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Computational Methods of Optimisation in Control

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1. Introduction

The facet that most distinguishes the modern approach to control system design from the classical frequency response method is the factor of optimisation in the design. There is a wide variety of ways in which this can be done and some useful methods will be reviewed. For systems of a size and complexity to be of industrial interest the amount of numerical work is on such a scale as to necessitate the use of computers, so that computational methods of optimisation become a subject of importance.

There already exists a catalogue of numerical techniques of use in optimising the design of control systems. However, the sphere of their usefulness is often quite restricted, so the problem becomes one of matching methods to the circumstances of the design. Great skill is required in doing this to ensure that considerations of error propagation or numerical stability do not vitiate the solution. Particularly is this so when applying the methods to dynamic systems where responses vary with time. Additional difficulties can arise here due to limitations of particular computers; either from the scale of storage required or from the duration of computation. However, the most encouraging recent developments show how methods developed originally for static (steady state) surface exploration and restricted to this case may be extended by skilful employment of functional analysis to handle the dynamic case.

2. Numerical Optimisation Methods

2.1 Surface exploration for stationary points

Many of the commonly used gradient methods, commencing with an initial guess, search for a stationary point by making changes in the independent variable in proportion to measured partial derivatives to obtain the next guess and so on. These methods result in geometric decays in the independent variables as they approach a stationary point for a second-degree or quadratic surface. The steepest descent method, in particular, corresponds to making vector changes in adjustment in proportion to successive local gradient vectors. The coefficients of the correction controlling matrix have values only on the diagonal and are identical. There is a resulting cross-coupling between the components of the independent variable in a multi-variable case.

Newton's method, where it can be applied, results in a superior characteristic. It leads to a controlling matrix consisting of elements derived from the inverse of the matrix of second partials. This can be seen very simply for a quadratic surface as follows:-

Surface, $V = \frac{1}{2} < \underline{x}, A\underline{x} > + < b,\underline{x} > + c$

Gradient, $V_x = A\underline{x} + b = g$

$V_{xx} = A$ (matrix of 2nd partials)

for $g = 0$ we require $\underline{x} = - A^{-1}.b$.

With Newton's method in the quadratic case multi-dimensional corrections are made in a single step (Newton-Raphson). A modified version of Newton's method has scaled down coefficients in the controlling matrix allowing correction in a number of steps, again with geometric decay to a stationary point. Successive adjustments proceed along a straight line in multi-dimensional space from the initial guess to the stationary point. Cross-coupling among the ordinates is eliminated altogether with this method; in this sense it is an ideal method but it can be extremely sensitive to inaccuracies in the computation due to round-off error, for example. In its favour, however, it does have the very desirable second-order convergence characteristic.

Both methods have limitations if the work is to be carried out in a step-by-step fashion. In Newton's method the calculation of the inverse of the A matrix of second partial derivatives is often not computationally feasible. The steepest descent method is liable to a form of numerical instability giving poor convergence; in effect spiralling into a minimum instead of taking the direct path, if too few steps are taken.

Many numerical procedures have been devised to improve these methods for step-by-step treatment. These vary from the straightforward as in Rosenbrock's method[1] which defines a new set of orthogonal vectors at each step to ensure search direction lies along the direction of fastest advance and being simple enables constraint boundaries to be preserved by erection of penalty barriers in a straightforward way, through a series of methods that do not require the calculation of derivatives at all (2),(3) and are thus necessarily rather cumbersome and slow but have the virtue of requiring very little storage facility in computation, to the series with quadratic convergence that seem to be clearly the most powerful, namely the series based on the conjugate gradient idea, due to Hestenes and Stiefel[4]. Two notable improvements have been added to this last method since this work. First the idea of a variable metric introduced by Davidon[5] and second the improvements introduced by Fletcher and Powell[6] involving 'guestimation' and improvement of the inverse of second partials matrix rather than direct calculation. A recent contribution by Powell[7] has also shown how the constrained problem may be tackled using the method.

2.2 Conjugate gradients

The manner in which this technique differs from the more classical approach is shown quite simply. Let the multi-dimensional surface again be given by

$$V = \frac{1}{2} < \underline{x}, A\underline{x} > + < b, \underline{x} > + c$$

Then $V_x = A\underline{x} + b = g$.

Now Δ grad $= A.\delta\underline{x}_i$ from the original equation; thus the key relationship is

$$\underline{g_{i+1} - g_i = A.\delta\underline{x}_i}$$

Compare this with the Newton-Raphson approach in which Δ grad $= - g_i$ since g_{i+1} is required zero; so that $- g_i = A.\delta x_i$ or $\delta\underline{x} = - A^{-1} g_i$ involving the inverse of A.

Continuing with the conjugate gradient method; in Fig. 1 the last step taken in x, δx_{i-1} is shown in direction s_{i-1}. The new gradient is g_i which taken with δx_{i-1} defines a plane. The V = constant hyper-spheres define a set of ellipses on this plane shown in the diagram. Our task is so to choose the current search direction s_i that it goes through the minimum value of V in this plane. Let the direction that does this be composed from the current gradient g_i by the addition of a component proportional to the old search direction s_{i-1}, as shown in the diagram.

Thus $s_i = g_i + \beta_i s_{i-1}$

By a set of orthogonality conditions we can show that

$$\beta_i = \frac{< g_i, g_i >}{< g_{i-1}, g_{i-1} >}$$

to satisfy this. Finally then it only remains to select that magnitude of δx_i which reaches the minimum in this direction and this is a simple minimisation with respect to a single parameter, α say, thus

$$\underset{\alpha}{V_{min}} (x_i + \alpha . s_i)$$

It is the simplicity of this final minimisation step that gives the method such power. It will reach the minimum of any unconstrained n-dimensional surface in n steps and so cannot be improved upon for the amount of computation involved even by the Newton-Raphson method which although reaching the minimum of a quadratic function in one step requires the calculation of the Jacobian with n terms for even that one step.

