begin{aligned}
&\sum_{i=1}^{m}\sum_{q=1}^{K_i}\left[\sum_{s=1}^{n}(-1)^{K_1+K_2+\ldots+K_{i-1}+s+q} d_{qs}^{ii}\Psi_s^o\right]f_{qi}(t_j)=0\\
&\sum_{i=1}^{m}\sum_{q=1}^{K_i}\left[\sum_{s=1}^{n}(-1)^{K_1+K_2+\ldots+K_{i-1}+s+q} d_{qs}^{ii}\Psi_s^o\right]f_{qi}^{(1)}(t_j)=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\sum_{i=1}^{m}\sum_{q=1}^{K_i}\left[\sum_{s=1}^{n}(-1)^{K_1+K_2+\ldots+K_{i-1}+s+q} d_{qs}^{ii}\Psi_s^o\right]f_{qi}^{(r_j-1)}(t_j)=0\\
&\sum_{i=1}^{m}\sum_{q=1}^{K_i}\left[\sum_{s=1}^{n}(-1)^{K_1+K_2+\ldots+K_{i-1}+s+q} d_{qs}^{ii}\Psi_s^o\right]f_{qi}^{(r_j)}(t_j)\neq 0
\end{aligned}\right\}$$

Таким образом определенному набору нечетных чисел удовлетворяющих условию $r_1+r_2+\ldots+r_e\leq n-1$ и определенному фиксированному набору значений параметров $t_1, t_2, \ldots, t_e$ однозначно сопоставляется линейная система из условия:

$$\left.\begin{aligned}
&\sum_{s=1}^{n}\left[\sum_{i=1}^{m}\sum_{q=1}^{K_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q} d_{qs}^{ii} f_{qi}^{(o)}(t_1)\right](-1)^s\Psi_s^o=0\\
&\sum_{s=1}^{n}\left[\sum_{i=1}^{m}\sum_{q=1}^{K_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q} d_{qs}^{ii} f_{qi}^{(1)}(t_1)\right](-1)^s\Psi_s^o=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\sum_{s=1}^{n}\left[\sum_{i=1}^{m}\sum_{q=1}^{K_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q} d_{qs}^{ii} f_{qi}^{(r_1-1)}(t_1)\right](-1)^s\Psi_s^o=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\sum_{s=1}^{n}\left[\sum_{i=1}^{m}\sum_{q=1}^{K_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q} d_{qs}^{ii} f_{qi}(t_e)\right](-1)^s\Psi_s^o=0\\
&\sum_{s=1}^{n}\left[\sum_{i=1}^{m}\sum_{q=1}^{K_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q} d_{qs}^{ii} f_{qi}^{(1)}(t_e)\right](-1)^s\Psi_s^o=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\sum_{s=1}^{n}\left[\sum_{i=1}^{m}\sum_{q=1}^{K_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q} d_{qs}^{ii} f_{qi}^{(r_e-1)}\right](-1)^s\Psi_s^o=0
\end{aligned}\right\} \qquad (2.8)$$

$$\left.\begin{aligned}
&\sum_{s=1}^{n}\Big[\sum_{i=1}^{m}\sum_{q=1}^{K_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}\, d_{qs}^{\,si} f_{qi}^{(\tau_1)}(t_1)\Big](-1)^s\Psi_s^0\neq 0\\
&\sum_{s=1}^{n}\Big[\sum_{i=1}^{m}\sum_{q=1}^{K_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}\, d_{qs}^{\,ii} f_{qi}^{(\tau_2)}(t_2)\Big](-1)^s\Psi_s^0\neq 0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\sum_{s=1}^{n}\Big[\sum_{i=1}^{m}\sum_{q=1}^{K_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}\, d_{qs}^{\,ii} f_{qi}^{(\tau_l)}(t_l)\Big](-1)^s\Psi_s^0\neq 0
\end{aligned}\right\}$$