2.3 Constrained minimisation; linear programming

Thus powerful techniques exist for unconstrained minimisation of functions of many variables. The same happy state of affairs does not however exist for the problem of function minimisation under conditions of constraint, and in the practical world these problems assume far greater importance. As a result a very large amount of effort has been put into some aspects of this problem and particularly into the linear case with inequality constraints. This latter problem has given rise to the voluminous and now highly specialised field of linear programming[8]. From the control systems point of view this well-developed technique has not proved as readily applicable as we might have expected. The range of problems to which it can be directly applied is restricted to the time optimal linear case with control constraints or the minimal fuel linear case similarly constrained[9].

Various attempts have been made to adapt linear programming methods[10] to a wider range of control problems but with only limited success.

2.4 Mathematical programming

If linear programming techniques have not been particularly fruitful from a control point of view this has been by no means the end of the matter. A much smaller body of work exists on more general programming techniques which cover quadratic[11] and non-linear cases. The analogies that exist between aspects of mathematical programming and techniques of optimal control are so close that common ground must exist. An attempt to provide a unification of the two fields was made by Mitter[12] and a rapprochement was sought by means of an International Conference on Programming and Control[13], held in Colorado in 1965. A numerical survey of non-linear programming by Zoutendijk[14] was a notable contribution. He identifies four groups of methods. Two of these represent extensions of the simplex method, the method associated with linear programming and consist of the cutting plane methods described by Kelley[15] which will only work for convex problems and the methods of approximation programming developed by Griffith and Stewart[16] which do work for non-convex problems but for which no proof of convergence has ever been published. The remaining two are a) the methods of feasible directions of which Rosen's gradient projection method[17] is an example that has been successfully applied to many problems with a non-linear cost function and linear constraints, and b) interior point methods such as the method of sequential unconstrained minimisation (SUMT) of Fiacco and McCormick[18]. These last methods are heavyweights capable of handling non-convex problems, including non-linear constraints at the price of considerable computational complexity.

An improvement due to Zangwill[19] enables the method to work on both sides of the constraint region and consequently no longer represents an interior point method. It can also handle equality constraints as well as inequality constraints. However, the method has its numerical irritations if the initial multiplier values are chosen unfortunately, resulting at one extreme in a very lengthy computation and at the other in the possibility of premature convergence to a false minimum.

Powell[7] has recently introduced a new technique for tackling the non-linearly constrained minimisation problem that appears to be free of these numerical indispositions. It utilises Fletcher-Powell[6] unconstrained minimisation operations but adjoins a factor containing the constraint variables together with a new parameter. Unlike previous methods the effect of this is to alter the value of the surface function at its minimum point, but since the minimum occurs at the same point this is immaterial and the advantages in controlling convergence that flow from the presence of an extra free parameter are very significant. Thus a new function is minimised as follows:-

$$V(\underline{x}) = v(\underline{x}) + \sum_{i=1}^{m} r_i^{-1} (\psi_i(\underline{x}) + \Theta_i)^2$$

where ψ_i are the constraint variables; r_i^{-1} and Θ_i are parameters > 0 in which $V(\underline{x})$ does not approach $v(\underline{x})$ in value at the minimum although they both reach a minimum at the same $\underline{x}$ value. A very simple up-date procedure is used, in which $\Theta_i + \psi_i$ becomes the new Θ_i at the next step and convergence is speeded up by holding the product $r_i^{-1}.\Theta_i$ approximately constant. This is certainly the most numerically aimiable procedure for solving the constrained minimisation problem to date.

3. Steady-State Optimisation; An Application

In considering applications of these powerful numerical procedures it should be emphasised that there is a problem of significant intellectual content in dovetailing the statement of the technical problem with the mathematical formulation and the computational algorithm. It is all too easy to assume that a slight misfit will be unimportant, but it is particularly true with sophisticated optimisation procedures that posing ill-considered questions results in useless answers.

The current control interest in these procedures lies more in their use in function-space form for solving the dynamic case than in the more direct finite state form used for so-called static optimisation that has so far been described. However, there are a number of steady-state optimisation problems of interest for which the methods can be directly used in this form. A good example of this which illustrates the dovetailing dilemma well is the solution of the optimal load flow problem in an electric power supply system using mathematical programming techniques.

The technical problem has a number of special features, among them the requirement that generator loads are to be calculated in such a manner that the loss of any one line will not result in overload of any other line in the network. Such a loading schedule is said to be fully secure and it represents a complication of the constraint idea which has to be accommodated in the minimisation procedure. Again, although the optimum so calculated is a steady-state one the technical situation is not static, hence the time for computation allowable if the method is to be useful is five minutes at the outside. On the other hand, the economic benefits to be enjoyed from such a calculation increase rapidly with size of network for which they can be made, so that a less accurate method that takes a shorter time to compute may pay off handsomely as compared with a fully accurate method requiring longer time to calculate. The implications of this type of trade-off for the general problem of real-time on-line control are obvious enough.

As an example of a large supply system we consider a network abstracted from the CEGB supergrid system. The network consists of 149 busses fed by 177 generators. Leaving aside for the moment the question of line security, the problem of economic scheduling for such a system represents a constrained optimisation problem; owing to the load limits these constraints are non-linear. The problem also involves minimisation of a non-linear function arising from the ohmic character of the network losses. Furthermore the power flows are expressed as functions of a complex variable since we are concerned with an a.c. network giving rise to in-phase and quadratic components of power.

Optimisation of load flows for a network on this scale represents a formidable computation both as far as magnitude of storage required and the time to perform the computation. Very roughly the storage requirement is $n^2 + 2m^2$ where n is number of nodes (326 in this case), and m is number of generators (177 in this case); thus storage is approximately 160K locations. The time of computation is difficult to estimate, but on an IBM 7094 II it would be something over an hour. This could be shortened by use of decomposition techniques which lend themselves to application in a power system network with its loosely coupled sectors, and might then be reduced to about a fifth of this, 12 minutes say, and this is without the security provision.

A very much smaller network for which these calculations without security have been carried out and reported(20) is the IEEE 30 line, 3 generator standard test system. The computation utilises the new Powell technique for constrained optimisation and time on an IBM 7090 including input-output, admittance matrix formulation and optimisation is just over 2 minutes. If the larger problem could be decomposed into five sub-problems of comparable size to this one (which it cannot realistically, due to the large number of generators involved) then the computation time would be expected to exceed 5 x 2 = 10 minutes by the time taken to determine the linking parameters between sectors. One must be wary of comparing computation times in a proportional way, however, since these computational procedures are iterative in nature and depend very heavily on the values used to initiate the computation. Factors of 5:1 in timings due to this cause are common and more specialised procedures may be even more sensitive in this respect.

Clearly very considerable simplification is necessary if the target time of 5 minutes is to be achieved for the large network. In order to explore possibilities one goes to the extreme of framing the load flow problem in terms suited for treatment by linear programming techniques.(21) One virtue of the use of this method is that the security provision can readily be allowed for, but the requirement of linearity imposes severe restrictions on realism, the principal one being that all line losses are neglected. Most of the time in the computation is used in nodal matrix update, calculating the new line flows and finding the line flows following outages in the security calculation; these operations require m^2, $m\ell$ and $2\ell^2$ multiplications respectively, where m is number of generators and ℓ is number of lines. The time for each multiplication is the basic multiply time, together with the time taken for other orders in the same loop which average 15 simple instructions. Thus each component calculation takes $(m^2 + m\ell + 2\ell^2)$ x (multiply time + 15 instructions time). The number of component calculations is approximately equal to the number of generators m, although the number of vertices that have to be calculated will depend on the luck of the draw in each case so that computation times can vary widely. For example, using an IBM 7094 the m components for a 110 line 30 generator system take 36 sec. to compute but a very lucky calculation could be as short as 1 sec, which illustrates the sort of difficulty we face with on-line control using these methods. This method is strongly dependent upon the number of generators, consequently for the large system with 149 busses and 177 generators the time using m cycles of calculation comes out at 12 minutes 50 seconds and there is still difficulty even with this rather crude method in guaranteeing the 5 minute requirement.

It is however possible that an intermediate method might happen to fit the bill by giving acceptable accuracy by taking some account of losses at the expense of some other feature of relatively less importance. An attempt has been made to do this using network flow techniques(22) in which the security requirement is played down and losses are introduced by 'costing' tie-line flows. The method allows piece-wise linear approximation of cost so that quadratic costing can be approximated in segments, and in fact for U.S. generators fitted with turbine governing valves this is closer than a quadratic law. The network flow method has increasing advantage for a large number of generators and is more consistent in its timings. However no fully satisfactory method of taking security into account has so far been found. In the table below a comparison is given between timings for plausible linear programming fully secure calculations and network flow calculations omitting security.

System		L.P. timings	N.F. timings
Busses	Generators		
23	24	2.48 sec	1.53 sec
56	43	19.80 sec	8.60 sec
149	177	12 min 50 sec	1 min 19.45 sec

Timings based on IBM 7094 II.

A promising compromise for the large network would be to use the network flow method for the initial calculation using the result as a 'starter' for an LP calculation consequently done in very favourable time for the method (down by a factor of 5 in the table) and used to achieve a secure but inaccurate schedule; then returning to network flow for a final accurate and secure result. The whole might reasonably be achieved in the 5 minutes allowable.

Many engineers are sceptical of the value of such exercises as this. For one thing it is difficult to express the nuances of numerical analysis in practical engineering terms. It is also regrettably quite difficult to express the advantage of their use in hard and fast economic terms. Operational curiosities may arise unless the engineering requirements are transcribed with some care into analytical and numerical terms. The new insight arising from such an exercise may in itself result in a major part of the gain. Coming to terms with the numerical age is one of the penances that has to be paid for enjoying its advantages. Control engineers more than most are at the spearhead of this movement and are inevitably deeply involved in the conversion process.

4. Optimisation Techniques in Control System Design

4.1 Parametric optimisation techniques

During the early days of control system design, in what may be dubbed the self-contained engineering phase, questions of optimisation hardly arose. The linear servomechanism was a purely single-input single-output electro-mechanical device in which design measures were concentrated on questions of relative stability and steady-state accuracy. The analysis was carried out in frequency response graphical terms on a cut and try basis, while transient behaviour was judged by responses to sudden displacement from equilibrum.

In practice the design frequently reduced to the evaluation of circuit parameters for one of a limited range of stabilising networks of simple and standard structure. The earliest attempts at optimisation involved equating the variations to zero of a suitable performance measure due to change of these circuit parameters and solving the resulting algebraic relationships. This involved expressing the performance measure as an algebraic function of system parameters, but for simple criteria such as integral of error squared this proved possible by means of standard integral forms. The method, while successful, was nevertheless peculiarly cumbersome due to the difficulty of solving the resulting sets of non-linear algebraic equations for the required parameter values.