Число уравнений (2.8) согласно цитированой в ~~§§~~3 Лемме не превышает $n-1$, так что действительно все векторы $\bar{\Psi}(0)$ – решения первых $\tau_1+\tau_2+\ldots+\tau_l$ уравнений образуют линейное подпространство в $\mathcal{T}^0$. Из этого линейного подпространства исключаются все векторы – решения линейной системы:

$$\begin{aligned}
&\sum_{s=1}^{n}\Big[\sum_{i=1}^{m}\sum_{q=1}^{K_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}\, d_q^{\,ii} f_{qi}^{(\tau_1)}(t_1)\Big](-1)^s\Psi_s^0=0\\
&\sum_{s=1}^{n}\Big[\sum_{i=1}^{m}\sum_{q=1}^{K_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}\, d_q^{\,ii} f_{qi}^{(\tau_2)}(t_2)\Big](-1)^s\Psi_s^0=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\sum_{s=1}^{n}\Big[\sum_{i=1}^{m}\sum_{q=1}^{K_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}\, d_q^{\,ii} f_{qi}^{(\tau_l)}(t_l)\Big](-1)^s\Psi_s^0=0
\end{aligned}$$

Число возможных наборов нечетных чисел $\tau_1, \tau_2, \ldots, \tau_l$ равняется $M[n:l]$.

Следовательно, если обозначить совокупность всех векторов $\bar{\Psi}(0)$, удовлетворяющих (2.8) через $\mathcal{T}_{ep}^0[t_1, t_2, \ldots, t_l, \tau_1, \tau_2, \ldots, \tau_l]$ то из приведенных россуждений ясно, что

$$\mathcal{T}_l^0=\bigcup_{\{0<t_1<t_2<\ldots<t_l\}}\Big\{\bigcup_{p=1}^{M[n:l]}\mathcal{T}_{ep}^0[t_1, t_2, \ldots, t_l;\tau_1,\tau_2,\ldots,\tau_l]\Big\}$$

Множество $\mathcal{T}_0^0$ также однозначно определено, поскольку:

$$\mathcal{T}_0^0=\mathcal{T}^0\setminus\bigcup_{l=1}^{n-1}\mathcal{T}_l^0$$

§ 8. Параметрическое представление допустимых экстремальных процессов $\Phi_{ext}[\bar{x}^0]$

Совокупность допустимых экстремальных процессов $\Phi_{ext}[\bar{x}^0]$ однозначно определена с помощью векторного дифференциального уравнения:

$$\dot{\bar{X}}=A\bar{X}+Bu_{ext}(t) \tag{3.1}$$

В (8.1) $u_{ext}(t)$ кусочно – постоянные функции (см. § 2), так что $u_{ext}(t)$ полностью задается через $\sigma=u_{ext}(0)$ и моменты $0<t_1<\ldots<t_l$ $(l\le n-1)$ разрыва. Для значений $t\neq t_1, t_2, \ldots, t_l$

$$U_{ext}(t) = \begin{cases} \sigma & \text{, если } 0 \le t < t_1 \\ (-1)^s \sigma & \text{, если } t_s < t < t_{s+1} \quad s=1,2,\dots,\ell-1 \\ (-1)^\ell \sigma & \text{, если } t_\ell < t \end{cases} \qquad (3.2)$$

Равенства (3.2) показывают, что для $t \neq t_1, t_2, \dots, t_\ell$ дифференциальное уравнение (3.1) является неоднородным бинейным дифференциальным уравнением.

Решения (3.1) для $t \neq t_1, t_2, \dots, t_\ell$ можно записать в следующем векторном виде:

$$\vec{x}(t) = \begin{cases} \sum\limits_{i=1}^{m} \sum\limits_{q=1}^{K_i} D_q^{i}(0)\, \vec{x}_q^{\,i}(t) + \dfrac{b}{a_0 a_n} \sigma \vec{e}_1, & \text{для } 0 \le t < t_1 \\ \sum\limits_{i=1}^{m} \sum\limits_{q=1}^{K_i} D_q^{i}(s)\, \vec{x}_q^{\,i}(t) + (-1)^s \dfrac{b}{a_0 a_n} \sigma \vec{e}_1, & \text{для } t_s < t < t_{s+1},\ s=1,2,\dots,\ell-1 \\ \sum\limits_{i=1}^{m} \sum\limits_{q=1}^{K_i} D_q^{i}(s)\, \vec{x}_q^{\,i}(t) + (-1)^\ell \dfrac{b}{a_0 a_n} \sigma \vec{e}_1, & \text{для } t_\ell < t \end{cases}$$