4.2 Variational optimisation

The first optimal method of control system design was based on the work of Wiener, using spectral density functions. This enabled input signals to be represented statistically and by subsequent extension of the method enabled r.m.s. constraints to be placed on system variables. The method was optimal in the sense that it utilised a criterion function based on mean square values and minimisation was carried out using the calculus of variations. However, it would not today be recognised as an example of optimal control. It was a good try at the real thing in its day. It was never very useful for multivariable systems owing to a difficulty in spectral factorisation required by the method. It is only recently by a quite different route through a study of quadratic minimisation using minimum realisations of state variable forms that a satisfactory solution to this has been found[23].

The route to optimal control as we know it, lay through the use of the state-vector formulation of control problems. The state-vector form of analysis makes it possible to readily adopt the viewpoint of variational calculus and furthermore to take advantage of the classical analysis of dynamic systems developed so extensively during the nineteenth century. The state-vector approach is a time-domain formulation that can handle non-linear time-varying characteristics subject to constraints. It is based on the use of matrix calculus, linear algebra and vector differential equations. Thus it lends itself very naturally to the analysis of multivariable systems. Since the technique of analysis is so radically different from those used previously in control system design, it has given rise to the concept of "modern control theory" by which tag it is identified as the new approach.

4.3 Minimum realisation of transfer functions

However, the state-variable analysis, while allowing new possibilities, exacts a certain price for the privilege. For example, for the multivariable case we may express dynamic relationships:-

$$\begin{aligned} \underline{\dot{x}} &= A\underline{x} + B\underline{u} \qquad &&\text{where } \underline{x} \text{ is an n-vector} \\ & &&\phantom{\text{where }} \underline{u} \text{ is an m-vector} \\ \underline{y} &= C\underline{x} &&\text{and } \underline{y} \text{ is an r-vector} \end{aligned} \qquad (1)$$

If all matrix positions are occupied we have $(n^2 + nm + nr)$ state coefficients to consider, but it is quite possible that a particular set of transfer functions of a multivariable system to be represented by less coefficients; particularly since many state variable realisations give the same set of transfer functions. The problem then in converting from transfer functions to state variable form is to ensure that the necessary but minimum number of coefficients are found, that is, the minimal realisation discussed by Kalman[24]. The diagram in Fig. 2 shows, using the controllable and observable concepts in control, that the realisation satisfying both these requirements has only $n(m + r)$ coefficients and this is the minimum number. An algorithm for obtaining the minimal realisation which proves to be far from straightforward[25] has been given by Mayne; the algorithm is also useful in the spectral factorisation problem mentioned above. It forms a necessary step in optimisation of practical multi-variable systems.

Many people have complained that the state-vector formulation has removed the possibility of retaining an engineering intuitive approach to control system design. This is not really so; the method is a lot less familiar and much less work has been put into systematising design procedures. For example, the counterpart of parameter optimisation of transfer functions discussed earlier lies in recent work on modal control[26] using the state-vector method. Using the equations (1) for the system, let the eigenvalues of A be $(\lambda_1, \ldots \lambda_n)$ assumed to be distinct then there exists an equivalent system defined by $\underline{x} = P.\underline{z}$ where the columns of P are the linearly independent eigenvalues of A. Thus:-

$$\dot{\underline{z}} = \Lambda.\underline{z} + b.\underline{u} \qquad \text{where } \quad = P^{-1}A.P \qquad (2)$$

$$\underline{y} = c.\underline{z} \qquad\qquad = \text{diag}\,(\lambda_1 \ldots \lambda_n)$$

If we wish to adjust the first s of the eigenvalues to new ones so that the resultant closed-loop system $\underline{x}' = (A - bc)^{-1}\underline{x}$ has eigenvalues $(\lambda_1^d \ldots \lambda_s^d, \lambda_{s+1} \ldots \lambda_n)$ where $\lambda_1^d \ldots \lambda_s^d$ are the new ones. We must choose c in a certain manner. It is possible to show[26] that values of the first s components of c, c_j say, have to take values given by a simple algebraic function of λ_j, λ_j^d and b_j; $j = 1 \ldots s$. If, on the other hand, the c's are fixed and cannot be chosen freely, then we have a difficult problem the solution to which has yet to be worked out.

One way round this difficulty which usually arises due to some components of the state-variable being unobservable involves the use of 'observers'[27]. As shown in Fig.3 the 'observer' regenerates an estimate of the state variable $\hat{x}$. This technique allows close control of the overall characteristic. The eigenvalues λ_i^f of the observer can be controlled by choice of Δ; the controller K is then chosen so that the system $x' = (A - BK)x$ has satisfactory eigenvalues λ_i^c; it can then be shown that the resulting overall system has eigenvalues $\lambda_i^f . \lambda_i^c$.

4.4 Quadratic minimisation

The popularity of quadratic measures as a criterion of performance lies in more than the mathematical ease with which such measures can be handled. The quadratic measure has a very fundamental role in any linear energy dependent system. The minimisation of quadratic measures leads us to such fundamental relationships as Kirchhoff's laws in linear electric networks. Two conclusions from this might be expected. First that much might already be known about handling the minimisation relationships and second that the resulting relationships would be fundamental in the analysis of linear systems. Both surmises are true, but if one expected them to be either direct or as a consequence easy to compute, one would be wrong.

This can perhaps be illustrated by consideration of Wiener's linear filter method. The dynamic behaviour of any linear system is described by means of linear differential equations and the solution of such equations may be greatly simplified by employment of a transform method; Wiener favoured Fourier transforms in the handling of which he was highly expert. But by their very nature the transforms, involving as they do integration over finite time epochs, yield steady-state solutions in a certain sense. They do not naturally lend themselves to finite time epoch solutions nor can they readily handle systems whose parameters are time-varying in nature and both of these cases are of interest in control systems. For these cases we have to solve the differential equations themselves directly. One thing, however, might be hoped for since the problem is linear and the measure quadratic so that the minimisation operation is also linear; this is the possibility of an algebraic solution. It turns out that the infinite time case that Wiener was concerned with is the only one that allows this and for this case we are better off with Wiener's method although it leads to difficulties with the multivariable system. For all other cases we have to resort to computational methods of optimisation.

A general form of quadratic performance index for the linear multivariable system is given by:-

$$V = \frac{1}{2}\int_{t_o}^{t_1} (\underline{x}'Q_1\underline{x} + 2\underline{x}'S\underline{u} + \underline{u}_1'R\underline{u}_1)dt$$

where Q_1 and S are non-negative definite and R is positive definite.

Regrouping the integrand as follows:-

$$L = \underline{x}'(Q_1 - SR^{-1}S')\underline{x} + (\underline{u}_1 + S'R^{-1}\underline{x})R(\underline{u}_1' + \underline{x}'SR^{-1})$$

$$= (\underline{x}'Q\underline{x} + \underline{u}'R\underline{u}) : \text{where } Q = Q_1 - SR^{-1}S > 0$$
$$\text{and } u_1 = \underline{u} + S'R^{-1}\underline{x}$$

leading to the more simple integral form with the same conditions on the matrices Q and R as above:-

$$V = \frac{1}{2}\int_{t_o}^{t_1} (\underline{x}'Q\underline{x} + \underline{u}'R\underline{u})dt$$

The linear dynamic behaviour is expressed by

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

The condition minimising V is given by the customary partial differential equation;-

$$\min \left[\frac{1}{2}(\underline{x}'Q\underline{x} + \underline{u}'R\underline{u}) + \frac{\partial V}{\partial \underline{x}} \cdot (A\underline{x} + B\underline{u})\right] = 0$$

from which the implicit minimising value of $\underline{u}$ is derived as

$$\underline{u} = -B'R^{-1}\frac{\partial V}{\partial \underline{x}}$$

With the advantage of hindsight we now generate a new positive definite symmetric matrix P, in place of the embarrassing partial $\frac{\partial V}{\partial \underline{x}}$, which we can show has always stable eigenvalues. The solution to the problem then rests on the properties, characteristics and values of this matrix. These are revealed to us as the solution of the following matrix Riccati differential equation:-

$$-\dot{P} = A'P + PA + Q - PBR^{-1}B'P \; ; \quad P(t_1) = 0.$$

Clearly for the infinite time case we are concerned with the solution of the r.h.s. of this equation equated to zero which is an algebraic equation but with many non-linear terms. It is easier to find the steady-state solution of the d.e. Various computer programs have been written to do this, but more efficient iterative procedures have been found by Kleinman[28] which will work just as well for the finite time case. Rearranging the equation:-

$$-P' = (A^*)'P + PA^* + Q + PBR^{-1}B'P \; ; \quad \text{where } A^* = A - BR^{-1}B'P$$

For a steady-state solution we have:-

$$0 = A'P + PA + Q + PBR^{-1}B'P \; ;$$

where recursively $\begin{cases} P_k = R^{-1}B'P_{k-1}; \; k=1,2\ldots \\ A_k = A - BP_k \end{cases}$ and where P_o

is chosen such that matrix $A_o = A - BP_o$ has eigenvalues with negative real parts (i.e. $P_o > 0$), then:-

$$P > P_{k+1} > P_k \ldots \; ; \; k=0,1,\ldots$$

and $$\lim_{k \to \infty} P_k = P \; .$$

Thus the method is one of successive substitutions for which a proof of convergence has been given by Wonham The procedure is the same as applying Newton's method to the solution of the original steady-state equation but with the advantage of guaranteeing monotonic convergence. In addition the iterations P_k are quadratically convergent unlike methods based on solution of the d.e. itself for its steady-state solution which only have linear convergence. P_k requires the solving of an n(n+1)/2 dimensional linear vector equation. Kleinman states 10th order variable cases require about 10 iterations to reach P.

This state-variable formulation of the linear multi-variable problem has many advantages. It can deal with the finite-time case; it can handle time-varying coefficients in the matrices. There are efficient computational methods of solution. Being in a linear form enables considerable extensions of the method to be explored. Some work has been done in trying to cope with the constrained form of control variable[(29)] and a method has been given for handling piece-wise linear control[(30)]. As so often with linear methods considerable extension is possible for approximate solution of problems that are not strictly linear at all. Also extension of ideas from single-input single-output linear systems to the multivariable case. Anderson[(31)] has shown how the idea of return difference may be generalised in this way and used to reduce the sensitivity of multivariable systems. He also shows that a system whose sensitivity to process parameter variations has been reduced is also optimal for some quadratic loss function. This is one area of optimal control studies where general relationships can be obtained; in general they cannot. It seems possible that with further development this manageable method will see extensive practical application for multivariable systems.

4.5 <u>Optimisation techniques for general linear systems</u>

There are many control problems in which the system is linear but performance cannot be realistically expressed in terms of a quadratic measure. Time optimal control is an example in which given end point conditions have to be reached in minimum time, or again terminal time is decided by reaching a specified terminal manifold. The simplifications obtained for the special case of quadratic minimisation no longer apply. Even the proof of the necessary conditions of optimality take us into deep water. The vehicle of the calculus of variations looks increasingly incommodious. It is now almost standard practice to take up the viewpoint of reachable convex sets rather than to deal with the derivatives needed in the calculus of variations.