$D_q^{i}(s)$, $s = 0,1,2,\dots,\ell$ произвольные константы соответствующие s-тому интервалу,

$\vec{x}_q^{\,i}(t) = \left[\sum\limits_{r=0}^{q-1} \frac{t^r}{r!} \vec{h}_{q-r}^{(i)}\right] e^{\lambda_i t}$ где, если $\vec{h}_{q-r}^{(i)} = (h_{q-r,1}^{(i)}, h_{q-r,2}^{(i)}, \dots, h_{q-r,n}^{(i)})$

то $h_{q-r,1}^{(i)} = 1$, $h_{q-r,k}^{(i)} = \lambda_i^{k-1} \left[\sum\limits_{\ell=0}^{m-1} \binom{k-1}{\ell} \frac{1}{\lambda_i^{\ell}}\right]$ $k = 2,3,\dots,n$.

а $\vec{e}_1 = (1,0,0,\dots,0)$ n - мерный вектор [6].

Лемма 1. Экстремальные процессы $\Phi_{ext}[\vec{x}^0]$ непрерывны и в точках разрыва соответствующих $u_{ext}(t)$.

Доказательство

Лемма 1 непосредственно следует из представления экстремальных процессов в следующей векторной форме:

$$\vec{x}(t) = \sum_{i=1}^{n} \vec{\varphi}_i(t)\left[x_i^0 + \int^t \psi_i^{\,n}(x)\, u_{ext}(x)\, dx\right]$$

где через $\psi_i^{\,n}$ обозначена i-тая компонента [1,7] $\vec{\psi}^{\,n}$

Непрерывность экстремальных процессов и в моменты разрыва $t_1, t_2, \dots, t_\ell$ соответствующих $u_{ext}(t)$ полностью определяет константы $D_q^{i}(s)$, $s = 0,1,2,\dots,\ell$, $i = 1,2,\dots,m$, $q = 1,\dots,K_i$ через $x_1^0, x_2^0, \dots, x_n^0$, σ и $t_1, t_2, \dots, t_s$.

В Приложение I поясняется как выводятся следующие формулы для $D_q^{i}(s)$:

$$D_q^{1i}(s) = \frac{1}{\prod\limits_{\substack{j=1\\ m\geq j\neq i}}^{m} (\lambda_j-\lambda_i)^{k_j k_i}} \left[W_q^{1i}(x_1^0, x_2^0, \dots, x_n^0, \sigma) + (-1)^{\sum_{j=1}^{i-1} k_j} (-1)^{1+q}\, 2\sigma \frac{b}{a_0 a_n} \sum_{u=1}^{s} \Delta_q^{1i}(t_u) e^{-\lambda_i t_u} \right] \quad (3.3)$$

Символы $W_q^{1i}(x_1^0, x_2^0, \dots, x_n^0, \sigma)$ и Δ_q^{1i} – определители которые данны в Приложении I.

§ 4. Нахождение начальных значений вспомагательных неизвестных, для которых соответствующее екстремальное управление является оптимальным

Поставленная в § 1 задача полностью решается доказательством Леммы 8 и Теоремы 2.

Лемма 8. Если собственные значения матрицы A реальные числа $\lambda_1, \lambda_2, \dots, \lambda_m$ с соответотвующей кратностью $k_1, k_2, \dots, k_m$ $(k_1+k_2+\dots+k_m=n)$ и удовлетворяют условию $\lambda_1<\lambda_2<\dots<\lambda_m<0$, то из $2n$ трансцендентных систем[1]

$$\left.\begin{aligned}
&(-1)^{\sum_{j=1}^{i-1} k_j}(-1)^{1+q}\frac{a_0 a_n}{b} W_q^{1i}(x_1^0, x_2^0, \dots x_n^0, \sigma) + (-1)^1 \Delta_q^{1i}(t^*) e^{-\lambda_i t^*} = 0\\
&i=1,2,\dots,m, \quad q=1,2,\dots,k_i
\end{aligned}\right\} \quad (4.1)$$

$$\left.\begin{aligned}
&(-1)^{\sum_{j=1}^{i-1} k_j}(-1)^{1+q}\frac{a_0 a_n}{b} W_q^{1i}(x_1^0, x_2^0, \dots x_n^0, \sigma) + 2\sum_{u=1}^{s}(-1)^{u-1}\Delta_q^{1i}(t_u) e^{-\lambda_i t_u} + (-1)^s \Delta_q^{1i}(t^*) e^{-\lambda_i t^*} = 0\\
&i=1,2,\dots,m, \quad q=1,2,\dots,k_i
\end{aligned}\right\}$$

только одна имеет решение $t_1, t_2, \dots, t_s, t^*$ которое удовлетворяют условию $0<t_1<t_2<\dots<t_s<t^*$ и это решение является единственным.