The proof of Pontriagin's Maximum Principle by Halkin[32] uses convexity theory and a Lyapunov theorem to show that the reachable set is convex. This disposes of the matter for the continuous time case but makes clear the dilemma for the discrete time case where the convexifying effect of continuous time is missing. Thus only a much weaker statement can be made in such cases and that covering a restrictive class of linear convex systems only.

Halkin goes on to attempt a proof for the non-linear case using Brouwer fixed point theory, taking us deep into the territory of the topologists where we may feel a need for professional allies. Increasingly in our subject, however, we can only avoid going so far afield to the neglect of where the subject is leading us. This theme comes much nearer home in the next section.

5. Optimisation Techniques for Non-linear Systems

It is when considering the design of non-linear control systems that numerical optimisation procedures really come into their own. The long history of attempts to solve non-linear problems by classical closed-form analytic methods shows that general methods are unlikely and that even when analytical methods can be found they can only cover a highly specialised range of problems, often of no very great practical interest. Thus numerical methods have to be resorted to and it is customary in view of the substantial amount of numerical manipulation involved to make use of computers for this.

While it was natural to seek to use the calculus of variations for these more difficult non-linear cases, the limitations of the method soon made themselves felt. The resulting two-point-boundary-value problem had to be solved by numerical procedures. The simplest version using first variations has poor convergence properties arising from the small range of variations for which the approximation is valid. It was not possible in any case to find the solution for a constrained control variable, although convergence properties could be markedly improved by the use of second variations[33]. However a heavy price had to be paid for this in terms of increased amount of computation required and the necessity to solve extra so-called 'accessory' equations.

A wide range of gradient methods was proposed mainly dividing into two main groups; namely i) those that employed successive approximation and sought to improve on a nominal control trajectory satisfying boundary conditions[34], and ii) those that embodied the implicit minimising conditions for optimality into the equations and either selected or sought to force these to match the given boundary conditions[35]. Only the matching force procedures were really starters in this group.

For a while the situation became a free-for-all in which a rash of algorithms for both types of approach were suggested with gay abandon and very little thought given to the computational implications of the recommendations. Many of them were of doubtful numerical integrity. Some were shown to be successful on selected problems but these might have been cases of lucky (or careful) selection. Very few were applied to a range of problems of practical interest or complexity.

In the last three years a very much more healthy state of affairs is coming about. The quest has all along been to find methods with good quadratic convergence characteristics. This has meant looking at steepest descent methods as a special case of Newton's method and in turn Newton's method as a special case of successive approximation. Thus attention has turned to the convergence properties of general iterative procedures. A recently translated essay by Kantarovich[36] contains all the known theorems regarding convergence of Newton's method of linear space and function space. Useful work on convergence proofs by Zangwill[37] and Polak[38] is noteworthy, although the practical significance of these results remains to be worked out. It is clear, however, that the serious study of the convergence properties of algorithms for optimal control is going to bring order out of the previous chaos, and in the process is turning up problems of quite impressive and fundamental mathematical interest. The field of control in one of its quite practical aspects of computational procedures for optimal control is proving to yield matters of respectable concern to pure mathematicians. It is the unprecedented range of applied and theoretical knowledge from elementary mechanics to theories of convergence algorithms that so singularly distinguishes the topic of control and perhaps gives some explanation of why its full development has not occurred earlier.

5.2 Efficient function space optimisation procedure

Evidence of real progress may be noted in the way in which computational techniques developed for finite space application such as conjugate directions and variable metric or the introduction of constraints using interior point methods (Fiacco and McCormack, SUMT) have been successfully transcribed into more powerful function space methods. As an example we show how the functional analysis approach allows the use of the conjugate gradient method for determination of dynamic optimal control[39]. The analogy with the finite dimensional form is indicated in brackets:-

$$\text{Performance index} : V = \int L \, dt$$

$$\text{Dynamics} : \dot{\underline{x}} = f(\underline{x},\underline{u},t)$$

$$\text{'Gradient'} \ (g) \ \text{is} \ H_u = L_u + \lambda . f_u \ .$$

Thus change in control $\underline{u}$ at the ith iteration is given by:-

$$\delta\underline{u}_i = H_{u_i} + \frac{< H_{u_i}, H_{u_i} >}{< H_{u_{i-1}}, H_{u_{i-1}} >} . \delta\underline{u}_{i-1}$$

$$(s_i) = (g_i) + (\beta_i) . (s_{i-1})$$

where $< H_{u_i}, H_{u_i} >$ is interpreted as $\int < H_{u_i}, H_{u_i} > dt$ then the change in performance index is given by

$$\delta V = \int < H_{u_i}, \delta u_i > dt \ .$$

Application of the method proceeds as follows:-

(1) Integrate adjoint set of equations to obtain $\lambda(t)$

(2) Obtain current 'gradient' $H_{u_i} = L_{u_i} + \lambda . f_{u_i}$

(3) Integrate to obtain $< H_{u_i} , H_{u_i} >$

(4) Recall $< H_{u_{i-1}} , H_{u_{i-1}} >$ calculated on previous run

(5) Work out $\delta \underline{u}_i = H_{u_i} + \dfrac{< H_{u_i} , H_{u_i} >}{< H_{u_{i-1}} , H_{u_{i-1}} >} . \delta \underline{u}_{i-1}$

(6) Choose step length α such that

$$\min_{\alpha} V (\underline{u}_{i-1} + \alpha \delta \underline{u}_i)$$

Function space methods have a role to fill in other aspects of control technique; a good example is given in a Congress paper on modelling[40]. Another is the paper on Differential Dynamic Programming[41]. This method is a combined method of solving optimal control problems; that is to say, for local variations it is an embedding method based on dynamic programming (but applied to a possibly non-optimal trajectory); for global variations it is a successive approximation method.

The method is initiated by choice of a nominal control trajectory $\overline{\underline{u}}(t)$ which enables a nominal state-variable vector $\overline{\underline{x}}$ to be found. Using the Principle of Optimality it is now possible to adopt $\underline{u}^*$ (the implicit minimising $\underline{u}$) as the control variable, omitting for the moment its dependence upon the state-variable $\underline{x}$. A second order expansion is made about $\underline{u}^*$ and $\underline{x}$ in order to find how the actual dependence of control $\underline{u}$ on state $\underline{x}$ may be taken into account in correcting $\underline{u}$; thus an incremental relationship $\delta \underline{u} = \beta . \delta \underline{x}$ is established in which $\beta = - H_{uu}^{-1} (H_{ux} + f_u^T V_{xx})$. (Note that H_{uu}^{-1} is not required globally, only locally; in contradistinction to dynamic programming). Finally, then, the new control $\underline{u} = \overline{\underline{u}} + \delta \underline{u}^* + \beta \, \delta \underline{x}$ and the solution has a term dependent on feedback from states, has less equations than with the calculus of variations and has quadratic convergence properties.

The method has been applied to control constrained solutions which the calculus of variations cannot handle and has also been employed to solve bang-bang cases such as come out of Pontriagin's method; it can also deal with terminal manifold and implicit time cases. Proof of convergence has been given for a restricted class of problems.

6. Optimisation Techniques for Stochastic Systems

6.1 Estimation of parameters for linear discrete systems

Maximum likelihood methods of estimating the parameters of a dynamic system yield models with theoretically attractive properties. For the very simplest forms of model these lead to the well-known least squares method or some generalisation of this in which case the methods of quadratic minimisation are available. However, for industrially realistic models a more complex structure is necessary and computational hill-climbing of a non-quadratic function is required as part of the estimation procedure. The fairly simple case dealt with by Åström[42]:-

$$A(z^{-1})\ y(t) = B(z^{-1})\ u(t) + C(z^{-1})\ e(t)$$

yields an error $e = \frac{1}{C}\ (Ay - Bu)$ which is a non-linear function of A, B and C and is typical of the difficulty in using desirable estimation procedures. Bohlin[43] has used a slightly more general case as a successful basis for industrial applications:-

$$y(t) = \frac{B(z^{-1})}{A(z^{-1})}\ u(t-\tau) + \lambda.\frac{C(z^{-1})}{D(z^{-1})}\ .\ e(t)\ .$$

Techniques have been worked out and programmed for deciding the order of the polynomials in the delay operator, z^{-1}, for the functions A, B, C and D and for delay parameter, τ; for performing the necessary hill-climbing routine and finally for calculating a minimum variance control law.

6.2 Estimation of parameters for transition probability methods

Many of the difficulties of applying dynamic programming disappear if variables can be defined as a limited number of discrete states. The "horrors of the expanding grid" are then avoided, as indeed is the "curse of dimensionality" if the transitions from one state to another can in addition be considered to occur in a discrete fashion. Furthermore, if the probability of any such transfer is fixed, known and independent of any others then we are dealing with a first order Markov Process about which much is known and very considerable simplification of optimal strategies results. A simple converging iterative procedure has been given by Howard[44].

In most physical processes, however, these transition probabilities are not known àpriori and so have to be estimated as control action proceeds. Thus it is necessary during the course of assessing these transition parameters to take actions which seem in retrospect to be non-optimal from the control point of view. The ideal strategy which has been given by Riordon[45] is that strategy which extracts the maximum decrease in error probability as a result of its use and reveals the close relationship between the estimation procedure and the system cost function. He shows that the relationship is expressed as a simple equality constraint which is readily implemented as an on-line control strategy. An example of its application to the adaptive ordering of power generation has been given[46].

6.3 Optimal stochastic control

Understandably early attempts to solve this most general problem were made on the basis of linear systems with quadratic performance indices and assumptions of Wiener Gaussian noise (WGN) disturbances only. Resort was made to the separation theorem whereby a wider range of linear system and WGN problems could be replaced by a much smaller range depending on current value of the state, $\underline{x}$. The separation occurred between a filter to yield the best estimate, $\hat{\underline{x}}$ and a control system giving quadratic minimisation based on assumed knowledge of state. The latter we have already mentioned and the former takes a very similar form, differing only in detail; thus for the filter we have:-

$$\dot{\underline{x}} = A\underline{x} + B\dot{\underline{\omega}} \qquad \text{where } \dot{\omega} \text{ and } \dot{v} \text{ are WGN of intensity}$$
$$\dot{\underline{y}} = C\underline{x} + \dot{\underline{v}} \qquad Q \text{ and } R \text{ respectively.}$$

Then it can be shown that[47]:-

$$\dot{\hat{\underline{x}}} = A\underline{x} + \Delta(\underline{z} - C\underline{x}) \qquad \text{where } \underline{z} = \dot{\underline{y}} + \dot{\underline{v}}$$
$$\text{and } \Delta = PC'R^{-1}$$

P is given by solution of the matrix Riccati d.e.:-

$$\dot{P} = AP + PA' - PC'R^{-1}CP + BQB'$$

Once we depart from the linearity, however, difficulties increase enormously and in a recent stringent commentary on control systems subject to stochastic disturbances we have Wonham[47] asking the question, 'Is the complication of including stochastics worthwhile?' In general, he opines, only marginally unless noise level is high in which case the system is useless anyway! Others with more courage (foolhardiness?) remain unconvinced and are making steady inroads into solving the computational difficulties[49]. Monte Carlo techniques seem particularly suited to these problems, which are difficult to solve by other methods, This works well when the problem is reduced to determination of parameters of a parameterised control law. The antithetic variate method of variance reduction can yield substantial improvement in the accuracy of estimates and is simple to apply. It requires a linearised model of the non-linear system and can be regarded as a Monte Carlo procedure to improve such an approximate solution[50].