Доказательство:

Приведенная в §3 Лемма позволяет разделить совокупность экстремальных процессов $\Phi_{ext}[\vec{x}^0]$ на непресекающиеся подсовокупности $\Phi_{ext}^p[\vec{x}^0]$ $(p=0,1,2,\dots,n-1)$ в зависимости от числа точек разрыва определяющих их $u_{ext}(t)$. Согласно теореме о существовании, существует такой оптимальной процесс, которой проходит через начало координат и принадлежит одному из множеств $\Phi_{ext}^p[\vec{x}^0]$ $(p=0,1,2,\dots,n-1)$.

Если этот екстремальный процес принадлежит $\Phi_{ext}^p[\vec{x}^0]$, то согласно параметричному равенству (3.3), это будет тот процесс, для которого существует $t^*>0$, так что:

---

[1] Ю.Г.Антомонов, В 5 получил последную систему (4.1) для частного случая $m=n$, $k_1=k_2=\dots=k_n=1$

$$\left.\begin{aligned}&\sum_{i=1}^{m}\sum_{q=1}^{K_i} D_q^{ii}(p)\Big[\sum_{r=0}^{q-1}\frac{(t^*)^r}{r!}\Big]e^{\lambda_i t^*}+(-1)^p\frac{b}{a_0a_n}\sigma=0\\ &\sum_{i=1}^{m}\sum_{q=1}^{K_i} D_q^{ii}(p)\Big[\sum_{r=0}^{q-1}\frac{(t^*)^r}{r!}\Big(\lambda_i^{n-1}\sum_{j=1}^{q-r-1}\binom{k-1}{j}\frac{1}{\lambda_i^j}\Big)\Big]e^{\lambda_i t^*}=0\\ &k=2,3,\dots,n-1,n\end{aligned}\right\}\qquad(4.2)$$

Системы (4.2) линейны по отношению к $D_q^{ii}(p)$ и того же типа. что и системы из которых определяются $D_q^{ii}(p)$ в Приложении I. Они отличаются лишь свободными членами.

Это позволяет и здесь использовать выведенные в Предложении I определительные обозначения $\Delta_q^{ii}(t)$ т.е.

$$D_q^{ii}(p)=\frac{(-1)^{\sum_{j=0}^{i-1}K_j}}{\prod\limits_{\substack{i=1\\ m\geq j>i}}(\lambda_j-\lambda_i)^{K_jK_i}}(-1)^{i+q}\Delta_{iq}^{ii}(t^*)e^{\lambda_i t^*}\cdot(-1)^{p-1}\frac{b}{a_0a_n}\sigma \qquad (4.2')$$

В левой стороне равенства (4.2′) находятся уже определенные формулой (3.3) $D_q^{ii}(p)$. В них σ и $t_1,t_2,\dots t_p,\ t^*$ обозначены точки разрыва $u_{опт}(t)$. Очевидно, что $0<t_1<t_2<\dots<t_p<t^*$.

Полученные после замещения $D_q^{ii}(p)$ зависимости (4.2′) всущности представляют собой n - уравнения p-ой трансцендентной системы в (4.1).

Ввиду того, что неизвестно к какой $\Phi_{\sigma t}^{p}[\bar{x}^0]$ относится оптимальный процес, кроме того, не известно σ, то сначала для всякого $p=0,1,2,\dots,n-1$ и $\sigma=\pm1$ составляются $2n$ - системы (4.1) и определяется та, для которых бы одно решение $t_1,t_2,\dots,t_p,t^*$ удовлетворяет условию $0<t_1<\dots<t_p<t^*$. Единственость этой системы и ее решения обуславливается теоремой единственности $u_{опт}(t)$.

Пусть собственные значения $\lambda_1,\lambda_2,\dots,\lambda_m$ матрицы A в (1.1) имеют соответствующую кратность $K_1,K_2,\dots,K_m$ $(K_1+K_2+\dots+K_m=n)$ и $\lambda_1<\lambda_2<\dots<\lambda_m<0$

Пусть для произвольно выбранного и фиксированного состояния $\bar{x}^0\in X_n$ система

(4.3)

$$\left.\begin{array}{l}(-1)^{\sum\limits_{l=1}^{i-1}K_l}(-1)^{i+q}\dfrac{a_0a_n}{b}\sigma W_q^{ii}(x_1^0,x_2^0,\ldots,x_n^0,\sigma)+2\sum\limits_{l=1}^{p}(-1)^{l-1}\Delta_{iq}(t_l)e^{-\lambda_i t_l}+(-1)^p\Delta_{iq}^{ii}(t^*)e^{-\lambda_i t^*}=0\\ \qquad i=1,2,\ldots,m\ ;\quad q=1,2,\ldots,k_i\end{array}\right\}\quad(4.3)$$

является одной из трех трансцендентных систем (4.1), между решениями которой существует решение $t_1,t_2,\ldots,t_p,t^*$ так что $0<t_1<t_2<\ldots<t_p<t^*$.

Пусть $\tau_1,\tau_2,\ldots,\tau_p$ есть произвольное сочетание целых, нечетных и неотрицательных чисел, удовлетворяющих неравенству

$$\tau_1+\tau_2+\ldots+\tau_p\leq n-1 \tag{4.4}$$

С фиксированием $\tau_1,\tau_2,\ldots,\tau_p$ решения (4.3) однозначно определяют множество $T_{опт}[\bar{x}^0;\tau_1,\tau_2,\ldots,\tau_p]\subset T_p^0$ из всех начальных векторов $\bar{\Psi}(0)$, которые являются решением системы:

$$\left.\begin{array}{l}\operatorname{sign}\Psi_n^0-\sigma=0\\ \sum\limits_{s=1}^{n}\Big[\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{k_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}d_q^{ii}f_{qi}(t_1)\Big](-1)^s\Psi_s^0=0\\ \sum\limits_{s=1}^{n}\Big[\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{k_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}d_q^{ii}f_{qi}^{(1)}(t_1)\Big](-1)^s\Psi_s^0=0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \sum\limits_{s=1}^{n}\Big[\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{k_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}d_q^{ii}f_{qi}^{(\tau_1-1)}(t_1)\Big](-1)^s\Psi_s^0=0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \sum\limits_{s=1}^{n}\Big[\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{k_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}d_q^{ii}f_{qi}(t_p)\Big](-1)^s\Psi_s^0=0\\ \sum\limits_{s=1}^{n}\Big[\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{k_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}d_q^{ii}f_{qi}^{(\tau_p-1)}(t_p)\Big](-1)^s\Psi_s^0=0\\ \sum\limits_{s=1}^{n}\Big[\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{k_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}d_q^{ii}f_{qi}^{(\tau_1)}(t_1)\Big](-1)^s\Psi_s^0\neq0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \sum\limits_{s=1}^{n}\Big[\sum\limits_{i=1}^{m}\sum\limits_{q=1}^{k_i}(-1)^{K_1+K_2+\ldots+K_{i-1}+q}d_q^{ii}f_{qi}^{(\tau_p)}(t_p)\Big](-1)^s\Psi_s^0\neq0\end{array}\right\}\quad(4.5)$$

В (4.5) с $f_{qi}(t)$ и $f_{qi}^{(\nu)}(t)$ обозначены соответно функции $\Big[\sum\limits_{\tau=0}^{q-1}\dfrac{t^\tau}{\tau!}\Big]e^{-\lambda_i t}$ и ν-ая их производная. Возможные сочетания из p неотрицательных, нечетных чисел $\tau_1,\tau_2,\ldots\tau_p$ удовлетворяющие равенство (4.4) являются конечным числом $M[n;p]$.

Теорема 2

Множество $\bigcup\limits_{\{\tau_1,\tau_2,\ldots,\tau_p\}}T_{опт}^0[\bar{x}^0;\tau_1,\tau_2,\ldots,\tau_p]$ содержит все начальные вектора, которые определяют $u_{опт}(t)$ через равенством (2.1).