Wonham is possibly too knowledgeable of the theoretical intricacies of the field to be optimistic; as he says "the only cure for dynamic disturbance is tight feedback and large control force; with fixed constraints on computation capacity and control force little can be achieved by subtle changes in control logic". The gap, however, between what is currently achieved and what should desirably be achieved is so great that Wonham's 'little' is perhaps worth going for, even if it does imply an outrageous degree of crude approximation in the computation. The matter needs further study.

7. On-line computations

No survey on computational methods of control should finish without at least a mention of one of the most depressing gaps in our knowledge. Mention has already been made of the difficulties of dovetailing steady-state optimisation methods to the circumstances of technical problems. These problems are many times compounded when it is a question of dovetailing, in terms suitable for real-time on-line systems, the demands of dynamic optimising control algorithms with the realities of computer programming and performance. Currently the problems of writing programs for on-line control are solved in an ad-hoc manner. Ardous efforts go into unravelling the knotty tangles that arise due to the presence of difficult timing problems. Owing to the lack of background theory for these operations control theory has been little influenced towards accommodating the difficulties. It seems inevitable that these programming difficulties will in due course have repercussions on desirable forms of control system analysis and on desirable extensions to control theory itself[51].

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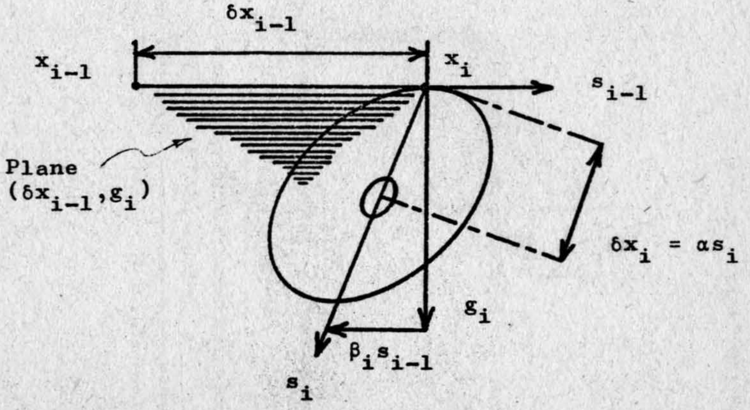

Fig.1 Conjugate Gradient Method

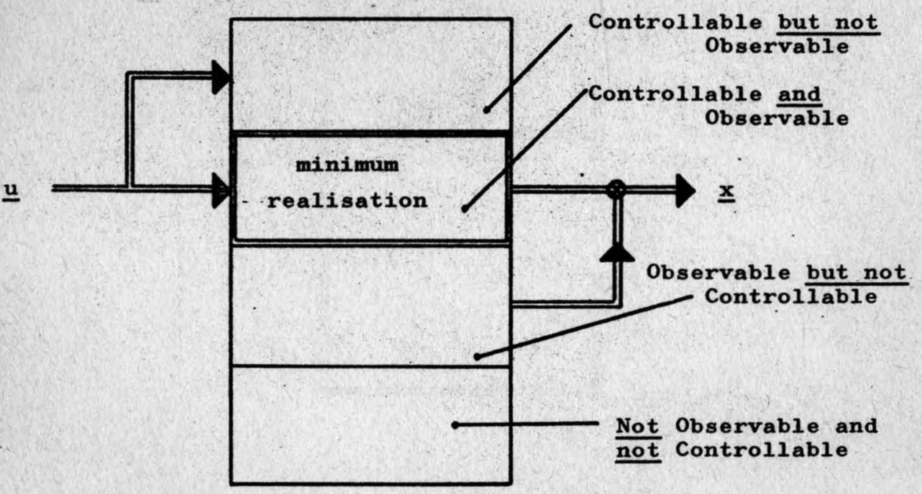

Fig.2 Minimum Realisation Concept

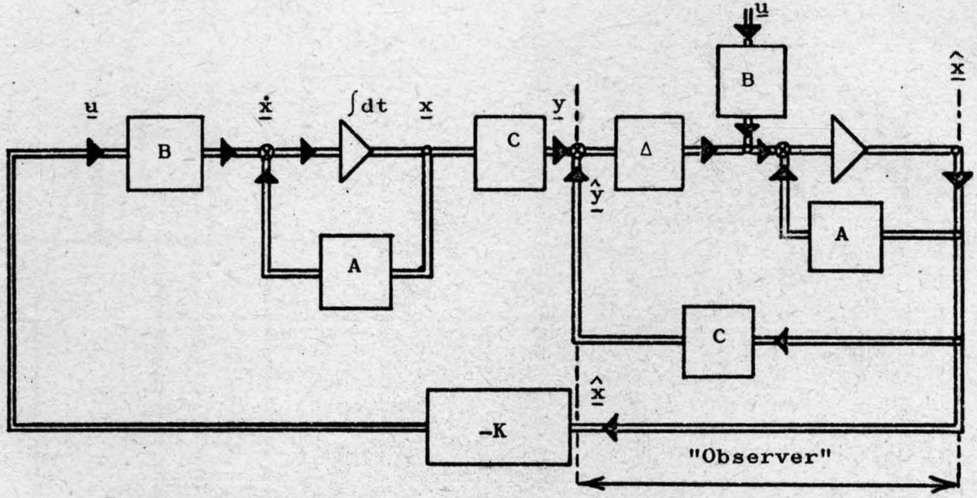

Fig.3 Use of 'Observers' for Regeneration of State Variable