Доказательство теоремы непосредственно следует из Теоремы 1, Леммы 2 и достаточности условия (2.1).

<u>Пример:</u>

Решим поставленную в § 1 задачу для объекта, закон движения которого описывается дифференциальным уравнением вида:

$$\frac{d^2x}{dt^2} + 3\ \frac{dx}{dt} + 2x = u(t)$$

Здесь $\lambda_1 = -2,\ \ \lambda_2 = -1,\ \ a_0 = 2,\ a_1 = 3,\ a_2 = 1,\ b = 1$

Пусть за начальное состояние $\bar{x}^0$ выбрана точка $(\frac{1}{2}, 0)$. Трансцендентные системы (4.1) для выбранного $\bar{x}^0$ имеют вид:

$$\sigma = 1 \qquad\qquad\qquad\qquad \sigma = -1$$

$$s = 0\begin{cases} (-1)^0 e^{t^*} = 0 \\ \frac{1}{2}(-1)^0 e^{2t^*} = 0 \end{cases} \qquad\qquad s = 0\begin{cases} 2 - (-1)^0 e^{t^*} = 0 \\ -1 - \frac{1}{2}(-1)^1 e^{2t^*} = 0 \end{cases}$$

$$s = 1\begin{cases} 2(-1)^0 e^{t_1} + (-1)^1 e^{t^*} = 0 \\ -(-1)^0 e^{2t_1} + \frac{1}{2}(-1)^2 e^{2t^*} = 0 \end{cases} \qquad\qquad s = 1\begin{cases} 2 - 2(-1)^0 e^{t_1} - (-1)^1 e^{t^*} = 0 \\ -1 + (-1)^0 e^{2t_1} - \frac{1}{2}(-1)^2 e^{2t^*} = 0 \end{cases}$$

Оказывается, что для $\sigma = -1$ и $s = 1$ соответствующая система имеет решение $e^{t_1} = 3$ и $e^{t^*} = 4$ т.е. $0 < \ln 3 = t_1 < \ln 4 = t^*$

Для выбранных $\lambda_1 = -2$ и $\lambda_2 = -1$ система (4.4) записывается одним равенством и двумя неравенствами:

$$\left.\begin{aligned} \psi_2^0 &< 0 \\ L:\ -4\psi_1^0 + 5\psi_2^0 &= 0 \\ -20\psi_1^0 + 11\psi_2^0 &\neq 0 \end{aligned}\right\}$$

решения которых лежат на утолщенной части прямой L (см. фиг.1).

§ 5. Приложение I

Непрерывность экстремальных траекторий и в точках разрыва $t_1, t_2, \ldots, t_\ell \ (\ell \leq n-1)$ $u_{ext}(t)$ требует согласования постоянных $D_q^{i}(s)$, $s = 0,1,2,\ldots,\ell$ для того что бы были справедливы равенства (3.3) и для значений $t_1, t_2, \ldots, t_\ell$. Непрерывность траекторий в $t_1, t_2, \ldots, t_\ell$ позволяет в (3.3) для любого t_s написать соответствующее равенство в векторном виде:

$$\sum_{i=1}^{m} \sum_{q=1}^{K_i} \left[D_q^{i}(s) - D_q^{i}(s-1) \right] \vec{x}_q^{\,i}(t_s) = 2(-1)^{s-1} \sigma \frac{b}{a_0 a_n} \vec{e}_1 \qquad (5.1)$$

Определитель перед неизвестными

$$D_q^{i}(s) - D_q^{i}(s-1), \quad \begin{matrix} i = 1,2,\ldots,m \\ q = 1,2,\ldots,K \end{matrix} \qquad (K_1 + K_2 + \ldots + K_m = n)$$

есть определитель Вронского $W(t_s)$, из фундаментальной системы решения однородного дифференциального уравнения:

$$\dot{\vec{x}} = A\vec{x}, \quad \text{для } t = t_s$$

$$W(t_s) = W(0)\, e^{-\int_0^{t_s} S_A\, dx}$$

так что

S_A обозначен след матрицы A, так что $S_A = -(K_1\lambda_1 + K_2\lambda_2 + \ldots + K_m\lambda_m)$ легко доказать, что

$$W(0) = \prod_{\substack{i=1 \\ m \geq j > i}}^{m} (\lambda_j - \lambda_i)^{K_j K_i}$$

Этот результат позволяет написать формулу:

$$W(t_s) = \prod_{\substack{i=1 \\ m \geq j > i}}^{m} (\lambda_j - \lambda_i)^{K_j K_i} e^{-\sum K_\nu \lambda_\nu t_s}$$

Так как для любого t_s, $W(t_s) \neq 0$, то $D_q^{i}(s) - D_q^{i}(s-1)$ однозначно определяются из (5.1) посредством формул:

$$D_q^{i}(s) - D_q^{i}(s-1) = \frac{(-1)^{\sum_{j=1}^{i-1} K_j}}{\prod_{\substack{i=1 \\ m \geq j > i}} (\lambda_j - \lambda_i)^{K_j K_i}} (-1)^{s+q}\, 2\sigma \frac{b}{a_0 a_n} \Delta_q^{i}(t_s)\, e^{-\lambda_i t_s} \qquad (5.2)$$

$\Delta_q^{i}(t_s)$ обозначены определители, которые получаются из дополнительного минора элементов $\omega_{r1}^{i}(t_s) = \left[\sum_{z=0}^{q-1} \frac{t_s^{z}}{z!} \right] e^{-\lambda_i t_s}$ в первой строке $W(t_s)$, после того как из столбцов будут вынесены за скобки экспоненты $e^{\lambda_i t_s}$.

Если в (5.2) вместо t_s поставить $t_{s-1}, t_{s-2}, \ldots, t_1$ и полученные равенства просуммируем получим для $D_q^{i}(s)$ следующие формулы:

$$D_q^{1i}(s) = D_q^{1i}(0) + \frac{(-1)^{\sum_{j=1}^{i-1} K_j}}{\prod\limits_{\substack{i=1 \\ m \geq j > i}}^{m} (\lambda_j - \lambda_i)^{K_j K_i}} (-1)^{K-q}\, 2\sigma \frac{B}{a_0 a_n} \sum_{\nu=1}^{s} \Delta_{1q}^{1i}(t_\nu)\, e^{-\lambda_i t_\nu} \tag{5.8}$$

Постоянные $D_q^{1i}(0)$ является линейными функциями $x_1^0, x_2^0, \ldots x_n^0, \sigma$. Их точный вид определяется линейной системой:

$$\left.\begin{aligned} &\sum_{i=1}^{m} \sum_{q=1}^{K_i} D_q^{1i}(0) + \sigma \frac{B}{a_0 a_n} = x_1^0 \\ &\sum_{i=1}^{m} \sum_{q=1}^{K_i} \left[\lambda_i^{K-1} \sum_{j=0}^{q-1} \binom{K-1}{j} \frac{1}{\lambda_i^j} \right] D_q^{1i}(0) = x_K^0 \\ &K = 2, 3, \ldots, n \end{aligned}\right|$$

посредством формул:

$$D_q^{1i}(0) = \frac{W_q^{1i}(x_1^0, x_2^0, \ldots, x_n^0, \sigma)}{\prod\limits_{\substack{i=1 \\ m \geq j > i}} (\lambda_j - \lambda_i)^{K_j K_i}} .$$

$W_q^{1i}(x_1^0, x_2^0, \ldots, x_n^0, \sigma)$ получается путем подстановки чисел $x_2^0, x_3^0 \ldots x_n^0, x_1^0 - \sigma \frac{B}{a_0 a_n}$ в i - тую группу $W(0)$.

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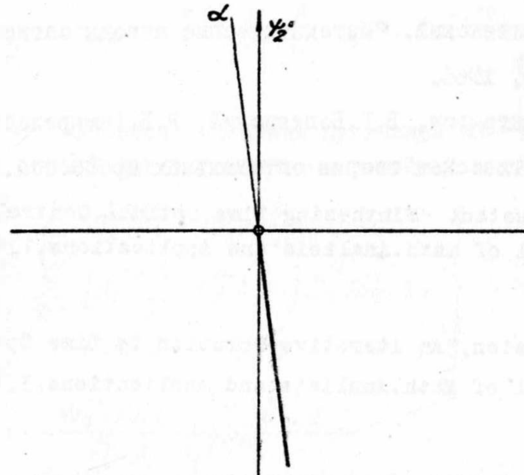

Фиг. 1